TOKYO J. MATH.
VOL. 26, NO. 1, 2003

Minimality and Hamiltonian Stability of Lagrangian Submanifolds in Adjoint Orbits

Hajime ONO

Tokyo Institute of Technology

(Communicated by R. Miyaoka)

Abstract. In this article, we improve and generalize the result of [O]; the one is the necessary and sufficient condition of the minimality of Lagrangian submanifolds in adjoint orbits, which are Hermitian symmetric spaces, and the other is the necessary and sufficient condition of the Hamiltonian stability of minimal Lagrangian submanifolds in adjoint orbits, which are not necessarily Hermitian symmetric spaces.

1. Introduction

Let (M, ω) be a compact Kähler manifold, and $L \subset M$ a compact minimal Lagrangian submanifold. Here we say a submanifold L minimal, if it has extremal volume under all smooth variations of L. In [Oh], Y.-G. Oh investigated the second variation of volume at L, and then defined the notion of "Hamiltonian stability"; L is called Hamiltonian stable if the second variation of volume is nonnegative for all Hamiltonian deformations of L. One of the main results in [Oh] is the following;

THEOREM 1.1 ([Oh]). *Let (M, ω) be a compact Kähler-Einstein manifold with $\rho = c\omega$, where ρ is the Ricci form of (M, ω). For a compact minimal Lagrangian submanifold $L \subset M$, this is Hamiltonian stable if and only if $\lambda_1(L) \geq c$, where $\lambda_1(L)$ is the first positive eigenvalue of the Laplacian Δ_L which acts on $C^\infty(L)$.*

In view of this theorem, it is an interesting problem to investigate $\lambda_1(L)$ for a compact minimal Lagrangian submanifold L in a Kähler-Einstein manifold. The well-known examples of Kähler-Einstein manifolds are orbits of the adjoint representation of a compact semisimple Lie group on its Lie algebra; let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\langle\ ,\ \rangle$ an Ad_G-invariant inner product on $\mathfrak{g}$, and M an adjoit orbit in $\mathfrak{g}$. Suppose that the Lie group G acts on M effectively. In this paper, when we say "adjoint orbit", we suppose that it satisfies this condition. Then M has the canonical complex structure J, and the canonical symplectic form F which is Kähler with respect to J (see [B] or Section 2 below). Note here that the 2-form ω associated with $\langle\ ,\ \rangle_{|M}$ and J, which is defined by $\omega(X, Y) = \langle JX, Y\rangle_{|M}$, is not always Kähler but Hermitian.

Received April 23, 2002

The main theorem of [O] is the following.

THEOREM 1.2 ([O]). *Let $(M^{2m}, J, \langle\ ,\ \rangle_{|M})$ be an adjoint orbit in $\mathfrak{g}$. Suppose that $\omega = \alpha F$ and that $\langle\ ,\ \rangle_{|M}$ is Kähler-Einstein with its Ricci form $\rho = c\omega$ for a positive constant c. Let $x : M \to \mathfrak{g}$ denote the inclusion. Then the following conditions are equivalent for a minimal Lagrangian submanifold L;*

(1) *L is Hamiltonian stable.*

(2) $\lambda_1(L) = c$.

(3) *All of the coordinate functions of $x_{|L}$, $x^i_{|L}$, are eigenfunctions of L with $c = \lambda_1(L)$.*

Note here that, by the assumption, $(M, J, \langle\ ,\ \rangle_{|M})$ is a Hermitian symmetric space.

For example, the adjoint orbit which satisfies the assumption of Theorem 1.2 is the orbit of

$$\mathfrak{su}(n) \ni \sqrt{-1}\begin{pmatrix} \lambda I_p & 0 \\ 0 & \mu I_{n-p} \end{pmatrix} \quad (\lambda, \mu \in \mathbf{R},\ \lambda - \mu > 0,\ p\lambda + (n-p)\mu = 0)$$

by the adjoint action of $SU(n)$, where $I_p \in \mathfrak{gl}(p)$ and $I_{n-p} \in \mathfrak{gl}(n-p)$ are the identity matrixes. It is diffeomorphic to the complex Grassmann manifold $\mathrm{Gr}_{n,p}(\mathbf{C})$.

In this paper, we improve this theorem as follows.

THEOREM 1.3. *Let $(M^{2m}, J, \langle\ ,\ \rangle_{|M})$ be an adjoint orbit in $\mathfrak{g}$. Suppose that $\omega = \alpha F$ and that $\langle\ ,\ \rangle_{|M}$ is Kähler-Einstein with its Ricci form $\rho = c\omega$ for a positive constant c. Denote its embedding by $x : M \to \mathfrak{g}$. Then the following conditions are equivalent for a Lagrangian submanifold L;*

(1) *$L \subset M$ is minimal.*

(2) $\Delta_L x_{|L} = c x_{|L}$.

(3) *The embedding $x_{|L} : L \to S^{\dim G-1}(\sqrt{m/c})$ is minimal.*

On the other hand, if the adjoint orbit $(M, \langle\ ,\ \rangle_{|M})$ does not satisfy the condition $\omega = \alpha F$, we cannot say about the minimality of Lagrangian submanifolds, but we can generalize Theorem 1.2 as follows.

THEOREM 1.4. *Let (M^{2m}, J) be an adjoint orbit in $\mathfrak{g}$ and $x : M \to \mathfrak{g}$ denote the embedding. Suppose that a G-invariant Kähler metric g is Kähler-Einstein with its Ricci form $\rho(X, Y) = cg(JX, Y)$ for a positive constant c and that $\Delta_{M,g} x = 2cx$. Then the following three conditions are equivalent, for a compact minimal Lagrangian submanifold $L \subset M$ with $\int_L x_{|L} dv = 0$.*

1. *L is Hamiltonian stable.*
2. $\lambda_1(L) = c$.
3. *All of the coordinate functions of $x_{|L}$, $x^i_{|L}$, are $c = \lambda_1(L)$-eigenfunctions.*

In this case, (M, J, g) is not necessarily a Hermitian symmetric space.

Recently, by Goldstein [G], the following theorem was proved.

THEOREM 1.5 ([G]). *Let (M, ω) be a Kähler manifold, $L \subset M$ an oriented closed minimal Lagrangian submanifold and V a holomorphic vector field defined in a neighbourhood of L in M. Then*

$$\int_L \mathrm{div}_{\mathbf{C}} V dv = 0 .$$

Note here that, we call a vector field V holomorphic if $V - iJV \in \Gamma(T^{1,0}M)$ is the holomorphic section, or equivalently, if the endmorphism $X \mapsto \nabla_X V$ of $T_m M$ is J-linear. For a holomorphic vector field V, $\mathrm{div}_{\mathbf{C}} V = \mathrm{trace}_{\mathbf{C}}(X \mapsto \nabla_X V)$ is well-defined and $2\,\mathrm{Re}\,\mathrm{div}_{\mathbf{C}} V = \mathrm{div}\, V$. On the other hand, if (M, ω) is Kähler-Einstein with its Ricci form $\rho = c\omega$ and $\lambda_1(M) = 2c$, then $\mathrm{grad}\, u$ is holomorphic, for any first eigenfunction u of the Laplacian Δ_M. So, by Theorem 1.5, the condition $\int_L x_{|L} dv = 0$ automatically holds, for any oriented minimal Lagrangian submanifold.

Examples of adjoint orbits which satisfy the assumption of Theorem 1.4 are given in Section 4.

2. The adjoint orbits of compact semisimple Lie groups

In this section, we review the notion of the adjoint orbits, Chapter 8 of [B], for the preparation of the following sections.

Let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\langle\ ,\ \rangle$ an Ad_G- invariant inner product on $\mathfrak{g}$, and M an adjoit orbit in $\mathfrak{g}$. Suppose that the Lie group G acts on M effectively. In this paper, when we say "adjoint orbit", we suppose that it satisfy this condition. For $U \in \mathfrak{g}$, the fundamental vector field X_U, attached to U, is defined by

$$X_U(w) = [U, w] \quad (w \in M) \tag{2.1}$$

under the identification $\mathfrak{g} \simeq T_w\mathfrak{g} \supset T_w M$ $(w \in M)$. Since G acts on M transitively, any tangent vector in $T_w M$ is written as the value of a fundamental vector field, and we can identify

$$T_w M \simeq \mathrm{Image}(\mathrm{ad}_w : \mathfrak{g} \to \mathfrak{g}) =: M_w \quad (w \in M) .$$

Similarly, we have an identification

$$N_w M \simeq \mathrm{Ker}(\mathrm{ad}_w : \mathfrak{g} \to \mathfrak{g}) =: L_w \quad (w \in M) ,$$

where $N_w M$ is the normal space of M at $w \in M$.

Next, we will define the canonical complex structure J on M. For $w \in M$, let $G_w = \{g \in G |\, \mathrm{Ad}(g)w = w\}$, S_w the connected center of G_w, and $\mathfrak{s}_w$ the Lie algebra of S_w. Note that $w \in \mathfrak{s}_w$. Then M_w is preserved by Ad_{G_w} and ad_{L_w}. Since the restriction of the adjoint action of G_w on M_w to S_w is completely reducible, we have an Ad_{S_w}- invariant orthogonal direct sum decomposition

$$M_w = \sum_{j=1}^{m} E_{w,j} \quad (\dim M = 2m) , \tag{2.2}$$

where each $E_{w,j}$ is a real two dimensional vector space isomorphic, as an S_w representation space, to the irreducible representation $\Gamma_{a_j} : S_w \to \mathrm{GL}(2; \mathbf{R})$ defined by

$$\Gamma_{a_j}(\exp(s)) = \begin{pmatrix} \cos a_j(s) & -\sin a_j(s) \\ \sin a_j(s) & \cos a_j(s) \end{pmatrix} \quad (s \in \mathfrak{s}_w) . \tag{2.3}$$

Here $a_j \in \mathfrak{s}_w^*$ is the weight of Γ_{a_j} (via $\langle\ ,\ \rangle_{|\mathfrak{s}_w}$, a_j may be viewed as an element of $\mathfrak{s}_w$) and we choose a_j so that $a_j(w) > 0$. Then $E_{w,j}$ is oriented by the basis for which the action of S_w is represented by Γ_{a_j}. The almost complex structure J on TM is defined as

$$J_w X = \frac{1}{a_j(w)}[w, X] \quad (w \in M,\ X \in E_{w,j}) . \tag{2.4}$$

This almost complex structure is integrable and G- invariant, see [B]. We call J the canonical complex structure of M.

Each G- invariant Kähler form on M, compatible with J, is constructed as follows; let $\mathfrak{s}$ be the vector bundle

$$\mathfrak{s} = \bigcup_{w \in M} \mathfrak{s}_w \longrightarrow M .$$

For a G- invariant section σ of $\mathfrak{s}$, we define the 2-form

$$B_w^\sigma(X, Y) = \langle \sigma(w), [U, V] \rangle \quad (w \in M,\ X, Y \in T_w M) , \tag{2.5}$$

where $U, V \in \mathfrak{g}$ are satisfied with $X = [U, w]$, $Y = [V, w]$. This is the G- invariant, closed 2-form of type $(1, 1)$. Moreover, if σ satisfies $\langle a_j, \sigma(w) \rangle > 0$ for any $w \in M$ and j, B^σ is positive definite. Conversely, for any G- invariant Kähler form ω on M, compatible with J, there is a G- invariant section σ of $\mathfrak{s}$, which satisfies $\langle a_j, \sigma(w) \rangle > 0$ for any $w \in M$ and j, such that $\omega = B^\sigma$, see [B]. Note here that the restriction of the Ad_G- invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ to M is not, in general, Kähler compatible with the canonical complex structure J, but Hermitian.

We supply two examples of G- invariant Kähler forms on M. The first example is given as follows,

$$F_w(X, Y) = \langle w, [U, V] \rangle \quad (w \in M,\ X, Y \in T_w M) ,$$

where $U, V \in \mathfrak{g}$ are satisfied with $X = [U, w]$, $Y = [V, w]$ and w is viewed as the tautological section of $\mathfrak{s}$. We shall refer to this as the canonical symplectic form of M.

The other one is defined by the G- invariant section γ of $\mathfrak{s}$; for an orthonormal basis $\{e_j, J_w e_j\}$ of $(E_{w,j}, \langle\ ,\ \rangle)$, γ is given by

$$\gamma(w) = \sum_{j=1}^{m} [e_j, J_w e_j] .$$

In fact, for any G- invariant Kähler metric, we see that its Ricci form is equal to

$$\rho_w(X, Y) = \langle \gamma(w), [U, V] \rangle , \tag{2.6}$$

and ρ itself is the Kähler-Einstein form, see [B].

3. The proofs of Theorems 1.3 and 1.4

First we will prove Theorem 1.3.
In this section, we use the notations prepared in the previous section.
We proved the following lemma in [O].

LEMMA 3.1. *Let $M^{2m} \subset \mathfrak{g}$ be an adjoint orbit with $\omega = \alpha F$. II and H denote the second fundamental form and the mean curvature vector of $x : M \to \mathfrak{g}$ respectively. Then we have*

$$II(X, Y) = II(JX, JY) \quad (X \text{ and } Y \text{ are vector fields on } M)\,, \tag{3.1}$$

and

$$H_w = \frac{-\gamma(w)}{m\alpha} \quad (w \in M)\,. \tag{3.2}$$

On the other hand, we have the following fundamental fact.

LEMMA 3.2. *Let $x : (M^{2m}, J, g) \to (N, \bar{g})$ be an immersion from an almost Hermitian manifold (M, J, g) to an Riemannian manifold $(N, \bar{g})$. Suppose that the second fundamental form II of the immersion x satisfies $II(X, Y) = II(JX, JY)$. Then an m-dimensional totally real submanifold $L \subset M$ is minimal if and only if*

$$\tilde{\tau} = \frac{1}{2}\tau \quad (\text{on } L)\,, \tag{3.3}$$

where $\tilde{\tau}$ and τ are the tension fields of $x_{|L}$ and x respectively.

PROOF. Let $\bar{H}$ be the mean curvature vector of $L \subset M$ and $\{X_i\}_{i=1}^m$ an orthonormal basis of $(T_pL, g_{|L})$. Then we have

$$\begin{aligned}\tilde{\tau}_p &= m\bar{H}_p + \sum_{i=1}^m II_p(X_i, X_i)\\ &= m\bar{H}_p + \frac{1}{2}\sum_{i=1}^m (II_p(X_i, X_i) + II_p(JX_i, JX_i))\\ &= m\bar{H}_p + \frac{1}{2}\tau_p\,.\end{aligned}$$

□

PROOF OF THEOREM 1.3. Let $\tilde{H}$ denote the mean curvature vector of the embedding $L \subset \mathfrak{g}$. By Lemmas 3.1 and 3.2, a Lagrangian submanifold L is minimal if and only if

$$\begin{aligned}\Delta_L x_{|L}(w) &= -m\tilde{H}_w\\ &= -\frac{1}{2}(2mH_w)\end{aligned}$$

$$= \frac{\gamma(w)}{\alpha} \quad (w \in L)\,.$$

The assumption that $\omega = \alpha F$ is Kähler-Einstein with $\rho = c\omega$ induces $\gamma(w) = c\alpha w$. Note here that $x : M \to \mathfrak{g}$ is the natural embedding. So we can identify $x(w)$ and w as the element of $\mathfrak{g}$. Then the minimality of L is equivalent to $\Delta_L x_{|L} = cx$ on L. The proof of Theorem 1.3 has finished. □

Next we will prove Theorem 1.4. Without the assumption $\omega = \alpha F$, the second fundamental form of the embedding $x : M \to \mathfrak{g}$ does not satisfy (3.1). So in this case, we cannot say about the minimality of Lagrangian submanifolds as Theorem 1.3. But, by using the fundamental inequality which is proved in [CMN], we can generalize Theorem 1.2 to the case $\omega \neq \alpha F$. So we review the inequality in [CMN].

Let (M^m, g) be a closed Riemannian manifold and $\Delta_{M,g}$ the Laplacian of M acting on $C^\infty(M)$. We write its eigenvalues $0 = \lambda_0(M) < \lambda_1(M) < \lambda_2(M) < \cdots < \lambda_k < \cdots \uparrow \infty$.

For any smooth map $x : M \to (\mathbf{R}^n, \langle\ ,\ \rangle)$ from M to an Euclidean space, we have the spectral decomposition of x

$$x = x_0 + \sum_{k \geq 1} x_k\,,$$

where $x_0 = \int_M x dv_g / \operatorname{Vol}(M)$ and each coordinate component of x_k is in the $\lambda_k(M)$-eigenspace. If x is non-constant map, then there are $p, q \geq 1$ such that $x_p, x_q \neq 0$ and $x = x_0 + \sum_{k=p}^{q} x_k$ (if there are infinitely many nonzero x_k's in the spectral decomposition, we put $q = \infty$). The pair $[p, q]$ is called the order of the map x. The inequality which we want to use is the following.

LEMMA 3.3. *Let $x : M \to (\mathbf{R}^n, \langle\ ,\ \rangle)$ be a non-constant map of the order $[p, q]$ from a compact Riemannian manifold to an Euclidean space. Then we have*

$$\lambda_p(M)(x - x_0, x - x_0)_{L^2} \leq 2E(x)\,, \tag{3.4}$$

where $E(x)$ is the energy of the map x. The equality holds if and only if the order of the map x is $[p, p]$.

The proof is simple. So we omit it. (See [CMN].) By using Lemma 3.3, we can prove the following.

PROPOSITION 3.4. *Let (M^{2m}, J, g) be a compact almost Hermitian manifold of* $\dim_{\mathbf{R}} M = 2m$. *If there is a smooth immersion $x : M \to (\mathbf{R}^{k+1}, \langle\ ,\ \rangle)$ which satisfies the following conditions*

(1) $\operatorname{Image} x \subset S^k(r)$,

(2) *the energy density function $e(x)$ of x is constant,*

(3) *for any vector fields X and Y on M, $\langle x_*X, x_*Y\rangle = \langle x_*JX, x_*JY\rangle$.*

Define

$$\mathcal{L}_a^p(x) = \left\{ L^m \subset M; \begin{array}{c} \text{totally real, the order of } x_{|L} \text{ is } [p, q], \\ \langle (x_{|L})_0, (x_{|L})_0 \rangle = a^2 \end{array} \right\}.$$

Then we have

$$\lambda_p(L) \le \frac{e(x)}{r^2 - a^2} \tag{3.5}$$

for any $L \in \mathcal{L}_a^p(x)$. The equality holds if and only if the order of $x_{|L}$ is $[p, p]$.

PROOF. Let $L \in \mathcal{L}_a^p(x)$ and $\{X_i\}_{i=1}^m$ be an orthonormal basis of $(T_pL, g_{|L})$. Then, since x satisfies the condition (2) and (3), we have

$$\begin{aligned} 2E(x_{|L}) &= \int_L \sum_{i=1}^m \langle x_*X_i, x_*X_i\rangle dv_g \\ &= \int_L \frac{1}{2}\sum_{i=1}^m (\langle x_*X_i, x_*X_i\rangle + \langle x_*JX_i, x_*JX_i\rangle) dv_g \\ &= e(x)\,\mathrm{Vol}(L)\,. \end{aligned}$$

On the other hand, since x satisfies the condition (1), we have

$$\begin{aligned} (x_{|L} - (x_{|L})_0, x_{|L} - (x_{|L})_0)_{L^2} &= (x_{|L}, x_{|L})_{L^2} - ((x_{|L})_0, (x_{|L})_0)_{L^2} \\ &= (r^2 - a^2)\,\mathrm{Vol}(L)\,. \end{aligned}$$

So this proposition is in consequence of Lemma 3.3. □

As the corollary of Proposition 3.4, when (M, ω) is a Kähler-Einstein manifold, we give a condition for the Hamiltonian stability (this concept was induced in [Oh]) of a compact minimal Lagrangian submanifold; let (M, ω) be a Kähler manifold, $L \subset M$ a compact minimal Lagrangian submanifold and V a normal variation vector along L. Since L is Lagrangian, we can regard $(V \rfloor \omega)_{|L}$ as an 1-form on L. If $(V \rfloor \omega)_{|L}$ is exact, V is called a Hamiltonian variation vector. A smooth family $\{\iota_t\}$ of embeddings of L into M is called a Hamiltonian deformation, if its derivative is Hamiltonian. Note that Hamiltonian deformations leave Lagrangian submanifolds Lagrangian. We say that a compact minimal Lagrangian submanifold is Hamiltonian stable, if the second variation of volume is nonnegative for all Hamiltonian deformations of L. One of the main theorem proved in [Oh] is Theorem 1.1.

COROLLARY 3.5. *Let (X, ω) be a compact Kähler-Einstein manifold with $\rho = c\omega$ and $\lambda_1(M) = 2c$, where ρ is the Ricci form of (X, ω). If there is a smooth immersion $x : M \to (\mathbf{R}^{k+1}, \langle\ ,\ \rangle)$ which satisfies the conditions* (1), (2) *and* (3) *in Proposition* 3.4. *Moreover suppose that x satisfies $\Delta_M x = 2c(x - x_0)$. Then we have*

1. *If L is a Hamiltonian stable minimal Lagrangian submanifold, then $L \in \mathcal{L}_b^p(x)$ for some $b \ge \langle x_0, x_0\rangle^{1/2} = a$.*
2. *A compact minimal Lagrangian submanifold L in $\mathcal{L}_a^p(x)$ is Hamiltonian stable if and only if $\lambda_1(L) = c$.*

PROOF. By the condition (1), (2), $\Delta_M x = 2c(x - x_0)$ and Lemma 3.3, we have

$$c = \frac{e(x)}{r^2 - a^2}\,.$$

If $L \in \mathcal{L}_b^p(x)$ for some $b < a$, then, by Proposition 3.4

$$\lambda_1(L) \le \lambda_p(L) \le \frac{e(x)}{r^2 - b^2} < c\,.$$

So L is not Hamiltonian stable. The assertion 2 can be proved similarly. □

COROLLARY 3.6. *Let $(M^{2m}, J, \langle\ ,\ \rangle_{|M})$ be an adjoint orbit in $\mathfrak{g}$. Suppose that $\omega = \alpha F$ and that $\langle\ ,\ \rangle_{|M}$ is Kähler-Einstein with its Ricci form $\rho = c\omega$ for a positive constant c. Denote its embedding by $x : M \to \mathfrak{g}$. Then the following conditions are equivalent for a Lagrangian submanifold L;*

(1) *L is minimal and Hamiltonian stable.*

(2) *$\int_L x_{|L} dv = 0$ and $\lambda_1(L) = c$.*

PROOF. By Theorem 1.3, if L is minimal and Hamiltonian stable, then (2) holds. On the other hand, since $\Delta_M x = 2cx$ by the assumption, if $\int_L x_{|L} dv = 0$ and $\lambda_1(L) = c$, then the order of $x_{|L}$ is $[1, 1]$ by Proposition 3.4. So L is minimal and Hamiltonian stable by Theorems 1.2 and 1.3. □

Let $(M^{2m}, J, B^{\frac{\gamma}{c}})$ be an adjoint orbit in $\mathfrak{g}$ and $x : M \to \mathfrak{g}$ denote the embedding. Note here that the Kähler form $B^{\frac{\gamma}{c}}$ is Kähler-Einstein with $\rho = cB^{\frac{\gamma}{c}}$. Suppose that $\Delta_{M,g} x = 2cx$. In this case, the conditions (1) and (3) in Proposition 3.4 are correct. The condition (2) is confirmed by the following lemma.

LEMMA 3.7. *The energy density function $e(x)$ of the embedding $x : (M, J, B^\sigma) \to (\mathfrak{g}, \langle\ ,\ \rangle)$ is equal to $\sum_{i=1}^m \frac{(a_i(w))^2}{\langle a_i, \sigma(w)\rangle}$. This does not depend on $w \in M$.*

PROOF. By the definition of the energy density function, we have

$$e(x)(w) = \frac{1}{2}\sum_{i=1}^{2m} \langle Y_i, Y_i\rangle$$

for any orthonormal basis $\{Y_1, \cdots, Y_{2m}\}$ of $(T_w M, g^\sigma)$, where $g^\sigma(X, Y) = B^\sigma(X, JY)$.

LEMMA 3.8. *Let $\{e_1, J_w e_1, \cdots, e_m, J_w e_m\}$ be an orthonormal basis of $(T_w M, \langle\ ,\ \rangle_{|M})$ with $span_{\mathbf{R}}\{e_i, J_w e_i\} = E_{w,i}$. Then $\{X_1, J_w X_1, \cdots, X_m, J_w X_m\}$ is an orthonormal basis of $(T_w M, g^\sigma)$, where $X_i = \frac{a_i(w)}{\sqrt{\langle a_i, \sigma(w)\rangle}} e_i$.*

PROOF. For $X \in E_{w,i}$, by the difinition of J, we have

$$X = \left[\frac{J_w X}{a_i(w)}, w\right], \quad J_w X = \left[\frac{-X}{a_i(w)}, w\right] \tag{3.6}$$

So, for any $Z_i \in E_{w,i}$ and $Z_j \in E_{w,j}$ $(i \neq j)$, by (2.5)

$$\begin{aligned} g^{\sigma}_{w}(Z_i, Z_j) &= B^{\sigma}_{w}(Z_i, J_w Z_j) \\ &= \frac{1}{a_i(w)a_j(w)}\langle \sigma(w), [J_w Z_i, -Z_j]\rangle \\ &= \frac{1}{a_i(w)a_j(w)}\langle [J_w Z_i, \sigma(w)], Z_j\rangle \\ &= 0\,, \end{aligned}$$

where the third equality is derived from the Ad_G-invariance of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, and the fourth one is derived from $[J_w Z_i, \sigma(w)] \in E_{w,i}$ and $Z_j \in E_{w,j}$. Similarly, since $[\sigma(w), X_i] = \langle a_i, \sigma(w)\rangle J_w X_i$, we have

$$\begin{aligned} g^{\sigma}_{w}(e_i, e_i) &= B^{\sigma}_{w}(e_i, J_w e_i) \\ &= \frac{1}{(a_i(w))^2}\langle \sigma(w), [J_w e_i, -e_i]\rangle \\ &= \frac{1}{(a_i(w))^2}\langle [\sigma(w), e_i], J_w e_i\rangle \\ &= \frac{\langle a_i, \sigma(w)\rangle}{(a_i(w))^2}\langle e_i, e_i\rangle \\ &= \frac{\langle a_i, \sigma(w)\rangle}{(a_i(w))^2}\,. \end{aligned}$$

□

By Lemma 3.8, we have

$$\begin{aligned} e(x)(w) &= \frac{1}{2}\sum_{i=1}^{m}(\langle X_i, X_i\rangle + \langle J_w X_i, J_w X_i\rangle) \\ &= \sum_{i=1}^{m}\frac{(a_i(w))^2}{\langle a_i, \sigma(w)\rangle}\,. \end{aligned}$$

If a_i is the weight at $w \in M$, then $\mathrm{Ad}(g)a_i$ is the weight at $\mathrm{Ad}(g)w \in M$. So, by the G-invariance of σ and Ad_G-invariance of $\langle\ ,\ \rangle$, the right hand side is independent of $w \in M$. □

PROOF OF THEOREM 1.4. We have seen that the adjoint orbit $(M^{2m}, J, B^{\frac{\gamma}{c}})$ with $\Delta_{M,g}x = 2cx$ satisfies the conditions (1), (2) and (3) in Proposition 3.4. So, by Corollary 3.5, we have proved Theorem 1.4. □

4. The examples of adjoint orbits

In this section, we will see some examples of adjoint orbits which satisfy the assumption of Theorem 1.4. (But the concrete calculations are given in Appendix.)

PROPOSITION 4.1. *Let (M^{2m}, J, g^{σ}) be an adjoint orbit and $x : M \to \mathfrak{g}$ denote the embedding. Then the tension field τ of the embedding x is*

$$\tau_w = 2\sum_{i=1}^{m} \frac{a_i(w)}{\langle a_i, \sigma(w)\rangle}[J_w e_i, e_i], \tag{4.1}$$

where $\{e_1, J_w e_1, \cdots, e_m, J_w e_m\}$ is an orthonormal basis of $(T_w M, \langle\ ,\ \rangle_{|M})$ with $e_i \in E_{w,i}$.

PROOF. We compute the second fundamental form II of $(M, g^{\sigma}) \to (\mathfrak{g}, \langle\ ,\ \rangle)$. In particular, we only want to know $II(X, X)$. Let D and ∇ denote the Levi-Civita connections of $(\mathfrak{g}, \langle\ ,\ \rangle)$ and (M, g^{σ}) respectively. Then, by the definition of the second fundamental form, $II_w(X, X) = (D_X X)(w) - (\nabla_X X)(w)$. Since II is the tensor, $II_w(X, X) = II_w(X_U, X_U)$, where the fundamental vector field X_U ($U \in \mathfrak{g}$) satisfies $X(w) = X_U(w) = [U, w]$.

$$(D_{X_U} X_U)(w) = [U, [U, w]]. \tag{4.2}$$

On the other hand, for $\nabla_{X_U} X_U$, we have the following lemma.

LEMMA 4.2.

$$g^{\sigma}(\nabla_{X_U} X_U, X_V) = g^{\sigma}(X_{[V,U]}, X_U). \tag{4.3}$$

PROOF. In general, if X is a Killing vector field, then we have $g^{\sigma}(\nabla_Y X, Z) + g^{\sigma}(\nabla_Z X, Y) = 0$. Since X_U and X_V are Killing vector fields,

$$\begin{aligned} g^{\sigma}(X_{[V,U]}, X_U) &= g^{\sigma}([X_U, X_V], X_U) \\ &= g^{\sigma}(\nabla_{X_U} X_V, X_U) - g^{\sigma}(\nabla_{X_V} X_U, X_U) \\ &= g^{\sigma}(\nabla_{X_U} X_U, X_V). \end{aligned}$$

Note here that $[X_U, X_V]$ is the bracket of the vector fields. □

LEMMA 4.3. *Let $X_j \in E_{w,j}$. Then we have*

$$II_w(X_j, X_j) = \frac{1}{a_j(w)}[J_w X_j, X_j]. \tag{4.4}$$

PROOF. By using (3.6) and (4.2), $II_w(X_j, X_j)$ is computed as

$$\begin{aligned} II_w(X_j, X_j) &= (D_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}})(w) - (\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}})(w) \\ &= \frac{1}{a_j(w)}[J_w X_j, X_j] - (\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}})(w). \end{aligned}$$

Since the second term of the right hand side of the above equation is tangent to M at w, it is sufficient to prove that

$$g^{\sigma}_w(\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}}, X_k) = g^{\sigma}_w(\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}}, J_w X_k) = 0, \tag{4.5}$$

for any k.

$$
\begin{aligned}
g^{\sigma}_{w}(\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}}, X_k) &= g^{\sigma}_{w}(\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}}, X_{\frac{J_w X_k}{a_k(w)}}) \qquad \text{(by (3.6))} \\
&= g^{\sigma}_{w}(X_{\left[\frac{J_w X_k}{a_k(w)}, \frac{J_w X_j}{a_j(w)}\right]}, X_{\frac{J_w X_j}{a_j(w)}}) \qquad \text{(by Lemma 4.2)} \\
&= \frac{1}{(a_j(w))^2 a_k(w)} \langle \sigma(w), [[J_w X_j, J_w X_k], X_j] \rangle \\
&= \frac{1}{(a_j(w))^2 a_k(w)} \langle [J_w X_j, [\sigma(w), X_j]], J_w X_k \rangle ,
\end{aligned}
$$

where the third equality is derived from the definition of g^{σ}, and the fourth one is derived from the Ad_G-invariance of $\langle\ ,\ \rangle$. Since $\sigma(w) \in \mathfrak{s}_w$ and $X_j \in E_{w,j}$, we have $[\sigma(w), X_j] = \langle a_j, \sigma(w) \rangle J_w X_j$. So $g^{\sigma}_{w}(\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}}, X_k)$ equals to zero. Similarly $g^{\sigma}_{w}(\nabla_{X_{\frac{J_w X_j}{a_j(w)}}} X_{\frac{J_w X_j}{a_j(w)}}, J_w X_k)$ equals to zero. □

By Lemma 3.8 and Lemma 4.3, we have

$$
\begin{aligned}
\tau_w &= \sum_{i=1}^{m} (II_w(X_i, X_i) + II_w(J_w X_i, J_w X_i)) \\
&= 2 \sum_{i=1}^{m} \frac{1}{a_i(w)} [J_w X_i, X_i] \\
&= 2 \sum_{i=1}^{m} \frac{a_i(w)}{\langle a_i, \sigma(w) \rangle} [J_w e_i, e_i],
\end{aligned}
$$

where $\{e_1, J_w e_1, \cdots, e_m, J_w e_m\}$ is an orthonormal basis of $(T_w M, \langle\ ,\ \rangle_{|M})$ and $X_i = \frac{a_i(w)}{\sqrt{\langle a_i, \sigma(w) \rangle}} e_i$. Thus the proof of Proposition 4.1 has finished. □

For example, if the canonical symplectic form F is Kähler-Einstein with $\rho = cF$, then (M, F) satisfies the assumption in Theorem 1.4.

PROPOSITION 4.4. *Let (M^{2m}, J, F) be an adjoint orbit in $\mathfrak{g}$ and $x : M \to \mathfrak{g}$ denote the embedding. Suppose that the canonical symplectic form F is Kähler-Einstein with $\rho = cF$. Then we have $\Delta_M x = 2cx$. Thus, by Theorem* 1.4, *the following three conditions are equivalent, for a compact minimal Lagrangian submanifold $L \subset (M, F)$ with $\int_L x_{|L} dv = 0$.*

1. *L is Hamiltonian stable.*
2. $\lambda_1(L) = c$.
3. *All of the coordinate functions of $x_{|L}$, $x^i_{|L}$, are $c = \lambda_1(L)$-eigenfunctions.*

PROOF. The assumption $\rho = cF$ is equivalent to $\gamma(w) = cw$ for any $w \in M$. On the other hand, the tension field τ of the embedding $x : (M, J, F) \to (\mathfrak{g}, \langle\ ,\ \rangle)$ is

$$\tau_w = -2\gamma(w)$$

by Proposition 4.1. So $\Delta_M x = -\tau_x = 2cx$. □

The examples of adjoint orbits which satisfy $\gamma(w) = cw$ are as follows (see Appendix);

- Let $G = SU(n)$. The adjoint orbit of

$$\frac{\sqrt{-1}}{c}\begin{pmatrix} \beta_1 I_{p_1} & & & & \\ & \ddots & & & \\ & & \beta_l I_{p_l} & & \\ & & & \ddots & \\ & & & & \beta_q I_{p_q} \end{pmatrix} \in \mathfrak{su}(n)$$

satisfies $\gamma(w) = cw$, where $\sum p_i = n$ and $\beta_l = \sum_{i=l+1}^{q} p_i - \sum_{i=1}^{l-q} p_i$.

- Let $G = SO(2n)$. The adjoint orbits of

$$\frac{1}{c}\begin{pmatrix} \gamma_1 J_{p_1} & & & \\ & \ddots & & \\ & & \gamma_q J_{p_q} & \\ & & & 0 I_{2r} \end{pmatrix} \in \mathfrak{so}(2n) \quad \left(\sum p_i + r = n\right)$$

and

$$\frac{1}{c}\begin{pmatrix} \gamma_1' J_{p_1} & & & & \\ & \ddots & & & \\ & & \gamma_{q-1}' J_{p_q-1} & & \\ & & & \frac{p_q}{2} J_{p_q} & \\ & & & & -\frac{p_q}{2} J \end{pmatrix} \in \mathfrak{so}(2n) \quad \left(\sum p_i + 1 = n\right)$$

satisfy $\gamma(w) = cw$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}(2)\,, \quad J_p = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \in \mathfrak{gl}(2p)\,,$$

$$\gamma_i = \begin{cases} \frac{1}{2}(p_i - 1) + p_{i+1} + \cdots + p_q + r & (i = 1, \cdots, q-1)\,, \\ \frac{1}{2}(p_q - 1) + r & (i = q)\,, \end{cases}$$

and

$$\gamma_i' = \frac{1}{2}(p_i + 1) + p_{i+1} + \cdots + p_q\,.$$

On the other hand, we have an example which does not $\rho = dF$ for any constant $d > 0$, but $\Delta_M x = 2cx$;

Let $G = SO(4)$. The adjoint orbit of

$$\begin{pmatrix} \mu_1 J & 0 \\ 0 & \mu_2 J \end{pmatrix} \in \mathfrak{so}(4) \quad (\mu_1 > \mu_2 > 0)$$

does not $\rho = dF$ for any constant $d > 0$, but $\Delta_M x = 2cx$. (See Appendix.)

A. Appendix

In this Appendix, we compute γ and the tension field τ of the embedding $x : (M, J, B^\gamma = \rho) \to (\mathfrak{g}, \langle\ ,\ \rangle)$, when G is $SU(n)$ or $SO(n)$.

A.1. The case $G = SU(n)$

We use $\langle X, Y\rangle = -\operatorname{trace} XY$, $X, Y \in \mathfrak{su}(n)$, as the $\mathrm{Ad}_{SU(n)}$ invariant inner product on $\mathfrak{su}(n)$. Let M be the adjoint orbit of

$$w_0 = \sqrt{-1}\begin{pmatrix} \mu_1 I_{p_1} & & 0 \\ & \ddots & \\ 0 & & \mu_q I_{p_q} \end{pmatrix} \in \mathfrak{su}(n)\,,$$

where μ_i and p_j satisfy the following conditions;

$$\mu_1 > \cdots > \mu_q\,, \quad \sum_{j=1}^{q} p_j = n\,, \quad \sum_{j=1}^{q} p_j\mu_j = 0\,,$$

and $I_p \in \mathfrak{gl}(p)$ is the identity matrix. If $q = 2$, then we can see that $\langle\ ,\ \rangle_{|M}$ satisfies the assumption of Theorem 1.2. So we suppose that $q > 2$.

By simple calculation, we see that

$$M_{w_0} = \left\{ \begin{pmatrix} 0 & X_{12} & \cdots & X_{1q} \\ -{}^t\bar{X}_{12} & 0 & & \vdots \\ \vdots & & \ddots & \\ -{}^t\bar{X}_{1q} & \cdots & & 0 \end{pmatrix} ;\ X_{ij} \in M(p_i, p_j; \mathbf{C}) \right\},$$

$$S_{w_0} = \left\{ \begin{pmatrix} \exp(\sqrt{-1}\theta_1) I_{p_1} & & 0 \\ & \ddots & \\ 0 & & \exp(\sqrt{-1}\theta_q) I_{p_q} \end{pmatrix} ;\ \sum_{j=1}^{q} p_j\theta_j = 0 \right\},$$

and

$$\mathfrak{s}_{w_0} = \left\{ \sqrt{-1}\begin{pmatrix} \theta_1 I_{p_1} & & 0 \\ & \ddots & \\ 0 & & \theta_q I_{p_q} \end{pmatrix} ;\ \sum_{j=1}^{q} p_j\theta_j = 0 \right\}.$$

Next, we give an $\mathrm{Ad}_{S_{w_0}}$-decomposition of M_{w_0}.

For $1 \leq i < j \leq q$, $1 \leq k \leq p_i$, and $1 \leq l \leq p_j$, we prepare the following matrices;

$$X_{\{ij,kl\}} = \begin{pmatrix} x_{11} & \cdots & x_{1p_j} \\ \vdots & & \vdots \\ x_{p_i 1} & \cdots & x_{p_i p_j} \end{pmatrix} \in M(p_i, p_j; \mathbf{C}),$$

where

$$x_{ab} = \begin{cases} 1 & (a = k \text{ and } b = l), \\ 0 & (\text{otherwise}), \end{cases}$$

$$e_{\{ij,kl\}} = \frac{1}{\sqrt{2}} \begin{array}{c} \\ i \\ \\ j \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & j \end{array} \\ \begin{pmatrix} & & \vdots & & \vdots \\ \cdots & & \cdots & \cdots & X_{\{ij,kl\}} \\ & & \vdots & & \\ \cdots & -{}^t X_{\{ij,kl\}} & & & \end{pmatrix} \end{array} \in M_{w_0},$$

and

$$f_{\{ij,kl\}} = \frac{\sqrt{-1}}{\sqrt{2}} \begin{array}{c} \\ i \\ \\ j \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & j \end{array} \\ \begin{pmatrix} & & \vdots & & \vdots \\ \cdots & & \cdots & \cdots & X_{\{ij,kl\}} \\ & & \vdots & & \\ \cdots & {}^t X_{\{ij,kl\}} & & & \end{pmatrix} \end{array} \in M_{w_0},$$

where the blank parts of the matrices above are zero matrices. Then we have an $\mathrm{Ad}_{S_{w_0}}$ decomposition of M_{w_0} as follows;

$$M_{w_0} = \sum_{1 \leq i < j \leq q} \sum_{\substack{1 \leq k \leq p_i \\ 1 \leq l \leq p_j}} E_{w_0,\{ij,kl\}},$$

where

$$E_{w_0,\{ij,kl\}} = span_{\mathbf{R}}\{e_{\{ij,kl\}}, f_{\{ij,kl\}}\}.$$

Moreover we see, by simple matrix calculation, the following.

- $\{e_{\{ij,kl\}},\ f_{\{ij,kl\}}\}$ is an orthonormal basis of $(E_{w_0,\{ij,kl\}}, \langle\ ,\ \rangle)$.
- On $E_{w_0,\{ij;kl\}}$, the weight $a_{\{ij;kl\}}$ of S_{w_0} is

$$a_{\{ij,kl\}}\left(\sqrt{-1}\begin{pmatrix} \theta_1 I_{p_1} & & \\ & \ddots & \\ & & \theta_q I_{p_q} \end{pmatrix}\right) = \theta_i - \theta_j.$$

- The canonical complex structure J is given by

$$J_{w_0} e_{\{ij,kl\}} = \frac{1}{a_{\{ij,kl\}}(w_0)}[w_0, e_{\{ij,kl\}}] = f_{\{ij,kl\}}$$

at w_0.

-

$$\gamma(w_0) = \sqrt{-1}\begin{pmatrix} \beta_1 I_{p_1} & & & & \\ & \ddots & & & \\ & & \beta_l I_{p_l} & & \\ & & & \ddots & \\ & & & & \beta_q I_{p_q} \end{pmatrix} \in \mathfrak{s}_{w_0},$$

for $\beta_l = \sum_{j=l+1}^{q} p_j - \sum_{j=1}^{l-1} p_j$.

- For $1 \le i < j \le q$,

$$\begin{aligned} \alpha_{ij} &:= (a_{\{ij,kl\}}, \gamma(w_0)) \\ &= \beta_i - \beta_j \\ &= \begin{cases} p_i + p_{i+1} & (j = i+1), \\ p_i + 2p_{i+1} + p_{i+2} & (j = i+2), \\ p_i + 2(p_{i+1} + \cdots + p_{j-1}) + p_j & (j > i+2). \end{cases} \end{aligned} \tag{A.1}$$

By Proposition 4.1 and some calculations,

$$\tau_{w_0} = 2\sqrt{-1} \sum_{1 \le i < j \le q} \frac{\mu_i - \mu_j}{\alpha_{ij}} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \quad & i & \quad & j \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & -p_j I_{p_i} & & \vdots & \\ & & & \vdots & \\ \cdots & \cdots & \cdots & p_i I_{p_j} & \\ & & & & \end{pmatrix} \end{array}.$$

A.2. The case $G = SO(2n)$

We use $\langle X, Y \rangle = -\operatorname{trace} XY$, $X, Y \in \mathfrak{so}(2n)$ as an $\mathrm{Ad}_{SO(2n)}$ invariant inner product on $\mathfrak{so}(2n)$. Different from the $G = SU(n)$ case, we investigate adjoint orbits of the following types respectively; let

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}(2; \mathbf{R}), \quad J_p = \begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix} \in \mathfrak{gl}(2p; \mathbf{R}).$$

Case 1. The adjoint orbit $M^{SO(2n)}_{(p_1,\cdots,p_q;r)}$ of the following element;

$$w_1 = \begin{pmatrix} \mu_1 J_{p_1} & & & \\ & \ddots & & \\ & & \mu_q J_{p_q} & \\ & & & 0 I_{2r} \end{pmatrix} \in \mathfrak{so}(2n)\,,$$

where μ_i, p_i, and r satisfy the following conditions;

$$\mu_1 > \cdots > \mu_q > 0\,, \quad p_i > 0\,, \quad r > 0\,, \quad \sum_{i=1}^{q} p_i + r = n\,.$$

Case 2. The adjoint orbit $M^{SO(2n)}_{(p_1,\cdots,p_q)}$ of the following element;

$$w_2 = \begin{pmatrix} \mu_1 J_{p_1} & & \\ & \ddots & \\ & & \mu_q J_{p_q} \end{pmatrix} \in \mathfrak{so}(2n)\,,$$

where μ_i and p_i satisfy the following conditions;

$$\mu_1 > \cdots > \mu_q > 0\,, \quad p_i > 0\,, \quad \sum_{i=1}^{q} p_i = n\,.$$

Case 3. The adjoint orbit $\tilde{M}^{SO(2n)}_{(p_1,\cdots,p_q)}$ of the following element;

$$w_3 = \begin{pmatrix} \mu_1 J_{p_1} & & & \\ & \ddots & & \\ & & \mu_q J_{p_q} & \\ & & & -\mu_q J \end{pmatrix} \in \mathfrak{so}(2n)\,,$$

where μ_i and p_i satisfy the following conditions;

$$\mu_1 > \cdots > \mu_q > 0\,, \quad p_i > 0\,, \quad \sum_{i=1}^{q} p_i + 1 = n\,.$$

By simple calculation,

Case 1.

$$\operatorname{Image}\operatorname{ad}_{w_1} = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1q} & \tilde{X}_1 \\ \vdots & & \vdots & \vdots \\ -{}^t X_{1q} & \cdots & X_{qq} & \tilde{X}_q \\ -{}^t \tilde{X}_1 & \cdots & -{}^t \tilde{X}_q & 0 \end{pmatrix} ;\ \begin{matrix} X_{ii} \in \operatorname{Image}\operatorname{ad}_{J_{p_i}} \subset \mathfrak{so}(2p_i), \\ X_{ij} \in M(2p_i, 2p_j; \mathbf{R})\ (i < j), \\ \tilde{X}_i \in M(2p_i, 2r; \mathbf{R}) \end{matrix} \right\}\,,$$

where

$$\operatorname{Image}\operatorname{ad}_{J_p} = \left\{ \begin{pmatrix} 0 & A_{12} & \cdots & A_{1q} \\ -A_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -A_{1q} & \cdots & \cdots & 0 \end{pmatrix} ;\ A_{ij} \in \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} ;\ a, b \in \mathbf{R} \right\} \right\} .$$

$$S_{w_1} = \left\{ \begin{pmatrix} R_{p_1}(\theta_1) & & & \\ & \ddots & & \\ & & R_{p_q}(\theta_q) & \\ & & & I_{2r} \end{pmatrix} ; \theta_1, \cdots, \theta_q \in \mathbf{R} \right\} ,$$

where

$$R_p(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & & & 0 \\ \sin\theta & \cos\theta & & & \\ & & \ddots & & \\ & & & \cos\theta & -\sin\theta \\ 0 & & & \sin\theta & \cos\theta \end{pmatrix} \in SO(2p) .$$

$$\mathfrak{s}_{w_1} = \left\{ \begin{pmatrix} t_1 J_{p_1} & & & \\ & \ddots & & \\ & & t_q J_{p_q} & \\ & & & 0 \end{pmatrix} ; t_1, \cdots, t_q \in \mathbf{R} \right\} .$$

Case 2.

$$\operatorname{Image}\operatorname{ad}_{w_2} = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1q} \\ \vdots & & \vdots \\ -{}^t X_{1q} & \cdots & X_{qq} \end{pmatrix} ;\ \begin{array}{l} X_{ii} \in \operatorname{Image}\operatorname{ad}_{J_{p_i}} \subset \mathfrak{so}(2p_i), \\ X_{ij} \in M(2p_i, 2p_j; \mathbf{R})\ (i < j) \end{array} \right\} .$$

$$S_{w_2} = \left\{ \begin{pmatrix} R_{p_1}(\theta_1) & & \\ & \ddots & \\ & & R_{p_q}(\theta_q) \end{pmatrix} ; \theta_1, \cdots, \theta_q \in \mathbf{R} \right\} .$$

$$\mathfrak{s}_{w_2} = \left\{ \begin{pmatrix} t_1 J_{p_1} & & \\ & \ddots & \\ & & t_q J_{p_q} \end{pmatrix} ; t_1, \cdots, t_q \in \mathbf{R} \right\} .$$

Case 3.

$$\operatorname{Image}\operatorname{ad}_{w_3} = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1q} & \tilde{X}_1 \\ \vdots & & \vdots & \vdots \\ -{}^tX_{1q} & \cdots & X_{qq} & \tilde{X}_q \\ -{}^t\tilde{X}_1 & \cdots & -{}^t\tilde{X}_q & 0 \end{pmatrix} ;\ \begin{array}{c} X_{ii} \in \operatorname{Image}\operatorname{ad}_{J_{p_i}} \subset \mathfrak{so}(2p_i), \\ X_{ij} \in M(2p_i, 2p_j; \mathbf{R})\ (i < j), \\ \tilde{X}_i \in M(2p_i, 2; \mathbf{R})\ (1 \leq i \leq q-1), \\ \tilde{X}_q = {}^t(A_1, \cdots, A_q), \\ \text{where } A_j \in \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} ; a, b \in \mathbf{R} \right\} \end{array} \right\}.$$

$$\mathfrak{s}_{w_3} = \left\{ \begin{pmatrix} t_1 J_{p_1} & & & \\ & \ddots & & \\ & & t_q J_{p_q} & \\ & & & -t_q J \end{pmatrix} ; t_1, \cdots, t_q \in \mathbf{R} \right\}.$$

Next we give an Ad_{w_i}-decomposition of $\operatorname{Image}\operatorname{ad}_{w_i}$, $\gamma(w_i)$ and τ_{w_i} for each $i = 1, 2, 3$. We prepare the following matrices in $\mathfrak{gl}(2; \mathbf{R})$;

$$e_1 = I_2\,, \quad e_2 = J\,, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $1 \leq i \leq q$, and $1 \leq j < k \leq p_i$,

$$\tilde{e}^3_{(i;jk)} = \begin{array}{c} \\ \\ j \\ \\ k \\ \\ \end{array} \!\! \begin{array}{c} \begin{array}{ccccc} \quad & j & \quad & k & \quad \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & \vdots & \cdots & e_3 & \\ & \vdots & & & \\ \cdots & -e_3 & & & \\ & & & & \end{pmatrix} \end{array} \in \operatorname{Image}\operatorname{ad}_{J_{p_i}} \subset \mathfrak{so}(2p_i)\,,$$

and

$$\tilde{e}^4_{(i;jk)} = \begin{array}{c} \\ \\ j \\ \\ k \\ \\ \end{array} \!\! \begin{array}{c} \begin{array}{ccccc} \quad & j & \quad & k & \quad \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & \vdots & \cdots & e_4 & \\ & \vdots & & & \\ \cdots & -e_4 & & & \\ & & & & \end{pmatrix} \end{array} \in \operatorname{Image}\operatorname{ad}_{J_{p_i}} \subset \mathfrak{so}(2p_i)\,.$$

For $1 \leq i < j \leq q$, $1 \leq k \leq p_i$, $1 \leq l \leq p_j$, and $m = 1, 2, 3, 4$,

$$\tilde{e}^m_{(ij;kl)} = \begin{array}{cc} & \begin{matrix} & l \end{matrix} \\ k & \begin{pmatrix} & \vdots & \\ \cdots & e_m & \\ & & \end{pmatrix} \end{array} \in M(2p_i, 2p_j; \mathbf{R})\,.$$

For $1 \leq i \leq q$, $1 \leq j \leq p_i$, $1 \leq k \leq r$, and $m = 1, 2, 3, 4$,

$$\tilde{f}^m_{r(i;jk)} = \begin{array}{cc} & \begin{matrix} & k \end{matrix} \\ j & \begin{pmatrix} & \vdots & \\ \cdots & e_m & \\ & & \end{pmatrix} \end{array} \in M(2p_i, 2r; \mathbf{R})\,.$$

For $1 \leq i \leq q$, $1 \leq j \leq p_i$, and $m = 1, 2, 3, 4$,

$$\tilde{f}^m_{(i;j)} = j \begin{pmatrix} \vdots \\ e_m \\ \vdots \end{pmatrix} \in M(2p_i, 2; \mathbf{R})\,.$$

Case 1. In this case, let

$$e^3_{(i;jk)} = \frac{1}{2} \begin{array}{cc} & \begin{matrix} & i \end{matrix} \\ i & \begin{pmatrix} & \vdots & \\ \cdots & \tilde{e}^3_{(i;jk)} & \\ & & \end{pmatrix} \end{array} \in \operatorname{Image} \operatorname{ad}_{w_1}\,,$$

$$e^4_{(i;jk)} = \frac{1}{2} \begin{array}{cc} & \begin{matrix} & i \end{matrix} \\ i & \begin{pmatrix} & \vdots & \\ \cdots & \tilde{e}^4_{(i;jk)} & \\ & & \end{pmatrix} \end{array} \in \operatorname{Image} \operatorname{ad}_{w_1}\,,$$

$$e^m_{(ij;kl)} = \frac{1}{2} \begin{array}{cc} & \begin{matrix} & i & & j \end{matrix} \\ \begin{matrix} \\ i \\ \\ j \end{matrix} & \begin{pmatrix} & \vdots & & \vdots \\ \cdots & \vdots & \cdots & \tilde{e}^m_{(ij;kl)} \\ & \vdots & & \\ \cdots & -{}^t\tilde{e}^m_{(ij;kl)} & & \\ & & & \end{pmatrix} \end{array} \in \operatorname{Image} \operatorname{ad}_{w_1}\,,$$

and

$$f^m_{r(i;jk)} = \frac{1}{2}\begin{array}{c} \\ i \\ \\ \end{array}\begin{array}{c} \quad\quad\quad i \\ \left(\begin{array}{cccc} & \vdots & & \\ \cdots & \cdots & \cdots & \tilde{f}^m_{r(i;jk)} \\ & \vdots & & \\ & -{}^t\tilde{f}^m_{r(i;jk)} & & \end{array}\right) \end{array} \in \operatorname{Image} \operatorname{ad}_{w_1} .$$

Then we have an $\operatorname{Ad}_{S_{w_1}}$ decomposition of $\operatorname{Image}\operatorname{ad}_{w_1}$ as follows;

$$\begin{aligned} \operatorname{Image}\operatorname{ad}_{w_1} = & \sum_{1\le i\le q}\ \sum_{1\le j<k\le p_i} D_{w_1,(i;jk)} \\ & + \sum_{1\le i<j\le q}\ \sum_{\substack{1\le k\le p_i \\ 1\le l\le p_j}} (E^1_{w_1,(ij;kl)} + E^2_{w_1,(ij;kl)}) \\ & + \sum_{1\le i\le q}\ \sum_{\substack{1\le j\le p_i \\ 1\le k\le r}} (F^1_{w_1,(i;jk)} + F^2_{w_1,(i;jk)}) , \end{aligned}$$

where

$$D_{w_1,(i;jk)} = span_{\mathbf{R}}\{e^3_{(i;jk)}, e^4_{(i;jk)}\} ,$$

$$E^1_{w_1,(ij;kl)} = span_{\mathbf{R}}\{e^1_{(ij;kl)}, e^2_{(ij;kl)}\} , E^2_{w_1,(ij;kl)} = span_{\mathbf{R}}\{e^3_{(ij;kl)}, e^4_{(ij;kl)}\} ,$$

and

$$F^1_{w_1,(i;jk)} = span_{\mathbf{R}}\{f^1_{r(i;jk)}, f^2_{r(i;jk)}\} , F^2_{w_1,(i;jk)} = span_{\mathbf{R}}\{f^3_{r(i;jk)}, f^4_{r(i;jk)}\} .$$

Moreover we see, by simple matrix calculation, the following.

- $\{e^3_{(i;jk)}, e^4_{(i;jk)}\}$, $\{e^1_{(ij;kl)}, e^2_{(ij;kl)}\}$, $\{e^3_{(ij;kl)}, e^4_{(ij;kl)}\}$, $\{f^1_{r(i;jk)}, f^2_{r(i;jk)}\}$, and $\{f^3_{r(i;jk)}, f^4_{r(i;jk)}\}$ are orthonormal basises of $(D_{w_1,(i;jk)}, \langle\ ,\ \rangle)$, $(E^1_{w_1,(ij;kl)}, \langle\ ,\ \rangle)$, $(E^2_{w_1,(ij;kl)}, \langle\ ,\ \rangle)$, $(F^1_{w_1,(i;jk)}, \langle\ ,\ \rangle)$, and $(F^2_{w_1,(i;jk)}, \langle\ ,\ \rangle)$ respectively.
- On $D_{w_1,(i;jk)}$, $E^1_{w_1,(ij;kl)}$, $E^2_{w_1,(ij;kl)}$, $F^1_{w_1,(i;jk)}$, and $F^2_{w_1,(i;jk)}$, their weights, $a_{(i;jk)}$, $a^1_{(ij;kl)}$, $a^2_{(ij;kl)}$, $a^1_{(i;jk)}$, and $a^2_{(i;jk)}$ respectively, of the action of S_{w_1} are as follows; for

  $$X = \begin{pmatrix} t_1 J_{p_1} & & & \\ & \ddots & & \\ & & t_q J_{p_q} & \\ & & & 0 \end{pmatrix} \in \mathfrak{s}_{w_1} ,$$

  we have

  $$a_{(i;jk)}(X) = 2t_i , \quad a^1_{(ij;kl)}(X) = t_i - t_j ,$$

$$a^2_{(ij;kl)}(X) = t_i + t_j\,, \quad a^1_{(i;jk)}(X) = a^2_{(i;jk)}(X) = t_i\,.$$

-
$$\gamma(w_1) = \begin{pmatrix} \gamma_1 J_{p_1} & & & \\ & \ddots & & \\ & & \gamma_q J_{p_q} & \\ & & & 0 I_{2r} \end{pmatrix},$$

where

$$\gamma_i = \begin{cases} \frac{1}{2}(p_i - 1) + p_{i+1} + \cdots + p_q + r & (i = 1, \cdots, q-1), \\ \frac{1}{2}(p_q - 1) + r & (i = q). \end{cases}$$

By Proposition 4.1 and some calculations,

$$\tau_{w_1} = \begin{pmatrix} \tau_1 J_{p_1} & & & \\ & \ddots & & \\ & & \tau_q J_{p_q} & \\ & & & 0 I_{2r} \end{pmatrix},$$

where

$$\begin{aligned} \tau_i = \frac{\mu_i}{\gamma_i}(-(p_i - 1) - 2r) - \left(\sum_{j=i+1}^{q} \frac{\mu_i - \mu_j}{\gamma_i - \gamma_j} p_j - \sum_{j=1}^{i-1} \frac{\mu_j - \mu_i}{\gamma_j - \gamma_i} p_j \right) \\ - \left(\sum_{j=i+1}^{q} \frac{\mu_i + \mu_j}{\gamma_i + \gamma_j} p_j + \sum_{j=1}^{i-1} \frac{\mu_j + \mu_i}{\gamma_j + \gamma_i} p_j \right). \end{aligned} \tag{A.2}$$

Case 2. This case is interpreted as Case 1 with $r = 0$. So we have

$$\gamma(w_2) = \begin{pmatrix} \gamma_1 J_{p_1} & & \\ & \ddots & \\ & & \gamma_q J_{p_q} \end{pmatrix},$$

where

$$\gamma_i = \begin{cases} \frac{1}{2}(p_i - 1) + p_{i+1} + \cdots + p_q & (i = 1, \cdots, q-1), \\ \frac{1}{2}(p_q - 1) & (i = q)\,, \end{cases}$$

and

$$\tau_{w_2} = \begin{pmatrix} \tau_1 J_{p_1} & & \\ & \ddots & \\ & & \tau_q J_{p_q} \end{pmatrix},$$

where

$$
\begin{aligned}
\tau_i = &-\frac{\mu_i}{\gamma_i}(p_i - 1) - \left(\sum_{j=i+1}^{q} \frac{\mu_i - \mu_j}{\gamma_i - \gamma_j} p_j - \sum_{j=1}^{i-1} \frac{\mu_j - \mu_i}{\gamma_j - \gamma_i} p_j \right) \\
&- \left(\sum_{j=i+1}^{q} \frac{\mu_i + \mu_j}{\gamma_i + \gamma_j} p_j + \sum_{j=1}^{i-1} \frac{\mu_j + \mu_i}{\gamma_j + \gamma_i} p_j \right).
\end{aligned}
\tag{A.3}
$$

Case 3. In this case, let

$$f^m_{(i;j)} = f^m_{1,(i;j1)} \in \mathrm{Image}\,\mathrm{ad}_{w_3} \ .$$

Then we have an $\mathrm{Ad}_{S_{w_3}}$ decomposition of $\mathrm{Image}\,\mathrm{ad}_{w_3}$ as follows;

$$
\begin{aligned}
\mathrm{Image}\,\mathrm{ad}_{w_3} = &\sum_{1\le i\le q} \sum_{1\le j<k\le p_i} D_{w_3,(i;jk)} \\
&+ \sum_{1\le i<j\le q} \sum_{\substack{1\le k\le p_i \\ 1\le l\le p_j}} (E^1_{w_3,(ij;kl)} + E^2_{w_3,(ij;kl)}) \\
&+ \sum_{1\le i\le q-1} \sum_{1\le j\le p_i} (F^1_{w_3,(i;j)} + F^2_{w_3,(i;j)}) \\
&+ \sum_{1\le j\le p_q} F^1_{w_3,(q;j)} \,,
\end{aligned}
$$

where

$$D_{w_3,(i;jk)} = span_{\mathbf{R}}\{e^3_{(i;jk)}, e^4_{(i;jk)}\}\,,$$

$$E^1_{w_3,(ij;kl)} = span_{\mathbf{R}}\{e^1_{(ij;kl)}, e^2_{(ij;kl)}\}\,, \quad E^2_{w_3,(ij;kl)} = span_{\mathbf{R}}\{e^3_{(ij;kl)}, e^4_{(ij;kl)}\}\,,$$

and

$$F^1_{w_3,(i;j)} = span_{\mathbf{R}}\{f^1_{(i;j)}, f^2_{(i;j)}\}\,, \quad F^2_{w_3,(i;j)} = span_{\mathbf{R}}\{f^3_{(i;j)}, f^4_{(i;j)}\}\,.$$

By the similar calculation in Case 1, we see that

$$
\gamma(w_3) = \begin{pmatrix}
\gamma_1 J_{p_1} & & & & \\
& \ddots & & & \\
& & \gamma_{q-1} J_{p_{q-1}} & & \\
& & & \frac{p_q}{2} J_{p_q} & \\
& & & & -\frac{p_q}{2} J
\end{pmatrix},
$$

where $\gamma_i = \frac{1}{2}(p_i + 1) + p_{i+1} + \cdots + p_q$ $(i = 1, \cdots, q-1)$.

$$
\tau_{w_3} = \begin{pmatrix}
\tau_1 J_{p_1} & & & \\
& \ddots & & \\
& & \tau_q J_{p_q} & \\
& & & \tilde{\tau} J
\end{pmatrix},
$$

where

$$
\begin{aligned}
-\tau_i = {} & \frac{\mu_i}{\gamma_i}(p_i - 1) + \sum_{j=i+1}^{q} p_j \left(\frac{\mu_i - \mu_j}{\gamma_i - \gamma_j} + \frac{\mu_i + \mu_j}{\gamma_i + \gamma_j} \right) \\
& + \sum_{j=1}^{i-1} p_j \left(-\frac{\mu_j - \mu_i}{\gamma_j - \gamma_i} + \frac{\mu_j + \mu_i}{\gamma_j + \gamma_i} \right) \\
& + \frac{\mu_i + \mu_q}{\gamma_i + \gamma_q} + (1 - \delta_{iq}) \frac{\mu_i - \mu_q}{\gamma_i - \gamma_q} ,
\end{aligned}
\tag{A.4}
$$

and

$$
-\tilde{\tau} = \sum_{j=1}^{q-1} p_j \frac{\mu_j - \mu_q}{\gamma_j - \gamma_q} - \sum_{j=1}^{q} p_j \frac{\mu_j + \mu_q}{\gamma_j + \gamma_q}. \tag{A.5}
$$

PROPOSITION A.1. *In Case 2, let $q = 2$, and $p_1 = p_2 = 1$. Then we have $\tau_{w_2} = -2w_2$.*

Note that, in this case, ρ is not equal to dF, for any constant d.

PROOF. First we prove that ρ is not equal to cF, for any constant c. By the hypothesis, w_2 is

$$
w_2 = \begin{pmatrix} \mu_1 J & 0 \\ 0 & \mu_2 J \end{pmatrix} ,
$$

where $\mu_1 > \mu_2 > 0$. On the other hand,

$$
\gamma(w_2) = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} .
$$

So by the definitions of ρ and F, we have proved that ρ is not equal to dF, for any constant d. On the other hand, by the equation (A.3), we have $\tau_{w_2} = -2w_2$. $\square$

A.3. The case $G = SO(2n+1)$

In this case, we only write down the results; let M be the adjoint orbit of $w_0 \in \mathfrak{so}(2n+1)$, where

$$
w_0 = \begin{pmatrix} \mu_1 J_{p_1} & & & \\ & \ddots & & \\ & & \mu_q J_{p_q} & \\ & & & 0 \end{pmatrix} \quad \left(\mu_1 > \cdots > \mu_q , \ \sum_{i=1}^{q} p_i = n \right) .
$$

Then we have

$$\gamma(w_0) = \begin{pmatrix} \gamma_1 J_{p_1} & & & \\ & \ddots & & \\ & & \gamma_q J_{p_q} & \\ & & & 0 \end{pmatrix} \quad (\gamma_i = \frac{1}{2}p_i + p_{i+1} + \cdots + p_q)\,,$$

$$\tau_{w_0} = \begin{pmatrix} \tau_1 J_{p_1} & & & \\ & \ddots & & \\ & & \tau_q J_{p_q} & \\ & & & 0 \end{pmatrix},$$

where

$$\begin{aligned} -\tau_i = \frac{\mu_i}{\gamma_i} p_i + \sum_{j=i+1}^{q} p_j \left(\frac{\mu_i - \mu_j}{\gamma_i - \gamma_j} + \frac{\mu_i + \mu_j}{\gamma_i + \gamma_j} \right) \\ + \sum_{j=1}^{i-1} p_j \left(-\frac{\mu_j - \mu_i}{\gamma_j - \gamma_i} + \frac{\mu_j + \mu_i}{\gamma_j + \gamma_i} \right). \end{aligned} \tag{A.6}$$

References

[B] A. L. BESSE, *Einstein manifolds*, Erge. der Math. und Grenz. **3** (1987), Springer.

[C] B.-Y. CHEN, *Geometry of submanifolds and its applications*, Science University of Tokyo (1981).

[CMN] B.-Y. CHEN, J.-M. MORVAN, T. Nore, Energy, tension and finite type map, Kodai Math. J. **9** (1986), 406–418.

[G] E. GOLDSTEIN, Holomorphic vector fields and minimal Lagrangian submanifolds, preprint.

[Oh] Y.-G. OH, Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds, Invent. Math. **101** (1990), 501–519.

[O] H. ONO, Minimal Lagrangian submanifolds in adjoint orbits and upper bounds on the first eigenvalue of the Laplacian, to appear in J. Math. Soc. Japan.

[T] T. TAKAHASHI, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan **18** (1966), 380–385.

Present Address:
DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY,
OH-OKAYAMA, MEGRO-KU, TOKYO, 152–8551 JAPAN.