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New Trigonometric Identities and Generalized Dedekind Sums

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Abstract. We obtain new trigonometric identities. We show that the coefficients of Laurent expansions of the identities give rise to the relation between special values of Hurwitz zeta function and Bernoulli numbers. Then we look into in detail the parameterized cotangent sums appearing in the identities.

1. New trigonometric identities, Hurwitz zeta values and Bernoulli numbers

Our starting point is the following identity (reciprocity law for parameterized cotangent sums): for $p, q \in \mathbf{Z}^+$ such that $\gcd(p, q) = 1$, and for $z \in \mathbf{C}$,

$$\begin{aligned}(1.1)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(z+\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\cot\left(\frac{\mu p\pi}{q}\right)\cot\left(z+\frac{\mu\pi}{q}\right)\\ &=-\cot(pz)\cot(qz)+\frac{1}{pq}\csc^2(z)-1\,.\end{aligned}$$

This identity can be derived from Theorem 2.4 of Dieter [4]. His proof rests on clever use of properties of trigonometric functions. An alternative proof of the identity, which uses Dedekind symbols of Jacobi forms, has been given in [6].

One of purposes of this article is to obtain a collection of identities which are similar to the identity (1.1). They are formulated in the following theorem.

THEOREM 1.1. *Let p and q be positive integers with* $\gcd(p, q) = 1$, *and* $z \in \mathbf{C}$.

(1) *For p even, we have*

$$\begin{aligned}(1.2)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(z+\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(z+\frac{\mu\pi}{q}\right)\\ &=-\csc(pz)\cot(qz)+\frac{1}{pq}\csc^2(z)\,.\end{aligned}$$

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(2) *For both p and q odd, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\cot\left(z+\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(z+\frac{\mu\pi}{q}\right) \tag{1.3}$$
$$=-\csc(pz)\csc(qz)+\frac{1}{pq}\csc^2(z)\,.$$

(3) *For p odd, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(z+\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(z+\frac{\mu\pi}{q}\right) \tag{1.4}$$
$$=-\csc(pz)\cot(qz)+\frac{1}{pq}\csc(z)\cot(z)\,.$$

(4) *For p and q with odd parity, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\csc\left(z+\frac{\mu\pi}{p}\right)$$
$$+\frac{1}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(z+\frac{\mu\pi}{q}\right) \tag{1.5}$$
$$=-\csc(pz)\csc(qz)+\frac{1}{pq}\csc(z)\cot(z)\,.$$

We will prove the identities in Theorem 1.1 by means of complex analysis. Incidentally, this gives an alternative proof to the identity (1.1). The conditions imposed on p and q in Theorem 1.1 are essential. Indeed, without the conditions it is easy to come up with counter examples.

Secondly, we compare the coefficients of Laurent expansions of the identities in Theorem 1.1 to obtain the relations between special values of Hurwitz zeta function $\zeta(s,\alpha)$ and Bernoulli numbers B_n $(n=0,2,\dots)$. These relations are variations of the following relation (1.6) already obtained by Apostol [2, Theorem 2]: for $p,q\in\mathbf{Z}^{+}$ such that $\gcd(p,q)=1$, and for $n\in\mathbf{Z}$, $n>1$,

$$\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\cot\left(\frac{\mu p\pi}{q}\right)\zeta\left(2n-1,\frac{\mu}{q}\right) \tag{1.6}$$
$$=\frac{(-1)^{n-1}(2\pi)^{2n-1}}{pq(2n)!}\left\{\sum_{k=0}^{n}\binom{2n}{2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}+(2n-1)B_{2n}\right\}.$$

The equation (1.6) can also be derived from (1.1) while the following equations (1.7), (1.8), (1.9) and (1.10) are derived from (1.2), (1.3), (1.4) and (1.5) respectively.

THEOREM 1.2. *Assume that p and q are positive integers with* $\gcd(p,q)=1$, *and* $n>1$.

(1) *For p even, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\zeta\left(2n-1,\frac{\mu}{q}\right)$$
$$=\frac{(-1)^{n}(2\pi)^{2n-1}}{pq(2n)\,!}\left\{\sum_{k=0}^{n}\binom{2n}{2k}(1-2^{1-2k})B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}-(2n-1)B_{2n}\right\}. \tag{1.7}$$

(2) *For both p and q odd, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\zeta\left(2n-1,\frac{\mu}{q}\right)$$
$$=\frac{(-1)^{n-1}(2\pi)^{2n-1}}{pq(2n)\,!}\left\{\sum_{k=0}^{n}\binom{2n}{2k}(1-2^{1-2k})(1-2^{1-2n+2k})B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}\right.$$
$$\left.+(2n-1)B_{2n}\right\}. \tag{1.8}$$

(3) *For p odd, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\left\{\zeta(2n-1,\frac{\mu}{2p})-\zeta\left(2n-1,1-\frac{\mu}{2p}\right)\right\}$$
$$+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2q}\right)-\zeta\left(2n-1,1-\frac{\mu}{2q}\right)\right\}$$
$$=\frac{(-1)^{n-1}(2\pi)^{2n-1}}{2pq(2n)\,!}\sum_{k=0}^{n}\binom{2n}{2k}\{(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}$$
$$-(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}\}. \tag{1.9}$$

(4) *For p and q with odd parity, we have*

$$\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2p}\right)-\zeta\left(2n-1,1-\frac{\mu}{2p}\right)\right\}$$
$$+\frac{1}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2q}\right)-\zeta\left(2n-1,1-\frac{\mu}{2q}\right)\right\}$$
$$=\frac{(-1)^{n-1}(2\pi)^{2n-1}}{2pq(2n)\,!}\sum_{k=0}^{n}\binom{2n}{2k}\{(2^{2k}-2)(2^{2n-2k}-2)B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}$$
$$+(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}\}. \tag{1.10}$$

These relations between special values of Hurwitz zeta function and Bernoulli numbers are, we believe, to be new.

We also obtain a corollary to Theorem 1.1 which can be regarded as new variants of the following reciprocity law for the classical Dedekind sums (refer to, for example, [10] or [12]):

$$\begin{aligned}(1.11)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\cot\left(\frac{\mu p\pi}{q}\right)\cot\left(\frac{\mu\pi}{q}\right)\\ &=\frac{p^2+q^2+1-3pq}{3pq}.\end{aligned}$$

COROLLARY 1.3. *Let p and q be positive integers such that* $\gcd(p,q)=1$. *Then the following formulae hold*:

(1) *For p even*, *we have*

$$\begin{aligned}(1.12)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(\frac{\mu\pi}{q}\right)\\ &=\frac{-p^2+2q^2+2}{6pq}.\end{aligned}$$

(2) *For both p and q odd*, *we have*

$$\begin{aligned}(1.13)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(\frac{\mu\pi}{q}\right)\\ &=\frac{-p^2-q^2+2}{6pq}.\end{aligned}$$

(3) *For p odd*, *we have*

$$\begin{aligned}(1.14)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(\frac{\mu\pi}{q}\right)\\ &=\frac{-p^2+2q^2-1}{6pq}.\end{aligned}$$

(4) *For p and q with odd parity, we have*

$$\begin{aligned}(1.15)\qquad &\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(\frac{\mu\pi}{q}\right)\\ &=\frac{-p^2-q^2-1}{6pq}.\end{aligned}$$

PROOF. The formulae (1.12), (1.13), (1.14) and (1.15) in Corollary 1.3 can be obtained from the identities (1.2), (1.3), (1.4) and (1.5), respectively, by cancelling the poles and then taking $\lim_{z\to 0}$ in the both sides of the identities. □

REMARK 1.1. The last equation (1.15) seems to be especially appealing. Fukumoto–Furuta–Ue [7] have recently discovered that Fukumoto–Furuta's w-invariants of some 3-manifolds are presented in terms of the "alternating cosecant sums" discussed in the equation (1.15). In fact, these three authors have obtained the reciprocity law

$$\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\left(\frac{\mu\pi}{p}\right)+\frac{2}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)$$
$$+\frac{1}{q}\sum_{\mu=1}^{q-1}\cot\left(\frac{\mu p\pi}{q}\right)\cot\left(\frac{\mu\pi}{q}\right)+\frac{2}{q}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(\frac{\mu\pi}{q}\right)=-1$$

applying the index theorem of Atiyah-Singer and Kawasaki [9].

2. The parameterized cotangent sums

We will study the trigonometric sums involved in (1.1):

$$\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(z+\frac{\mu\pi}{p}\right).$$

DEFINITION 2.1. Let p, q be integers such that $\gcd(p, q) = 1$ and $p > 0$, and $x \in \mathbf{R}$. We define

$$D(p, q; x) = \frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(x+\frac{\mu\pi}{p}\right).$$

We call $D(p, q; x)$ a parameterized cotangent sum (refer to [8] or [12] for cotangent sums, refer to [3], [10] and [11] for similar sums).

The parameterized cotangent sum $D(p, q; x)$ is a kind of generalized Dedekind symbol (refer to [5] for detail). We can regard (1.1) as the reciprocity law of the generalized Dedekind symbol $D(p, q; x)$.

Let us recall generalized Dedekind sums $s_k(q, p)$ introduced by Apostol [1].

DEFINITION 2.2 (Apostol [1, 2]).

$$s_k(q, p) = \sum_{\mu=1}^{p-1}\frac{\mu}{p}\bar{B}_k\left(\frac{\mu q}{p}\right) = \sum_{\mu=1}^{p-1}\frac{\mu}{p}B_k\left(\frac{\mu q}{p}-\left[\frac{\mu q}{p}\right]\right).$$

Here $B_k(x)$ is k-th Bernoulli polynomial, $\bar{B}_k(x)$ k-th Bernoulli function, and $[x]$ is the greatest integer $\le x$. It was shown by Apostol [2, pp. 2,4] that

$$
\begin{aligned}
s_1(q,p) &= \frac{1}{4p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right),\\
s_{2n-1}(q,p) &= \frac{i(2n-1)\,!}{(2\pi i p)^{2n-1}}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right)\quad (n>1)\,.
\end{aligned}
\tag{2.1}
$$

We will prove

PROPOSITION 2.1. *The function $D(p,q;x)$ has following Taylor expansion at $x=0$:*

$$D(p,q;x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}p^{2n-2}}{(2n-1)\,!}s_{2n-1}(q,p)x^{2n-2}\,.$$

This proposition shows the sum $D(p,q;x)$ is a generating functin of Apostol generalized Dedekind sums. We can regard the generalized Dedekind sums as functions on $V := \{(p,q)\in \mathbf{Z}^+\times\mathbf{Z}\mid \gcd(p,q)=1\}$. Furthermore we can identify V with $\mathbf{Q}$ assigning $q/p\in\mathbf{Q}$ to $(p,q)\in V$. From this point of view we can regard the generalized Dedekind sums as functions on $\mathbf{Q}/\mathbf{Z}$.

We claim that the correspondence which maps any element $q/p \mod \mathbf{Z}$ in $\mathbf{Q}/\mathbf{Z}$ to the sequence of generalized Dedekind sums, $\{s_{2n-1}(q,p)\}_{n=1}^{\infty}$, is injective.

THEOREM 2.2. *Let p, q, p', q' be integers such that* $\gcd(p,q)=\gcd(p',q')=1$, $p>0$ *and* $p'>0$. *Then the following three conditions are equivalent*:

(1) $p=p'$ *and* $q\equiv q' \pmod p$,
(2) $s_{2n-1}(q,p)=s_{2n-1}(q',p')$ *for any* $n\in\mathbf{Z}^+$,
(3) $D(p,q;x)=D(p',q';x)$.

3. Proving Theorem 1.1

In this section we give proof of Theorem 1.1.

PROOF OF THEOREM 1.1 We will prove the identity (1.2) first. We let $f(z)$ and $g(z)$ denote the left and right hand side of (1.2), respectively. Namely

$$f(z)=\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(z+\frac{\mu\pi}{p}\right)+\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(z+\frac{\mu\pi}{q}\right)$$

and

$$g(z)=-\csc(pz)\cot(qz)+\frac{1}{pq}\csc^2(z)\,.$$

We claim that

$$f(z)=g(z)\,.$$

First note that both $f(z)$ and $g(z)$ are meromorphic functions, and both have simple poles at the points $z = -\mu\pi/p + n\pi$ ($\mu = 1, 2, \dots, p-1;\ n = 0, \pm1, \pm2, \dots$) and $z = -\mu\pi/q + n\pi$ ($\mu = 1, 2, \dots, q-1;\ n = 0, \pm1, \pm2, \dots$). Furthermore, the residues at these points are equal, that is,

$$\mathrm{Res}_{z=-\mu\pi/p+n\pi}(f) = \frac{1}{p}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right) = \mathrm{Res}_{z=-\mu\pi/p+n\pi}(g)\,,$$

and

$$\mathrm{Res}_{z=-\mu\pi/q+n\pi}(f) = \frac{1}{q}\csc\left(\frac{\mu p\pi}{q}\right) = \mathrm{Res}_{z=-\mu\pi/q+n\pi}(g)\,.$$

Next we investigate other poles. We may assume without loss of generality that $g(z)$ having no pole of order greater than 1. This is because the principal parts of $-\csc(pz)\cot(qz)$ and $(1/pq)\csc^2(z)$ at $z = n\pi$ ($n = 0, \pm1, \pm2, \dots$) are $-1/pq(z-n\pi)^2$ and $1/pq(z-n\pi)^2$, respectively and they cancel out in $g(z)$ (it is easy to see this at $z = 0$, then notice that $g(z)$ is periodic with period π from the hypothesis that p is even). Clearly $f(z)$ has no pole of order greater than 1. Thus we know the principal parts of $f(z)$ and $g(z)$ coincide at all of their poles so that $f(z) - g(z)$ is an entire function.

It is obvious that $f(z)$ and $g(z)$ are bounded on the set $R_1 := \{z \in \mathbf{C} \mid |\Im(z)| \geq 1\}$, because both $\cot(z)$ and $\csc(z)$ are bounded on R_1. Hence $f(z) - g(z)$ is also bounded on R_1. Now, since $f(z)-g(z)$ is bounded on the compact set $U := \{z \in \mathbf{C} \mid |\Im(z)| \leq 1,\ |\Re(z)| \leq \pi\}$, and $f(z) - g(z)$ is periodic with period π, we see that $f(z) - g(z)$ is bounded on the set $R_2 := \{z \in \mathbf{C} \mid |\Im(z)| \leq 1\}$. Noting that $\mathbf{C} = R_1 \cup R_2$, we see that $f(z) - g(z)$ is bounded on the complex plane $\mathbf{C}$. Thus we can conclude that $f(z) - g(z)$ is a bounded entire function on $\mathbf{C}$, and then it must be a constant by the well-known Liouville Theorem. This constant must be zero, as (noting $\lim_{z\to i\infty}\cot(z) = -i$, $\lim_{z\to i\infty}\csc(z) = 0$)

$$\lim_{z\to i\infty} f(z) = \frac{-i}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right) + \frac{-i}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right) = 0,\ \lim_{z\to i\infty} g(z) = 0\,.$$

Here we make use of the identities

$$\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right) = \sum_{\mu=1}^{p-1}(-1)^{p-\mu}\cot\left(\frac{(p-\mu)q\pi}{p}\right) = -\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)$$

and

$$\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right) = \sum_{\mu=1}^{q-1}\csc\left(\frac{(q-\mu)p\pi}{q}\right) = -\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right).$$

This proves our claim that $f(z) = g(z)$, and hence the identity (1.2).

Similarly (1.3), (1.4) and (1.5) can be proved; detailed verifications are left to the reader. □

4. Proving Theorem 1.2

In this section we give a proof of Theorem 1.2. We need the following lemma to prove Theorem 1.2.

LEMMA 4.1. *Let α be a real number such that $0 < \alpha < 1$. Then, for sufficiently small $y \in \mathbf{R}$, we have the following expansions at $y = 0$:*

$$\pi \cot(\pi(y+\alpha)) = \pi \cot(\pi\alpha) + \sum_{k=1}^{\infty}\{(-1)^k \zeta(k+1,\alpha) - \zeta(k+1,1-\alpha)\}y^k \tag{4.1}$$

$$\begin{aligned}\pi \csc(\pi(y+\alpha)) =& \pi \csc(\pi\alpha) + \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}\Bigg\{(-1)^k \zeta\left(k+1, \frac{\alpha}{2}\right) \\ & - (-1)^k \zeta\left(k+1, \frac{1+\alpha}{2}\right) + \zeta\left(k+1, \frac{1-\alpha}{2}\right) \\ & - \zeta\left(k+1, \frac{2-\alpha}{2}\right)\Bigg\} y^k .\end{aligned} \tag{4.2}$$

PROOF. We proved (4.1) in [6, Lemma 5.4]. Hence we will prove (4.2). Applying a well-known formula $\csc(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} (-1)^n/(x+n\pi)$ for $|x| < \pi$, we have

$$\begin{aligned}\pi \csc(\pi(y+\alpha)) &= \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^n}{y+n+\alpha} = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^n}{n+\alpha} \sum_{k=0}^{\infty} \left(\frac{-y}{n+\alpha}\right)^k \\ &= \sum_{k=0}^{\infty} \left\{ \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^n}{(n+\alpha)^{k+1}} \right\} (-y)^k \\ &= \pi \csc(\pi\alpha) + \sum_{k=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\alpha)^{k+1}} + (-1)^k \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1-\alpha)^{k+1}} \right\} (-y)^k \\ &= \pi \csc(\pi\alpha) + \sum_{k=1}^{\infty} \Bigg\{ (-1)^k \frac{1}{2^{k+1}} \left[\zeta(k+1, \frac{\alpha}{2}) - \zeta(k+1, \frac{1+\alpha}{2}) \right] \\ &\quad + \frac{1}{2^{k+1}} \left[\zeta(k+1, \frac{1-\alpha}{2}) - \zeta(k+1, \frac{2-\alpha}{2}) \right] \Bigg\} y^k.\end{aligned}$$

Here we applied the formula $\sum_{n=0}^{\infty} (-1)^n/(n+\alpha)^s = (1/2^s)\{\zeta(s, \alpha/2) - \zeta(s, (1+\alpha)/2)\}$. □

PROOF OF THEOREM 1.2. First we will prove (1.7). We expand both sides of (1.2) and compare their coefficients. Using the formula (4.1) in Lemma 4.1, for sufficiently small

$x \in \mathbf{R}$, we have

$$\begin{aligned}
&\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(x+\frac{\mu\pi}{p}\right)\\
=&\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)+\frac{1}{p}\sum_{k=1}^{\infty}\frac{1}{\pi^{k+1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\\
&\times\left\{(-1)^{k}\zeta\left(k+1,\frac{\mu}{p}\right)-\zeta\left(k+1,1-\frac{\mu}{p}\right)\right\}x^{k}\\
=&\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)\\
&+\frac{2}{p}\sum_{\substack{k=2\\ k\text{ even}}}^{\infty}\frac{1}{\pi^{k+1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(k+1,\frac{\mu}{p}\right)x^{k}\\
&\qquad\text{(because } \cot((p-\mu)q\pi/p)=-\cot(\mu q\pi/p)\text{)}\\
=&\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)\\
&+\sum_{n=2}^{\infty}\left\{\frac{2}{p\pi^{2n-1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right)\right\}x^{2n-2}.
\end{aligned} \tag{4.3}$$

Similarly we have, (noting $\csc((q-\mu)p\pi/q)=-\csc(\mu p\pi/q)$ for p even)

$$\begin{aligned}
&\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(x+\frac{\mu\pi}{q}\right)\\
&\quad=\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\cot\left(\frac{\mu\pi}{q}\right)\\
&\quad\quad+\sum_{n=2}^{\infty}\left\{\frac{2}{q\pi^{2n-1}}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\zeta\left(2n-1,\frac{\mu}{q}\right)\right\}x^{2n-2}.
\end{aligned} \tag{4.4}$$

On the other hand using the following formulae (by our convention, $0\,!=1$):

$$\cot(x)=\frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n}2^{2n}B_{2n}}{(2n)\,!}x^{2n}\right),\quad \csc(x)=-\frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n}(2^{2n}-2)B_{2n}}{(2n)\,!}x^{2n}\right),$$

$$\csc^{2}(x)=-\frac{1}{x^{2}}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n}(2n-1)2^{2n}B_{2n}}{(2n)\,!}x^{2n}\right)$$

we have

$$
\begin{aligned}
& -\csc(px)\cot(qx) + \frac{1}{pq}\csc^2(x) \\
=& \frac{1}{px}\left(\sum_{n=0}^{\infty} \frac{(-1)^n(2^{2n}-2)B_{2n}}{(2n)\,!}(px)^{2n}\right)\left(\frac{1}{qx}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)\,!}(qx)^{2n}\right) \\
& + \frac{1}{pq}\left(\frac{-1}{x^2}\sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)2^{2n}B_{2n}}{(2n)\,!}x^{2n}\right) \\
=& \frac{1}{pqx^2}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} \frac{(-1)^n(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}}{(2k)\,!\,(2n-2k)\,!}x^{2n}\right\} \qquad (4.5) \\
& - \sum_{n=0}^{\infty}\left\{\frac{(-1)^n(2n-1)2^{2n}B_{2n}}{pq(2n)\,!}\right\}x^{2n-2} \\
=& \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{pq(2n)\,!}\left\{\sum_{k=0}^{n}\binom{2n}{2k}(1-2^{1-2k})B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}\right. \\
& \left. -(2n-1)B_{2n}\right\}x^{2n-2}.
\end{aligned}
$$

Comparing the coefficients of x^{2n-2} in (4.3), (4.4) and (4.5), we have

$$
\begin{aligned}
& \frac{2}{p\pi^{2n-1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right) \\
& \quad + \frac{2}{q\pi^{2n-1}}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\zeta\left(2n-1,\frac{\mu}{q}\right) \\
& = \frac{(-1)^n 2^{2n}}{pq(2n)\,!}\left\{\sum_{k=0}^{n}\binom{2n}{2k}(1-2^{1-2k})B_{2k}B_{2n-2k}p^{2k}q^{2n-2k} - (2n-1)B_{2n}\right\}
\end{aligned}
$$

for $n \in \mathbf{Z}$ such that $n > 1$. This proves the identity (1.7).

Secondly we prove (1.8). Starting from (1.3), we obtain

$$
\begin{aligned}
& \frac{2}{p\pi^{2n-1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right) \\
& \quad + \frac{2}{q\pi^{2n-1}}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\zeta\left(2n-1,\frac{\mu}{q}\right) \\
& = \frac{(-1)^{n+1}2^{2n}}{pq(2n)\,!}\left\{\sum_{k=0}^{n}(1-2^{1-2k})(1-2^{1-2n+2k})\binom{2n}{2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}\right. \\
& \quad \left. + (2n-1)B_{2n}\right\}
\end{aligned}
$$

for $n \in \mathbf{Z}$ such that $n > 1$ by the similar argument. This implies (1.8).

Thirdly we will prove (1.9). We expand both sides of (1.4) and compare their coefficients. Using the formula (4.2) in Lemma 4.1, for sufficiently small $x \in \mathbf{R}$, we have

$$
\begin{aligned}
&\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(x+\frac{\mu\pi}{p}\right)\\
&=\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)\\
&\quad+\frac{1}{p}\sum_{k=1}^{\infty}\frac{1}{(2\pi)^{k+1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\left\{(-1)^{k}\zeta(k+1,\frac{\mu}{2p})\right.\\
&\quad\left.-(-1)^{k}\zeta(k+1,\frac{p+\mu}{2p})+\zeta\left(k+1,\frac{p-\mu}{2p}\right)-\zeta\left(k+1,\frac{2p-\mu}{2p}\right)\right\}x^{k}\\
&=\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)\\
&\quad+\frac{1}{p}\sum_{k=1}^{\infty}\frac{1}{(2\pi)^{k+1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\left\{(-1)^{k}\zeta\left(k+1,\frac{\mu}{2p}\right)\right.\\
&\quad\left.-(-1)^{k}\zeta\left(k+1,\frac{2p-\mu}{2p}\right)+\zeta(k+1,\frac{\mu}{2p})-\zeta\left(k+1,\frac{2p-\mu}{2p}\right)\right\}x^{k} \qquad (4.6)\\
&\qquad\qquad(\text{because } \cot((p-\mu)q\pi/p)=-\cot(\mu q\pi/p))\\
&=\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)\\
&\quad+\frac{2}{p}\sum_{\substack{k=2\\ k\text{ even}}}^{\infty}\frac{1}{(2\pi)^{k+1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\left\{\zeta\left(k+1,\frac{\mu}{2p}\right)\right.\\
&\quad\left.-\zeta\left(k+1,\frac{2p-\mu}{2p}\right)\right\}x^{k}\\
&=\frac{1}{p}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\csc\left(\frac{\mu\pi}{p}\right)\\
&\quad+\sum_{n=2}^{\infty}\frac{2}{p(2\pi)^{2n-1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2p}\right)\right.\\
&\quad\left.-\zeta\left(2n-1,1-\frac{\mu}{2p}\right)\right\}x^{2n-2}.
\end{aligned}
$$

Similarly we have, (noting $\csc((q-\mu)p\pi/q) = \csc(\mu p\pi/q)$ for p odd)

$$\begin{aligned}
&\frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(x+\frac{\mu\pi}{q}\right)\\
&= \frac{1}{q}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\csc\left(\frac{\mu\pi}{q}\right)+\sum_{n=2}^{\infty}\frac{2}{q(2\pi)^{2n-1}}\sum_{\mu=1}^{q-1}\csc\left(\frac{\mu p\pi}{q}\right)\\
&\quad\times\left\{\zeta(2n-1,\frac{\mu}{2q})-\zeta\left(2n-1,1-\frac{\mu}{2q}\right)\right\}x^{2n-2}.
\end{aligned} \tag{4.7}$$

For the right hand side of (1.9), we have

$$\begin{aligned}
&-\csc(px)\cot(qx)+\frac{1}{pq}\csc(x)\cot(x)\\
&=\frac{1}{px}\left(\sum_{n=0}^{\infty}\frac{(-1)^n(2^{2n}-2)B_{2n}}{(2n)\,!}(px)^{2n}\right)\left(\frac{1}{qx}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}B_{2n}}{(2n)\,!}(qx)^{2n}\right)\\
&\quad-\frac{1}{pqx^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n(2^{2n}-2)B_{2n}}{(2n)\,!}x^{2n}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}B_{2n}}{(2n)\,!}x^{2n}\right)\\
&=\frac{1}{pqx^2}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(-1)^n(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}}{(2k)\,!(2n-2k)\,!}x^{2n}\right\}\\
&\quad-\frac{1}{pqx^2}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(-1)^n(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}}{(2k)\,!(2n-2k)\,!}x^{2n}\right\}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{pq(2n)\,!}\sum_{k=0}^{n}\binom{2n}{2k}\{(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}\\
&\quad-(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}\}x^{2n-2}.
\end{aligned} \tag{4.8}$$

Comparing the coefficients of x^{2n-2} in (4.6), (4.7) and (4.8), we have

$$\begin{aligned}
&\frac{2}{p(2\pi)^{2n-1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\cot\left(\frac{\mu q\pi}{p}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2p}\right)-\zeta\left(2n-1,1-\frac{\mu}{2p}\right)\right\}\\
&\quad+\frac{2}{q(2\pi)^{2n-1}}\sum_{\mu=1}^{q-1}\left(\frac{\mu p\pi}{q}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2q}\right)-\zeta\left(2n-1,1-\frac{\mu}{2q}\right)\right\}\\
&=\frac{(-1)^n}{pq(2n)\,!}\sum_{k=0}^{n}\binom{2n}{2k}\{(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}-(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}\}
\end{aligned}$$

for $n\in\mathbf{Z}$ such that $n>1$. This proves the identity (1.9).

Finally we will prove (1.10). Starting from the equation (1.5), we have

$$\begin{aligned}
&\frac{2}{p(2\pi)^{2n-1}}\sum_{\mu=1}^{p-1}(-1)^{\mu}\csc\left(\frac{\mu q\pi}{p}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2p}\right)-\zeta\left(2n-1,1-\frac{\mu}{2p}\right)\right\}\\
&\quad+\frac{2}{q(2\pi)^{2n-1}}\sum_{\mu=1}^{q-1}(-1)^{\mu}\csc\left(\frac{\mu p\pi}{q}\right)\left\{\zeta\left(2n-1,\frac{\mu}{2q}\right)-\zeta\left(2n-1,1-\frac{\mu}{2q}\right)\right\}\\
&=-\frac{(-1)^n}{pq(2n)\,!}\sum_{k=0}^{n}\binom{2n}{2k}\{(2^{2k}-2)(2^{2n-2k}-2)B_{2k}B_{2n-2k}p^{2k}q^{2n-2k}\\
&\quad+(2^{2k}-2)2^{2n-2k}B_{2k}B_{2n-2k}\}
\end{aligned}$$

for $n\in\mathbf{Z}$ such that $n>1$ by the similar argument. This proves the identity (1.10) and completes the proof of Theorem 1.2.

5. Proving Theorem 2.2

Finally we give proofs of Proposition 2.1 and Theorem 2.2.

PROOF OF PROPOSITION 2.1. We make use of the formula (4.1) as we did in the proof of Theorem 1.2.

$$\begin{aligned}
D(p,q;x)&=\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(x+\frac{\mu\pi}{p}\right)\\
&=\frac{1}{p}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\cot\left(\frac{\mu\pi}{p}\right)+\sum_{n=2}^{\infty}\left\{\frac{2}{p\pi^{2n-1}}\sum_{\mu=1}^{p-1}\cot\left(\frac{\mu q\pi}{p}\right)\zeta\left(2n-1,\frac{\mu}{p}\right)\right\}x^{2n-2}\\
&=4s_1(q,p)+\sum_{n=2}^{\infty}\left\{\frac{(-1)^{n-1}2^{2n}p^{2n-2}}{(2n-1)\,!}s_{2n-1}(q,p)\right\}x^{2n-2}\quad\text{(by (2.1))}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}p^{2n-2}}{(2n-1)\,!}s_{2n-1}(q,p)x^{2n-2}\,.
\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 2.2. (1)⇒(2). If $q\equiv q'(\mod p)$, then, by Definition 2.2, it is clear that $s_{2n-1}(q,p)=s_{2n-1}(q',p)$ for any $n\in\mathbf{Z}^+$. This implies (2).

(2)⇒(3). Assume $s_{2n-1}(q,p)=s_{2n-1}(q',p')$ for any $n\in\mathbf{Z}^+$. Then $D(p,q;x)=D(p',q';x)$ by Proposition 2.1.

(3)⇒(1). Assume that $D(p,q;x)=D(p',q';x)$. The simple poles of $D(p,q;x)$ and $D(p',q';x)$ which are the nearest to 0 are $x=\pm\pi/p$ and $x=\pm\pi/p'$ respectively. Since they should coincide, we obtain $\pi/p=\pm\pi/p'$. Then, by the assumption that $p,p'>0$, we know $p=p'$. Hence we can assume that $D(p,q;x)=D(p,q';x)$. Let us consider the residues

$$\mathrm{Res}_{z=-\pi/p} D(p,q;z) = \frac{1}{p}\cot\left(\frac{q\pi}{p}\right) \text{ and } \mathrm{Res}_{z=-\pi/p} D(p,q';z) = \frac{1}{p}\cot\left(\frac{q'\pi}{p}\right).$$

Since they should coincide, we obtain $\cot(q\pi/p) = \cot(q'\pi/p)$. This equation implies $q \equiv q' (\mod p)$ as required. □

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