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Iwasawa Theory for Extensions with Restricted p-Ramification

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1. Introduction

Let p be a prime number. For an algebraic number field K of finite degree, consider an (arbitrary) subset T of $P(K)$, where $P(K)$ is the set of all primes above p of K:

$$T \subset P(K)\,.$$

Let K_∞ be the cyclotomic $\mathbf{Z}_p$-extension of K and $T_\infty \subset P(K_\infty)$ the set of primes above T of K_∞. Then, by $\mathcal{M}_{T_\infty}(K_\infty)$, we denote the maximal abelian p-extension of K_∞ unramified outside T_∞. We call such an extension "the extension with restricted p-ramification".

Since $\Gamma := \mathrm{Gal}(K_\infty/K)$ acts on the Galois group

$$\mathcal{Y}_{T_\infty}(K_\infty) := \mathrm{Gal}(\mathcal{M}_{T_\infty}(K_\infty)/K_\infty)$$

by conjugation, it is regarded as a module over the power series ring $\Lambda := \mathbf{Z}_p[[T]]$ in the usual manner. This is finitely generated over Λ.

In this article, we investigate the following question: What are the Λ-rank of $\mathcal{Y}_{T_\infty}(K_\infty)$ and the μ-invariant of its Λ-torsion part $\mu(\mathcal{Y}_{T_\infty}(K_\infty)_{\Lambda-tor})$?

When $T = \varnothing$ (empty set), it is well known that $\mathcal{Y}_\varnothing(K_\infty)$ has Λ-rank zero by a result of Iwasawa and that it is conjectured that its μ-invariant vanishes. This is verified when K is an abelian field by Ferrero and Washington [FeWa]. It is also known that $\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) = r_2$ if $T = P(K)$, where r_2 is the number of complex primes of K. The μ-invariant of the Λ-torsion part of $\mathcal{Y}_{T_\infty}(K_\infty)$ is also conjectured to be zero and proved if K is abelian.

In case of CM-fields, the answer to the above question is known completely (cf. [JaMa]. See also Theorem 4.5 below).

On the other hand, for a general base field K and $T \subset P(K)$, we have a trivial lower bound of the Λ-rank (Proposition 2.3):

$$\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) \geq r_2 - \sum_{v \in P(K)-T} [K_v : \mathbf{Q}_p]\,.$$

However, we do not know how the Λ-rank should be in general. We give the following partial result by applying the methods of Ax and Brumer.

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THEOREM 1.1 (Theorem 6.2). *Assume there exists a subfield $k \subset K$ such that K/k is Galois and $K \cap k_\infty = k$. Let $G := \mathrm{Gal}(K/k)$. Assume that there exists a prime $u \in P(k)$ such that*

$$T' := \{v \in P(K) \mid \ v|u\}$$

is contained in T. Then we have

$$\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) \leq \Bigl(\sum_{v \in T}[K_v : \mathbf{Q}_p]\Bigr) - \Bigl(\sum_{\chi \in \Delta_{K/k}} \deg \chi\Bigr) - \delta\,.$$

Here, $\Delta_{K/k}$ is the set of the distinct irreducible characters of G over $\bar{\mathbf{Q}}$ which appear in the $\bar{\mathbf{Q}}[G]$-module $\mathcal{E}_K \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}$ where $\mathcal{E}_K$ is the group of global units of K. We put

$$\delta = \begin{cases} 0 & \text{if } \Delta_{K/k} \text{ contains the trivial character}\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Note that the right hand side in the Theorem is larger than or equal to the above trivial lower bound.

Next, we consider the special case where $K = \mathbf{Q}(\sqrt[3]{a})$ ($a \in \mathbf{Z}$, cube free), as a first example of the case of non abelian base fields.

THEOREM 1.2 (Theorem 7.3, Proposition 7.8). (i) *Let $K = \mathbf{Q}(\sqrt[3]{a})$. Let p be an odd prime such that $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in K where $K_{\mathfrak{p}_1} = \mathbf{Q}_p$ and $[K_{\mathfrak{p}_2} : \mathbf{Q}_p] = 2$. Let $T = \{\mathfrak{p}_2\}$. Then $\mathcal{Y}_{T_\infty}(K_\infty)$ is Λ-torsion.*

(ii) *Further, there is a sufficient condition for the vanishing of $\mu(\mathcal{Y}_{T_\infty}(K_\infty))$.*

The another reason why we consider this special example is that $\mathcal{Y}_{T_\infty}(K_\infty)$ in the above theorem for $p = 3$ is related with the Selmer group $\mathrm{Sel}_{p^\infty}(E/\mathbf{Q}_\infty)$ of a certain elliptic curve $E/\mathbf{Q}$ concerning with the μ-invariant ([Ha1], [Ha2]).

There is another application of the theory of $\mathcal{Y}_{T_\infty}(K_\infty)$. Let $\lambda_p(K)$, $\mu_p(K)$ and $\nu_p(K)$ be the classical Iwasawa invariants of K. (See §8 for the definition.) We give a criterion for the vanishing of these invariants for special non abelian fields. This is a generalization of a result of Fukuda-Komatsu([FuKo]).

THEOREM 1.3 (Theorem 8.1). *Let K be a number field. Assume that there are exactly two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of K above p such that $K_{\mathfrak{p}_1} = \mathbf{Q}_p$ and that they are totally ramified in K_∞. Then the following are equivalent.*

(i) $\lambda_p(K) = \mu_p(K) = \nu_p(K) = 0$

(ii) $\mathrm{Cl}(K)[p^\infty] = 0$ *and* $(1 + p\mathbf{Z}_p) = \overline{\mathcal{E}_K \cap (1 + p\mathbf{Z}_p)}$.

Here, $\mathrm{Cl}(K)[p^\infty]$ *is the p-part of the ideal class group of K and $\mathcal{E}_K$ is the group of global units of K which we embed in $K_{\mathfrak{p}_1} = \mathbf{Q}_p$.*

The outline of this article is as follows. From §2 to §4 we give general facts on $\mathcal{Y}_{T_\infty}(K_\infty)$ in the context of the classical Iwasawa theory. After seeing an application of the methods of

Ax and Brumer in §5, we give an upper bound of the Λ-rank of $\mathcal{Y}_{T_\infty}(K_\infty)$ (Theorem 6.2) in §6. Then, in §7 we consider a special case where $K = \mathbf{Q}(\sqrt[3]{a})$ and apply the above results to this case. In §8, we give a proof of Theorem 1.3.

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2. Extensions with restricted p-ramification

In this section, we define some notions related to extensions with restricted p-ramification. Then we recall known results and easy consequences.

Let p be a prime. For an algebraic number field K, let $P(K)$ be the set of all primes of K above p. For an arbitrary subset

$$T \subset P(K),$$

we denote by $\mathcal{M}_T(K)$ the maximal abelian p-extension of K which is unramified outside T. Let

$$\mathcal{Y}_T(K) := \mathrm{Gal}(\mathcal{M}_T(K)/K).$$

We also denote by $\mathcal{M}'_T(K)$ the maximal subfield of $\mathcal{M}_T(K)$ all of whose primes above $P(K) - T$ are completely decomposed and put

$$\mathcal{Y}'_T(K) := \mathrm{Gal}(\mathcal{M}'_T(K)/K).$$

Let K be a number field of finite degree and K_∞ the cyclotomic $\mathbf{Z}_p$-extension of K. For $T \subset P(K)$, let $T_\infty \subset P(K_\infty)$ be the set of primes above T. Then

$$\Gamma = \mathrm{Gal}(K_\infty/K)$$

acts on $\mathcal{Y}_{T_\infty}(K_\infty)$ and $\mathcal{Y}'_{T_\infty}(K_\infty)$ by conjugation in the usual way. Therefore $\mathcal{Y}_{T_\infty}(K_\infty)$ and $\mathcal{Y}'_{T_\infty}(K_\infty)$ are endowed with the action of

$$\Lambda := \mathbf{Z}_p[[\Gamma]].$$

By fixing a topological generator of Γ, we identify Λ with the power series ring $\mathbf{Z}_p[[T]]$ in the usual manner.

For $T = \varnothing$ (empty set), we put

$$H(K) := \mathcal{M}_\varnothing(K), \quad H'(K) := \mathcal{M}'_\varnothing(K)$$

and

$$A(K) := \mathcal{Y}_\varnothing(K), \quad A'(K) := \mathcal{Y}'_\varnothing(K).$$

For $T = P(K)$, we put $M(K) := \mathcal{M}_{P(K)}(K)$ and

$$\mathfrak{X}(K) := \mathcal{Y}_{P(K)}(K)\,.$$

We define similarly $A(K_\infty)$, $A'(K_\infty)$ and $\mathfrak{X}(K_\infty)$ for K_∞. These modules are well studied in Iwasawa theory.

THEOREM 2.1 (Iwasawa [Iw], see also [Wa]). (i) *$\mathfrak{X}(K_\infty)$ is a finitely generated Λ-module and* $\mathrm{rank}_\Lambda(\mathfrak{X}(K_\infty)) = r_2$*, where r_2 is the number of complex infinite primes of K.*

(ii) *Both $A(K_\infty)$ and $A'(K_\infty)$ are Λ-torsion. Further, $\mu(A(K_\infty)) = \mu(A'(K_\infty))$.*

Note that

$$H(K_\infty) \subset \mathcal{M}_T(K_\infty) \subset M(K_\infty) \quad \text{and} \quad H'(K_\infty) \subset \mathcal{M}'_T(K_\infty) \subset M(K_\infty)\,.$$

Thus $\mathcal{Y}_T(K_\infty)$ and $\mathcal{Y}'_T(K_\infty)$ are finitely generated over Λ.

Let v be a prime of K dividing p and w a prime of K_∞ above v. Denote K_v by the completion of K at v and by $K_{\infty,w}$ the composite field $K_v K_\infty$ in $\overline{K_v}$ where we identify K_∞ with its image of the embedding $K_\infty \hookrightarrow \overline{K_v}$ corresponding to w. Let

$$\begin{cases} X(K_{\infty,w}) := \mathrm{Gal}(K_{\infty,w}^{\mathrm{ab},p}/K_{\infty,w}) \\ X'(K_{\infty,w}) := \mathrm{Gal}(K_{\infty,w}^{\mathrm{ab},p}/K_{\infty,w}^{\mathrm{ur},p}) \end{cases}$$

where $K_{\infty,w}^{\mathrm{ab},p}$ is the maximal abelian p-extension of $K_{\infty,w}$ and $K_{\infty,w}^{\mathrm{ur},p}$ is the maximal unramified p-extension of $K_{\infty,w}$. Then $\Gamma_w := \mathrm{Gal}(K_{\infty,w}/K_v)(\cong \mathbf{Z}_p)$ acts on $X(K_{\infty,w})$ and $X'(K_{\infty,w})$ by conjugation and thus these are $\Lambda_w := \mathbf{Z}_p[[\Gamma_w]]$-modules. Further,

$$\bigoplus_{w|v} X(K_{\infty,w}) \cong \Lambda \otimes_{\Lambda_w} X(K_{\infty,w}) \quad \text{and} \quad \bigoplus_{w|v} X'(K_{\infty,w}) \cong \Lambda \otimes_{\Lambda_w} X'(K_{\infty,w})$$

as Λ-modules and it is known that

THEOREM 2.2 ([Iw] Theorem 25).

$$\mathrm{rank}_\Lambda\biggl(\bigoplus_{w|v} X(K_{\infty,w})\biggr) = \mathrm{rank}_\Lambda\biggl(\bigoplus_{w|v} X'(K_{\infty,w})\biggr) = [K_v : \mathbf{Q}_p]\,.$$

Further, $\mu((\bigoplus_{w|v} X(K_{\infty,w}))_{\Lambda\text{-}tor}) = 0$. Here, $(\bigoplus_{w|v} X(K_{\infty,w}))_{\Lambda\text{-}tor}$ is the maximal Λ-torsion submodule of $\bigoplus_{w|v} X(K_{\infty,w})$.

In particular, if p is odd and K does not contain the group of p-th roots of unity μ_p, then $\bigoplus_{w|v} X(K_{\infty,w}) \cong \Lambda^{\bigoplus [K_v:\mathbf{Q}_p]}$.

Since we have the exact sequences

$$\bigoplus_{v\in P(K)-T}\biggl(\bigoplus_{w|v} X'(K_{\infty,w})\biggr) \to \mathfrak{X}(K_\infty) \to \mathcal{Y}_{T_\infty}(K_\infty) \to 0 \quad \text{and} \tag{1}$$

$$\bigoplus_{v\in P(K)-T}\Bigl(\bigoplus_{w|v} X(K_{\infty,w})\Bigr)\to \mathfrak{X}(K_\infty)\to \mathcal{Y}'_{T_\infty}(K_\infty)\to 0\,, \tag{2}$$

we have

PROPOSITION 2.3.

$$\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) \geq r_2 - \sum_{v\in P(K)-T}[K_v:\mathbf{Q}_p]\,.$$

We also have

$$\bigoplus_{v\in P(K)-T}\Bigl(\bigoplus_{w|v} X(K_{\infty,w})/X'(K_{\infty,w})\Bigr)\to \mathcal{Y}_{T_\infty}(K_\infty)\to \mathcal{Y}'_{T_\infty}(K_\infty)\to 0\,. \tag{3}$$

Because $X(K_{\infty,w})/X'(K_{\infty,w})$ is isomorphic to $\mathbf{Z}_p$, we have:

PROPOSITION 2.4.

$$\mathrm{rank}_\Lambda \mathcal{Y}_{T_\infty}(K_\infty) = \mathrm{rank}_\Lambda \mathcal{Y}'_{T_\infty}(K_\infty) \quad \textit{and} \quad \mu(\mathcal{Y}_{T_\infty}(K_\infty)_{\Lambda-tor}) = \mu(\mathcal{Y}'_{T_\infty}(K_\infty)_{\Lambda-tor})\,.$$

REMARK 2.5. When K is abelian over an imaginary quadratic field k in which p splits and when T is the set of all the primes above one of the primes of k dividing p, $\mathcal{Y}_{T_\infty}(\tilde{K})$ has been considered in relation with the Iwasawa theory of CM-elliptic curves. Here $\tilde{K}/K$ is a certain $\mathbf{Z}_p$-extension which is not cyclotomic (cf. [Co]).

3. The extensions over finite number fields

Let p, K and $T\subset P(K)$ be as in §2. Let

$$Z_T(K) := \mathrm{Gal}(\mathcal{M}_T(K)/H(K))\,.$$

Then, we have the exact sequence

$$0\to Z_T(K)\to \mathcal{Y}_T(K)\to A(K)\to 0\,. \tag{4}$$

Assume $[K:\mathbf{Q}]\leq\infty$. Let $\mathcal{E}_K$ be the group of global units of K. For a prime v of K, let $\mathcal{U}_v$ be the group of principal units in K_v. Put

$$\mathcal{U}_{K,T} := \prod_{v\in T}\mathcal{U}_v.$$

Then we have the following:

PROPOSITION 3.1. *$Z_T(K)\cong \mathcal{U}_{K,T}/(\overline{\mathcal{E}_K\cap\mathcal{U}_{K,T}})$. Here, $\overline{\mathcal{E}_K\cap\mathcal{U}_{K,T}}$ is the topological closure of $\mathcal{E}_K\cap\mathcal{U}_{K,T}$ in $\mathcal{U}_{K,T}$.*

PROOF. The proof goes exactly on similar line to the proof of [Wa] Theorem 13.4. Let $\tilde{H}$ be the maximal unramified abelian extension of K and $\tilde{M}_T$ the maximal abelian extension of K unramified outside T. By the class field theory,

$$\mathrm{Gal}(\tilde{M}_T/\tilde{H}) \cong \left(K^\times \prod_{v:all} U_v\right) \Big/ \left(\overline{K^\times \prod_{v\notin T} U_v}\right).$$

Here $(K^\times \prod_{v:all} U_v)$ and $(\overline{K^\times \prod_{v\notin T} U_v})$ are considered as subgroups of the idele group of K, where U_v denotes the group of whole local units of K_v. The right hand side is isomorphic to $\prod_{v\in T} U_v/(\overline{K^\times \prod_{v\notin T} U_v} \cap \prod_{v\in T} U_v)$. By the same argument as in [Wa] Lemma 13.5, $(\overline{K^\times \prod_{v\notin T} U_v} \cap \prod_{v\in T} U_v) = \overline{\mathcal{E}_K}$. By taking the p-part, we have the proposition. □

For $\mathcal{Y}'_T(K)$, we see the following: let $H_T(K)$ be the maximal unramified abelian p-extension of K whose primes above $P(K)-T$ are all completely decomposed. Let $A_T(K) :=$ $\mathrm{Gal}(H_T(K)/K)$. Then $A_T(K)$ is a quotient of $A(K)$ and $A'(K)$ is a quotient of $A_T(K)$. Let $Z'_T(K) := \mathrm{Gal}(\mathcal{M}'_T(K)/H_T(K))$. Then we have

$$1 \to Z'_T(K) \to \mathcal{Y}'_T(K) \to A_T(K) \to 1\,. \tag{5}$$

PROPOSITION 3.2. *Let $\mathcal{E}_{K,(P(K)-T)}$ be the group of $(P(K)-T)$-units of K. Then we have $Z'_T(K) \cong \mathcal{U}_{K,T}/(\overline{\mathcal{E}_{K,(P(K)-T)}} \cap \mathcal{U}_{K,T})$.*

For the proof, let $\tilde{H}_T$ be the maximal unramified abelian extension of K whose primes above $P(K)-T$ are all completely decomposed and $\tilde{M}'_T$ the maximal abelian extension of K which is unramified outside T and all of whose primes above $P(K)-T$ are completely decomposed. Then

$$\begin{aligned}\mathrm{Gal}(\tilde{M}'_T/\tilde{H}_T) &\cong \left(K^\times \prod_{v\notin P(K)-T} U_v \prod_{v\in P(K)-T} K_v^\times\right) \Big/ \left(\overline{K^\times \prod_{v\notin P(K)} U_v \prod_{v\in P(K)-T} K_v^\times}\right)\\ &\cong \prod_{v\in T} U_v \Big/ \left(\left(\overline{K^\times \prod_{v\notin P(K)} U_v \prod_{v\in P(K)-T} K_v^\times}\right) \cap \prod_{v\in T} U_v\right).\end{aligned}$$

Thus we have the above fact in a similar manner as the proof of Proposition 3.1.

4. Relation between $\mathcal{Y}_{T_\infty}(K_\infty)$ and $\mathcal{Y}_{T_n}(K_n)$

First we prepare a Lemma on $\Lambda = \mathbf{Z}_p[[T]]$-modules. Let

$$\begin{cases} \omega_n := (1+T)^{p^n} - 1 \\ \nu_{m,n} := \omega_m/\omega_n \quad \text{for } m \geq n\,. \end{cases} \tag{6}$$

LEMMA 4.1. *Let $\{X_n\}_n$ be a projective system of Λ-modules. Let $X := \varprojlim X_n$ and assume $pr_n : X \to X_n$ are surjective for any $n \geq n_0$. Suppose that there exist Λ-modules D_n*

and that there exist commutative diagrams

$$\begin{array}{ccccccc} D_n & \longrightarrow & X/\omega_n & \longrightarrow & X_n & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \nu_{n+1,n}} & & \downarrow & & \\ D_{n+1} & \longrightarrow & X/\omega_{n+1} & \longrightarrow & X_{n+1} & \longrightarrow & 1 \end{array}$$

for any $n \geq n_0$, *whose rows are exact. Further, assume that the left vertical maps* $D_n \to D_{n+1}$ *are surjective. Then,* $X_n = X/\nu_{n,n_0} I$ *where* $I = \mathrm{Ker}(pr_{n_0})$.

PROOF. Let $I_n := \mathrm{Ker}(pr_n)$. Then the map $\nu_{n+1,n} : I_n/\omega_n X \to I_{n+1}/\omega_{n+1} X$ is surjective. Thus, we have $\nu_{n+1,n} I_n + \omega_{n+1} X = I_{n+1}$. Since $\nu_{n+1,n} I_n + \omega_{n+1} X = \nu_{n+1,n}(I_n + \omega_n X) = \nu_{n+1,n} I_n$, we have $\nu_{n+1,n} I_n = I_{n+1}$. □

Let K_n be the n-th layer of K_∞/K. Let T_n be the set of primes of K_n above T. Then we have

$$\mathcal{Y}_{T_\infty}(K_\infty) = \varprojlim_n \mathcal{Y}_{T_n}(K_n)$$

where the inverse limit is taken w.r.t. the natural restrictions. Let n_0 be the minimal number such that all of the primes in $P(K_{n_0}) - T_{n_0}$ are totally ramified in K_∞/K_{n_0}. Let

$$W_T := \mathrm{Gal}(\mathcal{M}_{T_\infty}(K_\infty)/K_\infty \mathcal{M}_{T_{n_0}}(K_{n_0})) . \tag{7}$$

This is a Λ-submodule of $\mathcal{Y}_{T_\infty}(K_\infty)$. We see that $\mathcal{Y}_{T_\infty}(K_\infty)/W_T$ is isomorphic to a submodule of $\mathcal{Y}_{T_{n_0}}(K_{n_0})$ which is a finitely generated $\mathbf{Z}_p$-module. Let ω_n and $\nu_{m,n}$ be the elements of Λ defined by (6).

PROPOSITION 4.2. *Assume* $T \neq P(K)$. *Then, for any* $n \geq n_0$, *we have*

$$\mathcal{Y}_{T_n}(K_n) \cong \mathcal{Y}_{T_\infty}(K_\infty)/\nu_{n,n_0} W_T .$$

If $n_0 = 0$ *and* $\sharp(P(K) - T) = 1$, *then*

$$\mathcal{Y}_{T_n}(K_n) \cong \mathcal{Y}_{T_\infty}(K_\infty)/\omega_n \mathcal{Y}_{T_\infty}(K_\infty)$$

for all n.

REMARK 4.3. When $T = \varnothing$, this is a well known result of Iwasawa ([Iw], [Wa] Lemma 13.18). On the other hand, when $T = P(K)$, it is also well known that $\mathrm{Gal}(\mathcal{M}_{T_n}(K_n)/K_\infty) \cong \mathcal{Y}_{T_\infty}(K_\infty)/\omega_n$.

PROOF. Let $\tilde{\mathcal{M}}_n$ be the subfield of $\mathcal{M}_{T_\infty}(K_\infty)/K_\infty$ corresponding to the subgroup $\omega_n \mathcal{Y}_{T_\infty}(K_\infty)$. This is the maximal subfield which is abelian over K_n. Then the following is exact:

$$1 \to \bigoplus_{\mathfrak{p}_n \in P(K_n) - T_n} T(\mathfrak{p}_n) \to \mathrm{Gal}(\tilde{\mathcal{M}}_n/K_n) \to \mathcal{Y}_{T_n}(K_n) \to 1 .$$

Here, $T(\mathfrak{p}_n)$ is the inertia group of $\mathfrak{p}_n$ in $\mathrm{Gal}(\tilde{\mathcal{M}}_n/K_n)$. For $n \geq n_0$, the restriction map $T(\mathfrak{p}_n) \to \mathrm{Gal}(K_\infty/K_n)(\cong \mathbf{Z}_p)$ is an isomorphism. Thus, for $\mathfrak{p}_{n+1} \in P(K_{n+1}) - T_{n+1}$ the image of $T(\mathfrak{p}_{n+1})$ by the restriction map

$$\phi_n : \mathrm{Gal}(\tilde{\mathcal{M}}_{n+1}/K_{n+1}) \to \mathrm{Gal}(\tilde{\mathcal{M}}_n/K_n)$$

is $pT(\mathfrak{p}_n)$ where $\mathfrak{p}_n = \mathfrak{p}_{n+1}|_{K_n}$. Consider the transfer map

$$\psi_n : \mathrm{Gal}(\tilde{\mathcal{M}}_n/K_n) \to \mathrm{Gal}(\tilde{\mathcal{M}}_{n+1}/K_{n+1})\,.$$

Then we see that the image of $T(\mathfrak{p}_n)$ by ψ_n is contained in $T(\mathfrak{p}_{n+1})$. Further, $\psi_n|_{T(\mathfrak{p}_n)}$ is an isomorphism. In fact, since $\phi_n \circ \psi_n = p$, $\phi_n(\psi_n(T(\mathfrak{p}_n))) = pT(\mathfrak{p}_{n+1}) = \phi_n(T(\mathfrak{p}_{n+1}))$. Thus, since ϕ_n is injective on $T(\mathfrak{p}_{n+1})$, $\psi_n(T(\mathfrak{p}_n)) = T(\mathfrak{p}_{n+1})$. Therefore we have the diagram for $n \geq n_0$

$$\begin{array}{ccccccc}
\bigoplus_{\mathfrak{p}_n \in P(K_n)-T_n} T(\mathfrak{p}_n) & \longrightarrow & \mathrm{Gal}(\tilde{\mathcal{M}}_n/K_n) & \longrightarrow & \mathcal{Y}_{T_n}(K_n) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{\mathfrak{p}_{n+1} \in P(K_{n+1})-T_{n+1}} T(\mathfrak{p}_{n+1}) & \longrightarrow & \mathrm{Gal}(\tilde{\mathcal{M}}_{n+1}/K_{n+1}) & \longrightarrow & \mathcal{Y}_{T_{n+1}}(K_{n+1}) & \longrightarrow & 1\,.
\end{array}$$

The vertical maps are transfers. The left vertical map is an isomorphism by the above. From this and the facts that $\mathrm{Gal}(\tilde{\mathcal{M}}_n/K_\infty) \cong \mathcal{Y}_{T_\infty}(K_\infty)/\omega_n$ and $\mathcal{M}_{T_n}(K_n) \cap K_\infty = K_n$ (since $T \neq P(K)$), we have the diagram

$$\begin{array}{ccccccc}
D_n & \longrightarrow & \mathcal{Y}_{T_\infty}(K_\infty)/\omega_n & \longrightarrow & \mathcal{Y}_{T_n}(K_n) & \longrightarrow & 1 \\
\downarrow & & \downarrow{\scriptstyle \nu_{n+1,n}} & & \downarrow & & \\
D_{n+1} & \longrightarrow & \mathcal{Y}_{T_\infty}(K_\infty)/\omega_{n+1} & \longrightarrow & \mathcal{Y}_{T_{n+1}}(K_{n+1}) & \longrightarrow & 1\,.
\end{array}$$

where $D_n := \mathrm{Ker}(\bigoplus_{\mathfrak{p}_n \in P(K_n)-T_n} T(\mathfrak{p}_n) \to \mathrm{Gal}(K_\infty/K_n))$. The left vertical map is an isomorphism. Thus we have Proposition 4.2 by Lemma 4.1. When $\sharp(P(K) - T) = 1$, then $D_n = 0$. Thus $\mathcal{Y}_{T_\infty}(K_\infty)/\omega_n \cong \mathcal{Y}_{T_n}(K_n)$. □

A result of the same type can be verified for $\mathcal{Y}'_{T_\infty}(K_\infty)$. In fact, in the above proof, if we replace $\mathcal{M}_{T_\infty}(K_\infty)$ by $\mathcal{M}'_{T_\infty}(K_\infty)$ and $T(\mathfrak{p}_n)$ by the decomposition group $Z(\mathfrak{p}_n)$, we have:

PROPOSITION 4.4. *For any $n \geq n_0$, we have*

$$\mathcal{Y}'_{T_n}(K_n) \cong \mathcal{Y}'_{T_\infty}(K_\infty)/\nu_{n,n_0} W'_T$$

where $W'_T := \mathrm{Gal}(\mathcal{M}'_{T_\infty}(K_\infty)/K_\infty \mathcal{M}'_{T_{n_0}}(K_{n_0}))$. If $n_0 = 0$ and $\sharp(P(K) - T) = 1$, then

$$\mathcal{Y}'_{T_n}(K_n) \cong \mathcal{Y}'_{T_\infty}(K_\infty)/\omega_n$$

for all n.

In concluding this section, we note the following explicit formula when K is a CM field.

THEOREM 4.5 (cf. [JaMa]). *Assume p is odd and K is a CM-field. Let*

$$T_0 := \{v \in T \mid \sigma v \in T\}$$

where σ is the complex conjugation. Then,

$$\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) = \sum_{v \in T_0} [K_v : \mathbf{Q}_p]/2\,.$$

Further, we have $\mu(\mathcal{Y}_{T_\infty}(K_\infty)_{\Lambda-tor}) = 0$ if $\mu(\mathcal{Y}_\varnothing(L_\infty)) = 0$ where $L = K(\mu_p)$. Here, $\mathcal{Y}_{T_\infty}(K_\infty)_{\Lambda-tor}$ is the maximal Λ-torsion submodule of $\mathcal{Y}_{T_\infty}(K_\infty)$.

When K is abelian, we have $\mu(\mathcal{Y}_\varnothing(L_\infty)) = 0$ since $K(\mu_p)$ is abelian. This gives a complete answer to our problem in this case.

5. A bound for the $\mathbf{Z}_p$-rank of global units: an application of Ax and Brumer's method

In this section, we consider the $\mathbf{Z}_p$-rank of $\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T}}$ in $\mathcal{U}_{K,T}$ after the methods of Ax and Brumer ([Ax] and [Br]). We also use the formulation of [EKW].

Let K be an algebraic number field of finite degree. Let p be a prime and $T \subset P(K)$ a non-empty subset. Assume there exists a subfield $k \subset K$ such that K/k is Galois. Assume that there exists a prime $u \in P(k)$ such that

$$T' := \{v \in P(K) \mid \; v|u\} \tag{8}$$

is contained in T. Let G be the Galois group of K over k. We prove the following:

THEOREM 5.1. *Let K/k and $T \subset P(K)$ be as above. Let $\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T}}$ be the topological closure of $\mathcal{E}_K \cap \mathcal{U}_{K,T}$ in $\mathcal{U}_{K,T}$. Then,*

$$\mathrm{rank}_{\mathbf{Z}_p} \overline{\mathcal{E}_K \cap \mathcal{U}_{K,T}} \geq \sum_{\chi \in \Delta_{K/k}} \deg(\chi)\,.$$

Here, $\Delta_{K/k}$ is the set of the distinct irreducible characters of G over $\bar{\mathbf{Q}}$ which appears in the $\bar{\mathbf{Q}}[G]$-module $\mathcal{E}_K \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}$. That is,

$$\mathcal{E}_K \otimes_{\mathbf{Z}} \bar{\mathbf{Q}} \cong \bigoplus_{\chi \in \Delta_{K/k}} V_\chi^{n_\chi}$$

where V_χ is the irreducible $\bar{\mathbf{Q}}[G]$-module corresponding to χ with $n_\chi > 0$ and if $\chi \neq \chi'$ in $\Delta_{K/k}$ then $V_\chi \not\cong V_{\chi'}$.

REMARK 5.2. The same result is obtained by C. Maire in [Ma] by the same method. But we prove this here for the completeness.

To prove Theorem 5.1, we need the following:

LEMMA 5.3 ([EKW] Lemme 1). *Let X be a finite dimensional $\mathbf{C}_p[G]$-module. Let $A \subset X$ be a $\bar{\mathbf{Q}}[G]$-submodule. Let A^{cl} be the topological closure of A in X. Then* $\dim_{\mathbf{C}_p} A^{\mathrm{cl}} \geq \sum_{\chi \in \Delta_A} \deg(\chi)$. *Here, Δ_A is the set of the distinct irreducible characters of G over $\bar{\mathbf{Q}}$ which appears in A.*

PROOF. Let V_χ be an irreducible component of A corresponding to χ. Then $V_\chi \otimes_{\bar{\mathbf{Q}}} \mathbf{C}_p$ is irreducible over $\mathbf{C}_p$. Thus, the induced map $V_\chi \otimes \mathbf{C}_p \to X$ should be injective. If $\chi \neq \chi'$, then we see the intersection of the images of $V_\chi \otimes \mathbf{C}_p$ and $V_{\chi'} \otimes \mathbf{C}_p$ is 0. Thus we have the conclusion. □

Let $T' \subset T$ be as (8). For $v \in T'$, we denote the corresponding embedding by $\iota_v : K \hookrightarrow K_v$. Let $G_v := \mathrm{Gal}(K_v/k_u) \subset G$ where $v|u$. Let

$$\log_p : K_v^\times \to K_v$$

be the p-adic logarithm map. This is a G_v-homomorphism. Then we have the following theorem due to Brumer:

THEOREM 5.4 (Brumer[Br]). *Let $\mathcal{E}_K \to \mathbf{C}_p$ be the composition of the map $\iota_v|_{\mathcal{E}_K}$, $\log_p$ and the inclusion $K_v \hookrightarrow \mathbf{C}_p$. Then the induced map*

$$\mathcal{E}_K \otimes \bar{\mathbf{Q}} \to \mathbf{C}_p$$

is injective.

PROOF OF THEOREM 5.1. We note that $\mathcal{E}_K \cap \mathcal{U}_{K,T}$ is of finite index in $\mathcal{E}_K$. We see that the $\mathbf{Z}_p$-rank of $\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T}}$ in $\mathcal{U}_{K,T}$ is not less than that of $\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T'}}$ in $\mathcal{U}_{K,T'}$. Thus, we only need to prove Theorem for T'. For $v \in T'$, let $\mathcal{U}_v$ be the principal local units of K_v. Let

$$\ell_v : \mathcal{U}_v \to K_v \otimes_{k_u} \mathbf{C}_p$$

be the map defined by $u \mapsto \log_p(u) \otimes 1$. We have $K_v \otimes_{k_u} \mathbf{C}_p \cong \mathbf{C}_p[G_v]$ as a $\mathbf{C}_p[G_v]$-module and ℓ_v is a $\mathbf{Z}_p[G_v]$-module homomorphism. Here we consider $\mathbf{C}_p$ as a trivial G_v-module. Let

$$X := \bigoplus_{v \in T'} (K_v \otimes_{k_u} \mathbf{C}_p)$$

and

$$\theta := \bigoplus_{v \in T'} \ell_v : \mathcal{U}_{K,T'} \to X\,.$$

Then we have $X \cong \mathbf{C}_p[G]$ and θ is a $\mathbf{Z}_p[G]$-homomorphism. By Theorem 5.4, we see that the induced map

$$(\theta|_{(\mathcal{E}_K \cap \mathcal{U}_{K,T'})}) \otimes \bar{\mathbf{Q}} : (\mathcal{E}_K \cap \mathcal{U}_{K,T'}) \otimes \bar{\mathbf{Q}} \to X\,.$$

is injective. Since the above map is a G-homomorphism, we have

$$\dim_{\mathbf{C}_p} (\theta(\mathcal{E}_K \cap \mathcal{U}_{K,T'}) \otimes \bar{\mathbf{Q}})^{\mathrm{cl}} \geq \sum_{\chi \in \Delta_{K/k}} \deg(\chi)$$

by Lemma 5.3. Therefore we get the inequality

$$\mathrm{rank}_{\mathbf{Z}_p}\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T'}} \geq \sum_{\chi \in \Delta_{K/k}} \deg(\chi)$$

since θ is a $\mathbf{Z}_p$-homomorphism. □

Next, we recall the well-known structure of $\mathcal{E}_K \otimes \mathbf{Q}$ as a $\mathbf{Q}[G]$-module.

DEFINITION 5.5. Let K/k be a Galois extension and $G := \mathrm{Gal}(K/k)$. Let V_1 be the set of all real primes of k which remain real in K, V_2 the set of real primes of k which become complex in K and V_3 the set of all complex primes. For a prime u in V_2, choose v, a prime of K above u. Let $\mathrm{Gal}(K_v/k_u) = \langle\sigma_v\rangle \subset G$. We define a $\mathbf{Q}[G]$-module $M_{K/k}$ as

$$M_{K/k} := \left(\bigoplus_{u \in V_1 \cup V_3} \mathbf{Q}[G]\right) \oplus \left(\bigoplus_{u \in V_2} \mathbf{Q}[G/\langle\sigma_v\rangle]\right).$$

DEFINITION 5.6. Let K/k, G, V_1, V_2, V_3 and σ_v be as above. Let

$$r_1 : \mathbf{Q}[G] \to \mathbf{Q} \quad (\text{resp.} \quad r_2 : \mathbf{Q}[G/\langle\sigma_v\rangle] \to \mathbf{Q})$$

be the map defined by $\sum_\tau a_\tau \tau \mapsto \sum_\tau a_\tau$ (resp. $\sum_{\tau \in G/\langle\sigma_v\rangle} a_\tau \tau \mapsto \sum_{\tau \in G/\langle\sigma_v\rangle} a_\tau$). Let

$$\psi_K := \left(\sum_{u \in V_1 \cup V_3} r_1\right) + \left(\sum_{u \in V_2} r_2\right) : M_{K/k} \to \mathbf{Q}\,.$$

PROPOSITION 5.7 (see also [EKW]). *As $\mathbf{Q}[G]$-modules,*

$$\mathcal{E}_K \otimes \mathbf{Q} \cong \mathrm{Ker}(\psi_K)\,.$$

PROOF. Let us consider the regulator map

$$r_K : \mathcal{E}_K \to M_{K/k} \otimes \mathbf{R}$$

defined by

$$\epsilon \mapsto \left(\bigoplus_{u \in V_1 \cup V_3} \sum_\tau (|\log \epsilon^{(\tau v)}|\tau)\right) \oplus \left(\bigoplus_{u \in V_2} \sum_\tau (|\log \epsilon^{(\tau v)}|\tau)\right)$$

where $v|u$ and $\epsilon^{(\tau v)} \in \mathbf{R}$ or $\mathbf{C}$ is the image of ϵ under the embedding corresponding to τv. This is a G-homomorphism. Dirichlet's unit theorem states that $r_K \otimes \mathbf{R}$ is injective and

$$\mathcal{E}_K \otimes \mathbf{R} \cong \mathrm{Ker}(\psi_K) \otimes \mathbf{R}\,.$$

For $\mathbf{Q}[G]$-modules A and B, if $A \otimes \mathbf{R} \cong B \otimes \mathbf{R}$ as $\mathbf{R}[G]$-modules, then $A \cong B$ as $\mathbf{Q}[G]$-modules (cf. [ANT] Chapter IV p.110 Lemma for the proof of Proposition 12). Thus we have the Proposition. □

6. A bound for the Λ-rank of $\mathcal{Y}_{T_\infty}(K_\infty)$

Let M be a finitely generated Λ-module. As for the Λ-rank of M, we see the following: Let $\omega_n := (1+T)^{p^n} - 1$ and $\nu_{m,n} := \omega_m/\omega_n$ be the elements of Λ as (6).

LEMMA 6.1. *Let M be a finitely generated Λ-module. Then we have*

$$\mathrm{rank}_\Lambda(M) = \lim_{m\to\infty} \frac{1}{p^m}(\mathrm{rank}_{\mathbf{Z}_p}(M/\nu_{m,n}))$$

for any $n \geq 0$.

PROOF. By the structure theorem of Λ-modules, there exists

$$M \to \Lambda^r \oplus \left(\bigoplus_i \Lambda/p^{n_i}\right) \oplus \left(\bigoplus_j \Lambda/(f_j)^{e_j}\right)$$

with finite kernel and cokernel, where f_j's are irreducible distinguished polynomials. Thus we have

$$\begin{aligned}\mathrm{rank}_{\mathbf{Z}_p}(M/\nu_{m,n}) = r(\mathrm{rank}_{\mathbf{Z}_p}(\Lambda/\nu_{m,n})) + \sum_i \mathrm{rank}_{\mathbf{Z}_p}(\Lambda/(p^{n_i}, \nu_{m,n}))\\ + \sum_j \mathrm{rank}_{\mathbf{Z}_p}(\Lambda/(f_j^{e_j}, \nu_{m,n}))\,.\end{aligned}$$

We see $\mathrm{rank}_{\mathbf{Z}_p}(\Lambda/\nu_{m,n}) = p^m - p^n$, $\sum_j \mathrm{rank}_{\mathbf{Z}_p}(\Lambda/(f_j^{e_j}, \nu_{m,n})) \leq \sum_j \mathrm{rank}_{\mathbf{Z}_p}(\Lambda/(f_j)^{e_j})$ and that $(\Lambda/(p^{n_i}, \nu_{m,n}))$ is finite. Thus we have the lemma since $r = \mathrm{rank}_\Lambda(M)$. □

We now consider K/k and $T \subset P(K)$ satisfying the conditions stated at the beginning of §5. Let K_∞ (resp. k_∞) be the cyclotomic $\mathbf{Z}_p$-extension of K (resp. k). We further assume here that

$$K \cap k_\infty = k\,.$$

Then,

$$\mathrm{Gal}(K_\infty/k) \cong G \times \Gamma\,.$$

Let $T_\infty \subset P(K_\infty)$ be the set of primes above T. Then we see that

THEOREM 6.2. *Assume K/k and $T \subset P(K)$ satisfy the above conditions. Then we have*

$$\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) \leq \left(\sum_{v\in T}[K_v : \mathbf{Q}_p]\right) - \left(\sum_{\chi\in\Delta_{K/k}} \deg\chi\right) - \delta\,.$$

Here, $\Delta_{K/k}$ is the set of the distinct irreducible characters of G over $\bar{\mathbf{Q}}$ which appear in the $\bar{\mathbf{Q}}[G]$-module $\mathcal{E}_K \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}$. We put

$$\delta = \begin{cases} 0 & \text{if } \Delta_{K/k} \text{ contains the trivial character}\,,\\ 1 & \text{otherwise}\,.\end{cases}$$

PROOF. First, we consider the case where $T \neq P(K)$. Let $W_T \subset \mathcal{Y}_{T_\infty}(K_\infty)$ be the Λ-submodule defined by (7). We know that $\mathcal{Y}_{T_\infty}(K_\infty)/W_T$ is a finitely generated $\mathbf{Z}_p$-module. Thus, $\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) = \mathrm{rank}_\Lambda(W_T)$ and

$$\mathrm{rank}_{\mathbf{Z}_p}(W_T/\nu_{n,n_0}) = \mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_\infty}(K_\infty)/\nu_{n,n_0}W_T) - \mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_\infty}(K_\infty)/W_T)\,.$$

By Proposition 4.2, we have $\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_\infty}(K_\infty)/\nu_{n,n_0}W_T) = \mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_n}(K_n))$, where $T_n \subset P(K_n)$ is the set of primes above T. On the other hand,

$$\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_n}(K_n)) = \mathrm{rank}_{\mathbf{Z}_p}(\mathcal{U}_{K_n,T_n}/(\overline{\mathcal{E}_{K_n} \cap \mathcal{U}_{K_n,T_n}}))$$

by (4) and Proposition 3.1. We claim here that

$$\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{U}_{K_n,T_n}/(\overline{\mathcal{E}_{K_n} \cap \mathcal{U}_{K_n,T_n}})) \leq p^n\left(\left(\sum_{v\in T}[K_v:\mathbf{Q}_p]\right) - \left(\sum_{\chi\in\Delta_{K/k}}\deg\chi\right) - \delta\right) + \delta\,.$$

By this claim, we have

$$\begin{aligned}\mathrm{rank}_{\mathbf{Z}_p}(W_T/\nu_{n,n_0}) \leq p^n\left(\sum_{v\in T}[K_v:\mathbf{Q}_p] - \sum_{\chi\in\Delta_{K/k}}\deg\chi - \delta\right) + \delta \\ - \mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_\infty}(K_\infty)/W_T)\,.\end{aligned}$$

Thus by Lemma 6.1, we have

$$\mathrm{rank}_\Lambda(\mathcal{Y}_{T_\infty}(K_\infty)) = \mathrm{rank}_\Lambda(W_T) \leq \sum_{v\in T}[K_v:\mathbf{Q}_p] - \sum_{\chi\in\Delta_{K/k}}\deg\chi - \delta\,.$$

So it remains to prove the claim above.

Since $\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{U}_{K_n,T_n}) = p^n\sum_{v\in T}[K_v:\mathbf{Q}_p]$, we see from Theorem 5.1 that

$$\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{U}_{K_n,T_n}/(\overline{\mathcal{E}_{K_n} \cap \mathcal{U}_{K_n,T_n}})) \leq p^n\left(\sum_{v\in T}[K_v:\mathbf{Q}_p]\right) - \sum_{\chi\in\Delta_{K_n/k}}\deg\chi\,.$$

We calculate $\sum_{\chi\in\Delta_{K_n/k}}\deg\chi$. Since $\mathrm{Gal}(K_n/k) \cong G\times\Gamma/\Gamma^{p^n}$, we have

$$M_{K_n/k} = \left(\bigoplus_{v\in V_1\cup V_3}\mathbf{Q}[G\times\Gamma/\Gamma^{p^n}]\right)\oplus\left(\bigoplus_{v\in V_2}\mathbf{Q}[(G\times\Gamma/\Gamma^{p^n})/\langle\sigma_v\rangle]\right),$$

and $\mathcal{E}_{K_n}\otimes\mathbf{Q} \cong \mathrm{Ker}(\psi_{K_n})$ by Proposition 5.7. Let $(\Gamma/\Gamma^{p^n})^\wedge$ be the set of characters of Γ/Γ^{p^n}. Since Γ/Γ^{p^n} is abelian, we see that if $\Delta_{K/k}$ contains the trivial character, then

$$\Delta_{K_n/k} = \{\chi\otimes\chi' | \chi\in\Delta_{K/k},\, \chi'\in(\Gamma/\Gamma^{p^n})^\wedge\}\,.$$

If $\Delta_{K/k}$ does not contain the trivial character, then

$$\Delta_{K_n/k} = \{\chi\otimes\chi' | \chi\in\Delta_{K/k},\, \chi'\in(\Gamma/\Gamma^{p^n})^\wedge\}\cup\{1\otimes\chi' | \chi'\in(\Gamma/\Gamma^{p^n})^\wedge \text{ and } \chi'\neq 1'\}\,.$$

where 1 and $1'$ are the trivial characters of G and Γ/Γ^{p^n}. Since each χ' is of degree one, we have $\deg(\chi \otimes \chi') = \deg \chi$. Thus, $\sharp\Delta_{K_n/k} = (p^n \sharp\Delta_{K/k}) + \delta(p^n - 1)$ and

$$\sum_{\chi \in \Delta_{K_n/k}} \deg \chi = \left(p^n \sum_{\chi \in \Delta_{K/k}} \deg \chi\right) + \delta(p^n - 1)\,.$$

This proves the claim.

In the case where $T = P(K)$, we have

$$\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_\infty}(K_\infty)/\omega_n) = \mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_n}(K_n)) - 1$$

by Remark 4.3. We calculate $\mathrm{rank}_{\mathbf{Z}_p}(\mathcal{Y}_{T_n}(K_n))$ similarly as above, and get the same conclusion. Note, however, that our estimate for the Λ-rank is weaker than Iwasawa's *equality* (Theorem 2.1), in this case. □

7. Λ-torsionness and μ-invariant of $\mathcal{Y}_{T_\infty}(K_\infty)$ for $K = \mathbf{Q}(\sqrt[3]{a})$

In this section, we consider a special base field

$$K = \mathbf{Q}(\sqrt[3]{a})$$

where $a \in \mathbf{Z}$ and cube free. Let $k = \mathbf{Q}(\zeta_3)$ and

$$L = \mathbf{Q}(\sqrt[3]{a}, \zeta_3)\,,$$

the Galois closure of K. Let σ be a generator of $\mathrm{Gal}(L/K)$ and τ that of $\mathrm{Gal}(L/k)$. Then

$$G := \mathrm{Gal}(L/\mathbf{Q}) \cong \mathfrak{S}_3$$

and G is generated by σ and τ, satisfying $\sigma^2 = 1$, $\tau^3 = 1$ and $\sigma\tau = \tau^{-1}\sigma$.

Let p be an odd prime satisfying the following: p inerts in k and π splits in L where π is the unique prime of k above p. This is equivalent to the assumption that K has two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ above p.

Denote the primes above p in L by v_1, v_2 and v_3. We see that $L_{v_i} = \mathbf{Q}_p(\zeta_3)$. We may assume that $G_{v_i} := \mathrm{Gal}(L_{v_i}/\mathbf{Q}_p)$ is $\langle \tau^{i-1}\sigma\tau^{-(i-1)} \rangle$ in G. Then, we denote the primes of K above p by $\mathfrak{p}_1$ and $\mathfrak{p}_2$, where $\mathfrak{p}_1$ is below v_1 while $\mathfrak{p}_2$ is below v_2 and v_3. We have $K_{\mathfrak{p}_1} = \mathbf{Q}_p$ and $K_{\mathfrak{p}_2} = \mathbf{Q}_p(\zeta_3)$, $[K_{\mathfrak{p}_2} : \mathbf{Q}_p] = 2$.

Let L_∞ (resp. k_∞, K_∞ and $\mathbf{Q}_\infty$) be the cyclotomic $\mathbf{Z}_p$-extension of L (resp. k, K and $\mathbf{Q}$). Let $\Gamma = \mathrm{Gal}(L_\infty/L)$ and we identify this with $\mathrm{Gal}(K_\infty/K)$, $\mathrm{Gal}(k_\infty/k)$ and $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. We see that L_∞ is Galois over $\mathbf{Q}$ and $\mathrm{Gal}(L_\infty/\mathbf{Q}) \cong G \times \Gamma$. We identify $\mathrm{Gal}(L_\infty/\mathbf{Q}_\infty)$ with G. We easily see the following:

LEMMA 7.1. *The prime v_i (resp. $\mathfrak{p}_i$, π) is totally ramified in L_∞/L (resp. K_∞/K, k_∞/k).*

We write v_i (resp. $\mathfrak{p}_i$, π) again for the unique prime of L_∞ (resp. K_∞, k_∞) above v_i (resp. $\mathfrak{p}_i$, π).

REMARK 7.2. The reason why we consider this special case is that this situation appears in [Ha1],[Ha2] for $p = 3$. There, $\mathcal{Y}_{T_\infty}(K_\infty)$ plays an important role in relation with the μ-invariants of Selmer groups of certain elliptic curves.

7.1. The Λ-torsionness We will prove the following:

THEOREM 7.3. *Let K, L and p be as above. Put $T'_\infty = \{v_1\} \subset P(L_\infty)$ and $T_\infty = \{\mathfrak{p}_2\} \subset P(K_\infty)$. Then $\mathcal{Y}_{T'_\infty}(L_\infty)$ and $\mathcal{Y}_{T_\infty}(K_\infty)$ are Λ-torsion.*

For the proof, we need the following:

LEMMA 7.4. *$\mathcal{Y}_{T_\infty}(K_\infty)$ is Λ-torsion if and only if the kernel of the restriction map*

$$\mathrm{res}_{v_1} : X(L_{\infty,v_1}) \to \mathfrak{X}(L_\infty)$$

is Λ-torsion.

PROOF. The above map is a homomorphism of $\mathrm{Gal}(L/K) = \langle\sigma\rangle$-modules. For a $\langle\sigma\rangle$-module M, let $M^{(\sigma=\pm1)}$ be the maximum $\langle\sigma\rangle$-submodule of M on which σ acts as multiplication by ±1. Then

$$M = M^{(\sigma=1)} \oplus M^{(\sigma=-1)}\,.$$

We also see that the kernel of res_{v_1} is Λ-torsion if and only if so are the kernels of $\mathrm{res}_{v_1}^{(\sigma=\pm1)}$. We have the commutative diagram

$$\begin{array}{ccc} X(L_{\infty,v_1})^{(\sigma=-1)} & \xrightarrow{\mathrm{res}_{v_1}^{(\sigma=-1)}} & \mathfrak{X}(L_\infty)^{(\sigma=-1)} \\ \downarrow & & \downarrow \\ X(k_{\infty,\pi})^- & \xrightarrow{\mathrm{res}_\pi^-} & \mathfrak{X}(k_\infty)^-\,. \end{array}$$

Here $X(k_{\infty,\pi})^-$ and $\mathfrak{X}(k_\infty)^-$ are the minus parts of $X(k_{\infty,\pi})$ and $\mathfrak{X}(k_\infty)$, respectively, i.e., the maximum submodules on which the complex conjugation in $\mathrm{Gal}(k/\mathbf{Q})$ acts by (-1)-multiplication. Here, π is the unique prime of k_∞ above p. The cokernel of the bottom row is $A'(k_{\infty,\pi})^-$ which is Λ-torsion. We see that $\mathrm{rank}_\Lambda X(k_{\infty,\pi})^- = 1$ and $\mathrm{rank}_\Lambda \mathfrak{X}(k_\infty)^- = 1$ by Theorems 2.2 and 2.1, since $X(k_{\infty,\pi})^+ = X(k^+_{\infty,\pi})$ and $\mathfrak{X}(k_\infty)^+ = \mathfrak{X}(k^+_\infty)$ where $k^+ = \mathbf{Q}$. Thus the kernel of res_π^- is Λ-torsion. Since the left column is an isomorphism, the kernel of $\mathrm{res}_{v_1}^{(\sigma=-1)}$ is Λ-torsion.

On the other hand, we have another commutative diagram

$$\begin{array}{ccc} X(L_{\infty,v_1})^{(\sigma=1)} & \xrightarrow{\mathrm{res}_{v_1}^{(\sigma=1)}} & \mathfrak{X}(L_\infty)^{(\sigma=1)} \\ \downarrow & & \downarrow \\ X(K_{\infty,\mathfrak{p}_1}) & \xrightarrow{\mathrm{res}_{\mathfrak{p}_1}} & \mathfrak{X}(K_\infty)\,. \end{array}$$

The vertical maps are isomorphisms. By Theorem 2.1, $\mathrm{rank}_\Lambda(\mathfrak{X}(K_\infty)) = 1$. Since $K_{\mathfrak{p}_1} = \mathbf{Q}_p$, $X(K_{\infty,\mathfrak{p}_1}) \cong \Lambda$ by Theorem 2.2. Thus, $\mathrm{res}_{v_1}^{(\sigma=1)}$ is injective if and only if $\mathrm{res}_{\mathfrak{p}_1}$ is injective. We also see $\mathrm{res}_{\mathfrak{p}_1}$ is injective if and only if the cokernel of $\mathrm{res}_{\mathfrak{p}_1}$ is Λ-torsion. The cokernel of $\mathrm{res}_{\mathfrak{p}_1}$ is $\mathcal{Y}'_{T_\infty}(K_\infty)$. By Proposition 2.4, $\mathcal{Y}'_{T_\infty}(K_\infty)$ is Λ-torsion if and only if so is $\mathcal{Y}_{T_\infty}(K_\infty)$. This proves the claim. □

PROOF OF THEOREM 7.3. For the first assertion, we apply Theorem 6.2 to the extension L/K and $T' = \{v_1\} \subset P(L)$. Note here that T' is clearly $\mathrm{Gal}(L/K) = \langle\sigma\rangle$-stable.

We see that

$$M_{L/K} = \mathbf{Q} \oplus \mathbf{Q}[\langle\sigma\rangle]$$

since there exist two infinite primes of K one of which is the real prime becoming complex in L and the another of which is the complex prime. Thus we see

$$\mathrm{Ker}(\psi_L) \cong \mathbf{Q}[\langle\sigma\rangle]\,.$$

By Proposition 5.7, we have $\sum_{\chi\in\Delta_{L/K}} \deg\chi = 2$. On the other hand, $[L_{v_1} : \mathbf{Q}_p] = 2$. Thus by Theorem 6.2,

$$\mathrm{rank}_\Lambda \mathcal{Y}_{T'_\infty}(L_\infty) \le [L_{v_1} : \mathbf{Q}_p] - \sum_{\chi\in\Delta_{L/K}} \deg\chi = 0\,.$$

Here, $\delta = 0$ because $\Delta_{L/K}$ contains the trivial character. This proves the first assertion.

For the second, we consider the map

$$\sum_i \mathrm{res}_{v_i} : \bigoplus_i X(L_{\infty,v_i}) \to \mathfrak{X}(L_\infty)$$

which is a $G = \mathrm{Gal}(L/\mathbf{Q})$-module homomorphism. G acts on the set $\{X(L_{\infty,v_i})\}_i$ transitively. Thus, the kernel of $\mathrm{res}_{v_i} : X(L_{\infty,v_i}) \to \mathfrak{X}(L_\infty)$ is Λ-torsion for $i = 1$ if and only if so is for any i. Assume $\mathcal{Y}_{T_\infty}(K_\infty)$ is not Λ-torsion. Then the kernel of the map

$$\mathrm{res}_{v_1} : X(L_{\infty,v_1}) \to \mathfrak{X}(L_\infty)$$

is not Λ-torsion by Lemma 7.4. Thus the same happens for any i. Therefore we have

$$\mathrm{rank}_\Lambda(\mathrm{res}_{v_i}(X(L_{\infty,v_i})) \le 1$$

since $\mathrm{rank}_\Lambda X(L_{\infty,v_i}) = 2$ by Theorem 2.2. Thus

$$\mathrm{rank}_\Lambda(\mathrm{res}_{v_2}(X(L_{\infty,v_2})) + \mathrm{res}_{v_3}(X(L_{\infty,v_3}))) \le 2$$

in $\mathfrak{X}(L_\infty)$. Since $\mathrm{rank}_\Lambda(\mathfrak{X}(L_\infty)) = 3$ by Theorem 2.1, the cokernel of the map

$$X(L_{\infty,v_2}) \oplus X(L_{\infty,v_3}) \to \mathfrak{X}(L_\infty)$$

is not Λ-torsion. The cokernel is $\mathcal{Y}'_{T'_\infty}(L_\infty)$. By Proposition 2.4, $\mathcal{Y}'_{T'_\infty}(L_\infty)$ is Λ-torsion if and only if so is $\mathcal{Y}_{T'_\infty}(L_\infty)$. This contradicts the first assertion. □

7.2. A criterion for the vanishing of $\mu(\mathcal{Y}_{T_\infty}(K_\infty))$ We give a sufficient condition for the vanishing of the μ-invariant of $\mathcal{Y}_{T_\infty}(K_\infty)$ for p and K as in §7.1.

We first quote elementary lemmas on Λ-modules.

LEMMA 7.5 ([Gr] p. 123, Lemma for Proposition 10). *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of finitely generated Λ-modules. If A is a free Λ-module and B has no non-trivial finite Λ-submodule, then C also has no non-trivial finite Λ-submodule.

LEMMA 7.6 ([Go]). *Let M be a finitely generated Λ-torsion Λ-module. Assume M has no non-trivial finite Λ-submodule. Let*

$$e_n := \mathrm{ord}_p(\sharp(M/\omega_n)) .$$

Here, we set $e_n = \infty$ if $\sharp(M/\omega_n) = \infty$. Then, $\mu(M) = 0$ if there exists an $n \geq 0$ such that $e_{n+1} < \infty$ (which implies $e_n < \infty$) and $(e_{n+1} - e_n) < \varphi(p^{n+1})$. Here φ is the Euler φ-function.

Let us return to the situation in the previous subsection. Let $K = \mathbf{Q}(\sqrt[3]{a})$. Let p be an odd prime such that $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in K where $K_{\mathfrak{p}_1} = \mathbf{Q}_p$ and $[K_{\mathfrak{p}_2} : \mathbf{Q}] = 2$. Let $T_\infty = \{\mathfrak{p}_2\} \subset P(K_\infty)$.

PROPOSITION 7.7. *$\mathcal{Y}'_{T_\infty}(K_\infty)$ has no non-trivial finite Λ-submodule.*

PROOF. By (2), the sequence

$$X(K_{\infty,\mathfrak{p}_1}) \to \mathfrak{X}(K_\infty) \to \mathcal{Y}'_T(K_\infty) \to 1$$

is exact. By Theorem 2.2, $X(K_{\infty,\mathfrak{p}_1}) \cong \Lambda$ and by Theorem 2.1, $\mathrm{rank}_\Lambda \mathfrak{X}(K_\infty) = 1$. Since $\mathcal{Y}'_T(K_\infty)$ is Λ-torsion by Theorem 7.3, the left map should be an injection. Thus we have the Proposition by Lemma 7.5. □

Thus, we have the following:

PROPOSITION 7.8. *Let K and p be as above. Let $\mathrm{Cl}_{\{\mathfrak{p}_1\},n}[p^\infty]$ be the p-part of the $\mathfrak{p}_1$-ideal class group of K_n and $\mathcal{E}_{n,\{\mathfrak{p}_1\}}$ the group of global $\mathfrak{p}_1$-units of K_n. Put*

$$e_n := \mathrm{ord}_p(\,\mathcal{U}_{n,\mathfrak{p}_2}/\overline{\mathcal{U}_{n,\mathfrak{p}_2} \cap \mathcal{E}_{n,\{\mathfrak{p}_1\}}}) + \mathrm{ord}_p(\,\mathrm{Cl}_{\{\mathfrak{p}_1\},n}[p^\infty]) .$$

Then, if there exists an $n \geq 0$ such that $e_{n+1} < \infty$ and $(e_{n+1} - e_n) < \varphi(p^{n+1})$, then $\mu(\mathcal{Y}_{T_\infty}(K_\infty)) = 0$.

PROOF. We note that $\mu(\mathcal{Y}_{T_\infty}(K_\infty)) = 0$ if and only if $\mu(\mathcal{Y}'_{T_\infty}(K_\infty)) = 0$ by Proposition 2.4. By the class field theory, $\mathrm{Cl}_{\{\mathfrak{p}_1\},n}[p^\infty]$ is isomorphic to $A_{\{\mathfrak{p}_2\}}(K_n)$ defined before Proposition 3.2. By (5) and Proposition 3.2, $e_n = \mathrm{ord}_p(\sharp\mathcal{Y}'_{T_n}(K_n))$. By Proposition 4.4, we have $\mathcal{Y}'_{T_n}(K_n) = \mathcal{Y}'_{T_\infty}(K_\infty)/\omega_n$. Then, we have the Proposition by Lemma 7.6. □

REMARK 7.9. The reason why we consider $\mathcal{Y}'_{T_\infty}(K_\infty)$ instead of $\mathcal{Y}_{T_\infty}(K_\infty)$ is as follows: We have $\mathrm{rank}_{\mathbf{Z}_p}\mathcal{E}_{n,\{\mathfrak{p}_1\}} = 2p^n$. Since $\mathrm{rank}_{\mathbf{Z}_p}\mathcal{U}_{n,\mathfrak{p}_2} = 2p^n$, we can expect that $\mathcal{U}_{n,\mathfrak{p}_2}/\overline{\mathcal{U}_{n,\mathfrak{p}_2} \cap \mathcal{E}_{n,\{\mathfrak{p}_1\}}}$ is finite. (See the examples of the next subsection.)

7.3. Example Let $p = 3$. Let $K = \mathbf{Q}(\sqrt[3]{a})$ with $(3) = \mathfrak{p}_1\mathfrak{p}_2$. This occurs if and only if $b^2 \equiv c^2 \mod 9$ where b and c are square free integers which are relatively prime to each other satisfying $a = bc^2$. Thus, $a = 10, 17, 19, 26, 28 \ldots$, for example. We have the following:

PROPOSITION 7.10. *If* $p = 3$*, then* $\lambda(\mathcal{Y}'_{T_\infty}(K_\infty)) \geq 1$.

PROOF. By Proposition 3.2, $\mathcal{Y}'_{T_\infty}(K_\infty)$ contains $\varprojlim \mathcal{U}_{n,\mathfrak{p}_2}/\overline{\mathcal{U}_{n,\mathfrak{p}_2} \cap \mathcal{E}_{n,\{\mathfrak{p}_1\}}}$. We see that $\varprojlim \mathcal{U}_{n,\mathfrak{p}_2}$ contains $\varprojlim \mu_{3^n}$ since $K_{\mathfrak{p}_2} = \mathbf{Q}_3(\zeta_3)$. But $\mathcal{E}_{n,\{\mathfrak{p}_1\}}$ does not contain p-th roots of unity and hence we see that $\varprojlim \mathcal{U}_{n,\mathfrak{p}_2}/\overline{\mathcal{U}_{n,\mathfrak{p}_2} \cap \mathcal{E}_{n,\{\mathfrak{p}_1\}}}$ contains $\varprojlim \mu_{3^n}$. □

Thus if it happens that $e_0 = 1$, then $\lambda(\mathcal{Y}'_{T_\infty}(K_\infty)) = 1$ and $\mu(\mathcal{Y}'_{T_\infty}(K_\infty)) = 0$.

Let us see some examples. Let $a = 10$. Then $A_0 = 0$ and $\mathcal{U}_{0,\mathfrak{p}_2}/\overline{\mathcal{U}_{0,\mathfrak{p}_2} \cap \mathcal{E}_{0,\{\mathfrak{p}_1\}}} \cong \mathbf{Z}/3 \oplus \mathbf{Z}/3$. Thus, $\lambda \geq 2$ or $\mu > 0$ in this case. We see that $\sharp(\mathcal{U}_{0,\mathfrak{p}_2}/\overline{\mathcal{U}_{0,\mathfrak{p}_2} \cap \mathcal{E}_{0,\{\mathfrak{p}_1\}}}) \geq 9$ for $a = 17, 19, 26, 28, 44, 45$. (For the computation, we used Kash[Kash] and Pari[Pari].) Therefore we have to compute for $n \geq 1$ to determine whether $\mu(\mathcal{Y}'_{T_\infty}(K_\infty)) = 0$ or not.

8. An application to the vanishing of Iwasawa invariants

In this section, we give an application to the original Iwasawa invariants.

Let K be a number field of finite degree. Let $\lambda_p(K)$, $\mu_p(K)$ and $\nu_p(K)$ be the classical Iwasawa invariants of K. That is, for all sufficiently large n, we have

$$\sharp \mathrm{Cl}(K_n)[p^\infty] = p^{\lambda_p(K)n + \mu_p(K)p^n + \nu_p(K)}$$

where $\mathrm{Cl}(K_n)[p^\infty]$ is the p-Sylow subgroup of the ideal class group of K_n, the n-th layer of K_∞/K.

The following is a generalization of a criterion of Fukuda-Komatsu ([FuKo]).

THEOREM 8.1. *Let K be a number field. Assume that there are exactly two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of K above p such that $K_{\mathfrak{p}_1} = \mathbf{Q}_p$ and that they are totally ramified in K_∞. Then, $\lambda_p(K) = \mu_p(K) = \nu_p(K) = 0$ if and only if $A(K) = 0$ and $\mathcal{U}_{K,T''}/(\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T''}}) = 0$. Here, $T'' = \{\mathfrak{p}_1\}$.*

To prove this, we need the following:

LEMMA 8.2. $A(K_\infty) \cong \mathcal{Y}_{T''_\infty}(K_\infty)$ *where* $T''_\infty = \{\mathfrak{p}_1\} \subset P(K_\infty)$.

PROOF. By (4) and Proposition 3.1, the kernel of

$$\mathcal{Y}_{T''_\infty}(K_\infty) \to A(K_\infty)$$

is isomorphic to

$$\varprojlim \mathcal{U}_{K_n,T_n''}/\overline{\mathcal{E}_{K_n} \cap \mathcal{U}_{K_n,T_n''}}\,.$$

Since $K_{\mathfrak{p}_1} = \mathbf{Q}_p$,

$$\varprojlim \mathcal{U}_{K_n,T_n''}/\overline{\mathcal{E}_{K_n} \cap \mathcal{U}_{K_n,T_n''}}$$

is a quotient of

$$\varprojlim \mathcal{U}_{\mathbf{Q}_n,\pi_n}/\overline{\mathcal{E}_{\mathbf{Q}_n} \cap \mathcal{U}_{\mathbf{Q}_n,\pi_n}}$$

where π_n is the unique prime of $\mathbf{Q}_n$ above p. Since $A(\mathbf{Q}_\infty) = 0$ as is well known, we have

$$\varprojlim \mathcal{U}_{\mathbf{Q}_n,\pi_n}/\overline{\mathcal{E}_{\mathbf{Q}_n} \cap \mathcal{U}_{\mathbf{Q}_n,\pi_n}} \cong \mathfrak{X}(\mathbf{Q}_\infty)$$

by (4) and Proposition 3.1 for $K = \mathbf{Q}_\infty$ and $T = P(\mathbf{Q}_\infty) = \{\pi_\infty\}$. It is also well known that $\mathfrak{X}(\mathbf{Q}_\infty) = 0$. □

PROOF OF THEOREM 8.1. We note that $\lambda_p(K) = \mu_p(K) = \nu_p(K) = 0$ is equivalent to $A(K_\infty) = 0$. By the above Lemma and Nakayama's lemma, this is equivalent to

$$\mathcal{Y}_{T_\infty''}(K_\infty)/\omega_0 = 0\,.$$

By Proposition 4.2,

$$\mathcal{Y}_{T_\infty''}(K_\infty)/\omega_0 \cong \mathcal{Y}_{T''}(K)\,.$$

Thus, again by (4) and Proposition 3.1, we get our conclusion. □

EXAMPLE 8.3. Let

$$K = \mathbf{Q}(\sqrt[3]{a})$$

with $a \in \mathbf{Z}$, $a > 0$ and cube free. Let ε be the fundamental unit of K. Let p be an odd prime satisfying the condition of Theorem 8.1. Then, we see that

$$\mathcal{U}_{K,T''}/(\overline{\mathcal{E}_K \cap \mathcal{U}_{K,T''}}) = 0 \Leftrightarrow \varepsilon^{p-1} \not\equiv 1 \mod (\mathfrak{p}_1^2)$$

and the validity of the latter condition is easily computable. An odd prime p satisfies the condition of Theorem 8.1 if and only if either (A) $p = 3$ when $b^2 \equiv c^2 \mod 9$ where b and c are square free integers which are relatively prime to each other satisfying $a = bc^2$ or (B) $p \nmid 3a$ and $p \equiv 2 \mod 3$. In the case (B), we calculated $\varepsilon^{p-1} \mod (\mathfrak{p}_1^2)$ for $a = 2, 3, 5, 6, 10$ and for $3 < p < 1000$ by using Pari-GP[Pari] and Kash[Kash]. Then, we found that $A(K_\infty) \neq 0$ only when $a = 3$ and $p = 23$.

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