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On an Airy Function of Two Variables II

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0. Introduction

Consider an integral of the form

$$z_C(x, y) = \int_C \exp\left(-\frac{t^4}{4} + \frac{xt^2}{2} + yt\right)dt \,, \tag{0.1}$$

where x and y are complex variables. Here C is a path of integration such that the integrand vanishes at its terminal points. The Airy function

$$u(x) = \int_C \exp\left(-\frac{t^3}{3} - tx\right)dt$$

may be regarded as a confluent type of the Gauss hypergeometric function $F(\alpha, \beta, \gamma, x)$ ([3]). There exists an analogous relation between (0.1) and the Appell hypergeometric function $F_1(\alpha, \beta, \beta', \gamma, x, y)$ ([2]). The function (0.1) is called Pearcey's integral or an Airy function of two variables (cf.[4]). The integral $z_C(x, y)$ satisfies a system of partial differential equations of the form

$$\begin{aligned} \partial_x{}^2 u &= \frac{x}{2}\partial_x u + \frac{y}{4}\partial_y u + \frac{1}{4}u \,, \\ \partial_x \partial_y u &= \frac{x}{2}\partial_y u + \frac{y}{4}u \,, \\ \partial_y{}^2 u &= 2\partial_x u \,, \end{aligned} \tag{0.2}$$

whose solutions constitute a 3-dimensional vector space over $\mathbf{C}$ (cf.[2]). This system is equivalent to

$$dV = (P(x, y)dx + Q(x, y)dy)V \tag{0.3}$$

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with

$$P(x, y) = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & x/2 & y/4 \\ y/2 & 0 & x/2 \end{pmatrix}, \quad Q(x, y) = \begin{pmatrix} 0 & 0 & 1 \\ y/2 & 0 & x/2 \\ 0 & 2 & 0 \end{pmatrix},$$

$$V = {}^t\left(u, \partial_x u, \partial_y u\right).$$

System (0.2) or (0.3) possesses the singular loci $x = \infty$ and $y = \infty$ of irregular type. In [5] we studied the asymptotic behavior of linearly independent solutions of (0.2) near $y = \infty$. The purpose of this paper is to examine asymptotics near another singular point $x = \infty$.

In section 1, we recall some facts in [5], namely some relations among integrals satisfying (0.2) and convergent series expansions of three linearly independent solutions in $\mathbf{C}^2$. In section 2, we give asymptotic expansions of solutions which constitute another fundamental system of solutions by the saddle point method. In section 3, we calculate Stokes multipliers concerning them.

Throughout this paper we use the notation below:

(1) $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$,
where a is a complex number and k is a nonnegative integer;

(2) $\left(t : \infty \to \infty\, ;\ \arg t : \theta \to \theta'\right)$

denotes a path starting from $t = \infty$ and tending to $t = \infty$ along which $\arg t$ varies from θ to θ'.

1. Preliminaries

Denote the integrand of (0.1) by

$$f(x, y, t) = \exp\left(-\frac{t^4}{4} + \frac{xt^2}{2} + yt\right). \tag{1.1}$$

Consider the six paths given by

$$\begin{aligned} C_1 &= (t : \infty \to \infty\, ;\ \arg t : 0 \to \pi/2)\ , \\ C_2 &= (t : \infty \to \infty\, ;\ \arg t : \pi/2 \to \pi)\ , \\ C_3 &= (t : \infty \to \infty\, ;\ \arg t : \pi \to 3\pi/2)\ , \\ C_4 &= (t : \infty \to \infty\, ;\ \arg t : 3\pi/2 \to 2\pi)\ , \\ C_R &= (t : \infty \to \infty\, ;\ \arg t : \pi \to 0)\ , \\ C_I &= (t : \infty \to \infty\, ;\ \arg t : 3\pi/2 \to \pi/2) \end{aligned}$$

(cf. Fig. 1 , Fig. 2). Now we put

$$z_1 = \int_{C_1} f\,dt, \quad z_2 = \int_{C_2} f\,dt, \quad z_3 = \int_{C_3} f\,dt, \quad z_4 = \int_{C_4} f\,dt,$$

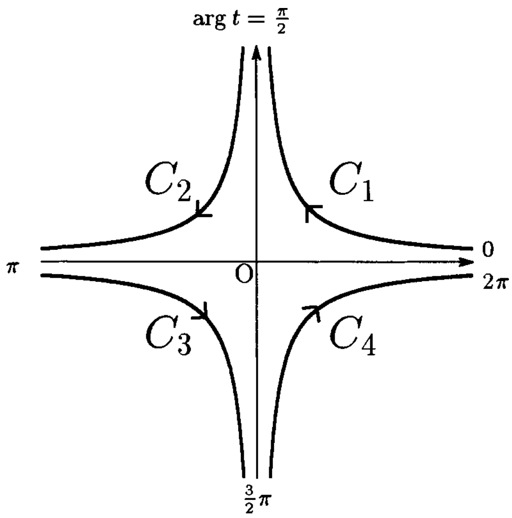

FIGURE 1.

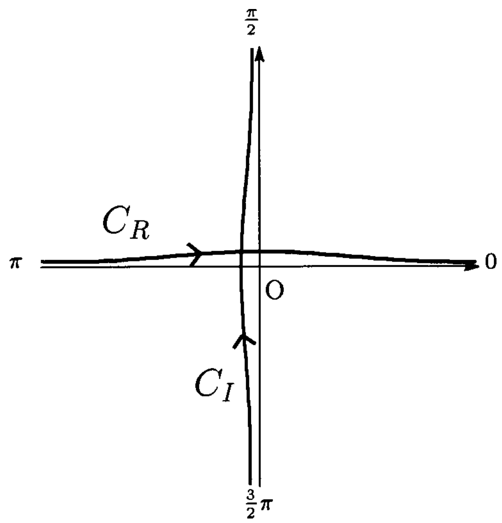

FIGURE 2.

$$z_R = \int_{C_R} f\,dt, \quad z_I = \int_{C_I} f\,dt, \quad f = f(x, y, t)\,.$$

Clearly the integrand $f(x, y, t)$ vanishes at the terminal points of each path. It is easy to see that $z_4 = z_1 + z_2 + z_3$. Furthermore, we have the following two propositions.

PROPOSITION 1.1. ([5]). *Let* $Z(x, y)$ *and* $Z^*(x, y)$ *be column vectors given by*

$$Z(x, y) = {}^t\,(z_R, z_I, z_1)\ , \quad Z^*(x, y) = {}^t(z_1, z_2, z_3)\ .$$

Then we have

$$Z(x, y) = MZ^*(x, y)\ , \quad M = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\ .$$

PROPOSITION 1.2. ([5]). *We have*

$$\begin{aligned} z_2(x, y) &= iz_1\,(-x, iy)\ , \\ z_3(x, y) &= -z_1\,(x, -y)\ , \\ z_4(x, y) &= -iz_1\,(-x, -iy)\ , \\ z_I(x, y) &= iz_R\,(-x, iy)\ . \end{aligned}$$

The solutions z_R, z_I, and z_1 are expanded into convergent series in powers of x and y.

THEOREM 1.3. ([5]). *In the domain* $|x| < +\infty$, $|y| < +\infty$, *we have*

$$z_R = \sum_{\substack{j\ge 0\\ k\ge 0}} \frac{x^j y^k}{(1)_j(1)_k}(1+(-1)^k)2^{\frac{k-3}{2}}\Gamma\left(\frac{2j+k+1}{4}\right), \tag{1.2}$$

$$z_I = \sum_{\substack{j\ge 0\\ k\ge 0}} \frac{x^j y^k}{(1)_j(1)_k}(-)^j i^{k+1}(1+(-1)^k)2^{\frac{k-3}{2}}\Gamma\left(\frac{2j+k+1}{4}\right), \tag{1.3}$$

$$z_1 = \sum_{\substack{j\ge 0\\ k\ge 0}} \frac{x^j y^k}{(1)_j(1)_k}((-)^j i^{k+1}-1)2^{\frac{k-3}{2}}\Gamma\left(\frac{2j+k+1}{4}\right), \tag{1.4}$$

which are linearly independent.

By Proposition 1.2, we easily get convergent series expansions of z_2, z_3, and z_4, which are linearly independent.

2. Asymptotic Expansions near $x = \infty$

2.1. Saddle Points. We give asymptotic expansions of z_1, z_2, z_3, and z_4 near $x = \infty$, by the saddle point method. The saddle points of the function

$$h(t) = -\frac{t^4}{4} + \frac{xt^2}{2} + yt \tag{2.1}$$

are the roots of the equation $h'(t) = -t^3 + xt + y = 0$.

PROPOSITION 2.1. *Assume that* $|x^{-2/3}y| < r_0$, *where* r_0 *is a sufficiently small positive constant. Then the saddle points of* $h(t)$ *are given by*

$$t_j = (-)^j x^{\frac{1}{2}} \left\{ 1 + (-)^j \frac{1}{2}\left(\frac{y}{x^{\frac{3}{2}}}\right) - \frac{3}{8}\left(\frac{y}{x^{\frac{3}{2}}}\right)^2 + (-)^j \frac{1}{2}\left(\frac{y}{x^{\frac{3}{2}}}\right)^3 + \cdots \right\} \quad (j = 0, 1),$$

$$t_2 = -\frac{y}{x}\left\{ 1 + \left(\frac{y^2}{x^3}\right) + 3\left(\frac{y^2}{x^3}\right)^2 + 12\left(\frac{y^2}{x^3}\right)^3 + \cdots \right\}.$$

The branch of t_0 *is taken in such a way that* $\arg t_0 = 0$ *for* $x > 0$.

This proposition tells us an approximate location of t_j as x tends to ∞.

2.2. Results. We put

$$v = v(x, y) = \frac{t_0^2}{2}(x - 3t_0^2),$$

where $t_0 = t_0(x, y)$ is the saddle point given above. Note that

$$h(t_0) = \frac{x^2}{4}(1 + 4(x^{-\frac{3}{2}}y) + (x^{-\frac{3}{2}}y)^2 + O((x^{-\frac{3}{2}}y)^3)),$$

$$v = -x^2\left(1 + \frac{5}{2}(x^{-\frac{3}{2}}y) + \frac{1}{4}(x^{-\frac{3}{2}}y)^2 + O((x^{-\frac{3}{2}}y)^3)\right).$$

We get the following theorem.

THEOREM 2.2. *Let* r *be an arbitrary small positive constant. Then the integral* z_1 *admits the asymptotic expansion in powers of* v^{-1}:

$$z_1 \simeq \sqrt{\pi}\, i t_0 v^{-\frac{1}{2}} e^{h(t_0)} \sum_{m=0}^{\infty}\left(\sum_{k=0}^{2m} \frac{(-)^{m+k}(-k)_{2m-k}(\frac{1}{2})_{m+k}}{(1)_{2m-k}4^{2m-k}} w^k\right) v^{-m} \tag{2.2}$$

uniformly for $|x^{-2/3}y| < r$, *as* x *tends to* ∞ *through the sector* $|\arg x - \pi/2| < 3\pi/4 - \delta$. *Here* $w = t_0^4 v^{-1}$, *and* δ *is a positive constant depending on* r *and satisfying* $\delta \to 0$ *as* $r \to 0$.

Let $H_\nu(z)$ be the Hermite polynomial:

$$H_\nu(z) = (-1)^\nu e^{z^2}\left(\frac{d}{dz}\right)^\nu e^{-z^2} = \nu! \sum_{m=0}^{[\nu/2]} \frac{(-1)^m (2z)^{\nu-2m}}{m!(\nu-2m)!}$$

(cf. [1]p.193). The following is an asymptotic expansion in powers of x^{-1}.

THEOREM 2.3. *Let r' be an arbitrary small positive constant. Then the integral z_1 admits the asymptotic expansion in powers of x^{-1}:*

$$z_1 \simeq -\sqrt{\pi} x^{-\frac{1}{2}} \exp\left(\frac{x^2}{4} + x^{\frac{1}{2}} y + \frac{1}{4} x^{-1} y^2\right) \sum_{m=0}^{\infty} \phi_m\left(\frac{i}{2} x^{-\frac{1}{2}} y\right) x^{-m} \tag{2.3}$$

uniformly for $|y| < r'$, as x tends to ∞ through the sector $|\arg x - \pi/2| < 3\pi/4 - \delta$, where $\phi_m(u)$ is a polynomial given by

$$\phi_m(u) = \sum_{k=0}^{m} \frac{(-)^{m+k}(-k)_{m-k}}{(1)_k (1)_{m-k} 2^{3m}} e^{-\frac{m}{2}\pi i} H_{2k+m}(u)\,.$$

Denote by Z_1 the right-hand side of (2.3). Then by Proposition 1.2 we easily obtain the following.

COROLLARY 2.4. *The integrals z_2, z_3, and z_4 admit asymptotic representations uniformly for $|y| < r'$:*

$$z_2(x, y) \simeq Z_2 = i Z_1(-x, e^{\frac{1}{2}\pi i} y)$$

as x tends to ∞ through the sector $|\arg x + \pi/2| < 3\pi/4 - \delta$;

$$z_3(x, y) \simeq Z_3 = -Z_1(x, e^{\pi i} y)$$

as x tends to ∞ through the sector $|\arg x + 3\pi/2| < 3\pi/4 - \delta$;

$$z_4(x, y) \simeq Z_4 = -i Z_1(-x, e^{\frac{3}{2}\pi i} y)$$

as x tends to ∞ through the sector $|\arg x - 3\pi/2| < 3\pi/4 - \delta$. Here r' and δ are the constants given in Theorem 2.3.

The following is an asymptotic expansion of z_R in powers of x^{-2}.

THEOREM 2.5. *Let r'' be an arbitrary small positive constant. Then the integral z_R admits the asymptotic expansion in powers of x^{-2}:*

$$z_R \simeq (2\pi)^{\frac{1}{2}} i x^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x^{-1} y^2\right) \sum_{k=0}^{\infty} \frac{(-)^k}{(1)_k 2^{4k}} H_{4k}(2^{-\frac{1}{2}} x^{-\frac{1}{2}} y) x^{-2k} \tag{2.4}$$

uniformly for $|y| < r''$, as x tends to ∞ through the sector $|\arg x - \pi| < 3\pi/4 - \delta'$. Here δ' is a positive constant depending on r'' and satisfying $\delta' \to 0$ as $r'' \to 0$.

2.3. Proof of Theorem 2.2

2.3.1. Modification of the Path C_1. We need to modify the path C_1 into a suitable form, and examine conditions concerning x under which the modification is possible. The necessary properties of C_1 are listed below, in which ε is a small positive constant given later.

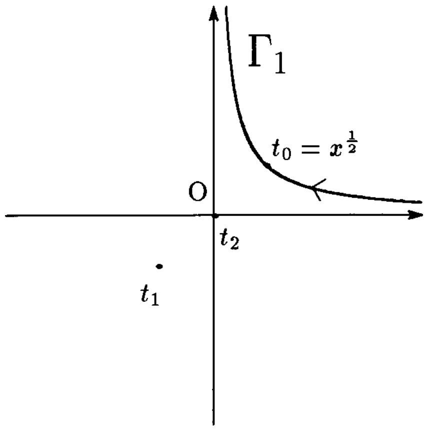

FIGURE 3.

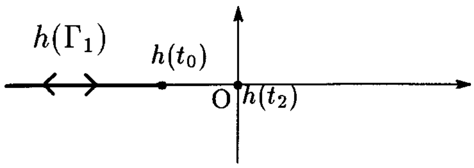

FIGURE 4.

(i) The path C_1 passes through the saddle point $t_0 = t_0(x, y)$, and lies outside the circles $|t - t_j| = |x|^{\varepsilon}$ $(j = 1, 2)$.

(ii) C_1 does not pass through the origin O.

(iii) $(d/d\rho)\mathrm{Re}(h(t)) \leq -c$ for $t \in C_1$, where c is a positive constant and ρ denotes the length of the part of the image curve $h\,(C_1)$ from $\tau = h(t_0)$ to $\tau = h(t)$.

(iv) The function $h(t) - h(t_0)$ is real-valued along C_1 inside the circle $|t - t_0| < |x|^{\varepsilon}$.

In the case where $y = 0$ and $\arg x = \pi/2$, the saddle points $t_0 = x^{1/2}$ and $t_1 = -x^{1/2}$ are symmetric with respect to the origin. The other saddle point t_2 is at the origin. Take the path C_1 to be the curve Γ_1 defined by

$$t = t(s) = \frac{t_0}{\sqrt{2}}\left(s\gamma + \frac{1}{s\gamma}\right), \quad \gamma = \frac{1+i}{\sqrt{2}} \quad (0 < s < +\infty) \tag{2.5}$$

(Fig. 3). Obviously (i) and (ii) are satisfied. Substituting (2.5) into (2.1), we get

$$h(t(s)) = \frac{t_0^4}{16}\left(s^2 + \frac{1}{s^2}\right)^2$$

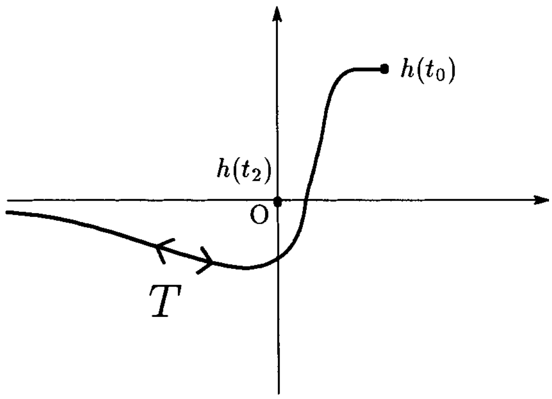

FIGURE 5.

with $h'(t_0) = -t_0^3 + xt_0 = 0$. Then the image $h(C_1)$ is the half line $T_1 : \tau \leq h(x^{1/2}) = x^2/4$ on the negative real axis in the τ-plane (Fig.4). If s moves from 0 to 1, then $\tau = h(t(s))$ monotonically increases from ∞ ($\arg\tau = \pi$) to $x^2/4$, and if s moves from 1 to $+\infty$, then τ monotonically decreases from $x^2/4$ to ∞ ($\arg\tau = \pi$). The properties (iii) and (iv) are also satisfied. Thus we have verified that Γ_1 is a desired path.

Next, we consider the case where (x, y) are located in the general position. We modify the image of Γ_1 continuously in the τ-plane in accordance with $\arg x$. In this case the saddle points t_j $(j = 0, 1)$ are close to the point $(-)^j x^{1/2}$, and their images $h(t_j)$ are close to $x^2/4$. The other saddle point t_2 is close to the origin, and its image is close to the origin. Note that, when t is on the circle $|t - t_j| = |x|^\varepsilon$, the function

$$h(t) - h(t_j) = (-3t_j^2 + x)(t - t_j)^2 - t_j(t - t_j)^3 - \frac{1}{4}(t - t_j)^4 \qquad (2.6)$$

satisfies $|h(t) - h(t_j)| \sim |x|^{1+2\varepsilon}$ as $x \to \infty$. We draw a curve T in the τ-plane starting from $\tau = h(t_0)$ and tending to $\tau = \infty$ ($\arg\tau = \pi$) with the properties (cf. Fig. 5):

(1) the curve T lies outside the circle $|\tau - h(t_2)| = 2|x|^{1+2\varepsilon}$;

(2) T does not pass through the origin in τ-plane;

(3) $(d/d\rho)\mathrm{Re}\,\tau \leq -c$ for $\tau \in T$, where ρ denotes the length of the part of the curve T from $h(t_0)$ to τ;

(4) the function $\tau - h(t_0)$ is real-valued along T inside the circle $|\tau - h(t_0)| < 2|x|^{1+2\varepsilon}$;

(5) T is obtained by a continuous modification of T_1 preserving properties (1) through (4).

Such a drawing of T is possible as long as

$$\left| \arg\left(\frac{1}{2}x^2\right) - \pi \right| < \frac{3}{2}\pi - \delta',$$

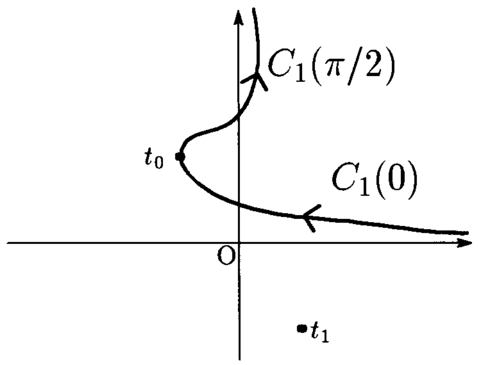

FIGURE 6.

namely

$$\left|\arg x - \frac{\pi}{2}\right| < \frac{3}{4}\pi - \delta\,, \tag{2.7}$$

where δ and δ' are sufficiently small positive constants (cf. Fig. 5).

Note that the mapping $\tau = h(t)$ is biholomorphic at each point in $\mathbf{C} - \{t_0, t_1, t_2\}$. There are two inverse images of T. One is the curve $C_1\,(0)$ starting from $t = \infty$ ($\arg t = 0$) and ending at $t = t_0$, which is the inverse image of the part of T starting from $\tau = \infty$ ($\arg \tau = \pi$) and ending at $\tau = h(t_0)$. Another is $C_1\,(\pi/2)$ starting from $t = t_0$ and tending to $t = \infty$ ($\arg t = \pi/2$), which is the inverse image of the remaining part of T (Fig. 6). Now we take the path C_1 to be the union of $C_1\,(0)$ and $C_1\,(\pi/2)$. By conditions (1) through (5), this new path is a continuous modification of Γ_1, and possesses properties (i) through (iv). Hence we obtain the desired path C_1, when (2.7) is satisfied.

2.3.2. Calculation of the Principal Part of the Integral z_1. We calculate the principal part of z_1. The major contribution comes from the integral near the saddle point $t = t_0$. Now we put $t = t_0(1+\sigma)$ in (2.1). Then

$$h\,(t_0(1+\sigma)) = h(t_0) - \frac{t_0^2}{2}(x - 3t_0^2)\sigma^2 - t_0^4\sigma^3 - \frac{t_0^4}{4}\sigma^4\,,$$

with $h'(t_0) = -t_0^3 + xt_0^2 + y = 0$. In the case where $y = 0$ and $\arg x = \pi/2$, if t moves on Γ_1 near $t = t_0$ in the direction $\arg t = 3\pi/4$, then σ moves near $\sigma = 0$ in the direction $\arg \sigma = \pi/2$. Let I_1 be the principal part. We get

$$I_1 = \int_{C_1, |\sigma| \le \theta(x)} f\,dt = t_0 e^{h(t_0)} \int_{C_1, |\sigma| \le \theta(x)} \exp(v\sigma^2) \exp\left(-t_0^4\sigma^3\left(1 + \frac{\sigma}{4}\right)\right) d\sigma\,,$$

where $\theta(x) = |x|^{-1+\varepsilon'}$ $\left(\varepsilon = -1/2 + \varepsilon'\right)$. For $|\sigma| \le \theta(x)$,

$$\exp\left(-t_0^4\sigma^3\left(1+\frac{\sigma}{4}\right)\right) = \sum_{k=0}^{\infty}\frac{(-)^k}{(1)_k}\left(1+\frac{\sigma}{4}\right)^k t_0^{4k}\sigma^{3k}$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-)^{k+l}(-k)_l}{(1)_k(1)_l 4^l}t_0^{4k}\sigma^{3k+l}\,.$$

Substituting this into I_1, we have

$$I_1 = t_0 e^{h(t_0)}\sum_{k=0}^{M}\sum_{l=0}^{M}\frac{(-)^{k+l}i^{3k+l}(-k)_l}{(1)_k(1)_l 4^l}t_0^{4k}\int_{C_1,|\sigma|\le\theta(x)}\sigma^{3k+l}e^{v\sigma^2}du + R\,, \tag{2.8}$$

where M is an arbitrary positive integer. Put $\sigma = iv^{-1/2}u$. Then u moves on the real axis. Hence

$$I_1 = it_0v^{-\frac{1}{2}}e^{h(t_0)}\sum_{k=0}^{M}\sum_{l=0}^{M}\frac{(-)^{k+l}i^{3k+l}(-k)_l}{(1)_k(1)_l 4^l}$$
$$\times t_0^{4k}v^{-\frac{3k+l}{2}}\int_{|u|\le|x|^{\varepsilon'}}u^{3k+l}e^{-u^2}du + R\,.$$

Here the error term is given by

$$R = O\left(t_0v^{-\frac{1}{2}}e^{h(t_0)}x^{-(M+1)}\int_{|u|\le|x|^{\varepsilon'}}u^{3(M+1)}e^{-u^2}du\right),$$

because $t_0 \sim x^{1/2}$, $v \sim -x^2$. Note that, for every positive integer p,

$$\int_{|u|\le|x|^{\varepsilon'}}u^pe^{-u^2}du = \begin{cases}\Gamma(\frac{p+1}{2}) + O(e^{-|x|^{\varepsilon'}}) & (p \text{ is even})\,,\\ O(e^{-|x|^{\varepsilon'}}) & (p \text{ is odd})\,.\end{cases} \tag{2.9}$$

Then we have

$$\begin{aligned}(it_0v^{-\frac{1}{2}}&e^{h(t_0)})^{-1}I_1\\ &= \sum_{k=0}^{M}\sum_{\substack{l=0\\ k+l=\text{even}}}^{M}\frac{(-)^{k+l}i^{3k+l}(-k)_l}{(1)_k(1)_l4^l}t_0^{4k}v^{-\frac{3k+l}{2}}\Gamma\left(\frac{3k+l+1}{2}\right) + O(x^{-(M+1)})\\ &= \sum_{m=0}^{N}\sum_{k=0}^{2m}\frac{(-)^{m+k}(-k)_{2m-k}(\frac{1}{2})_{m+k}\sqrt{\pi}}{(1)_{2m-k}4^{2m-k}}t_0^{4k}v^{-(m+k)} + O(x^{-(M+1)})\end{aligned} \tag{2.10}$$

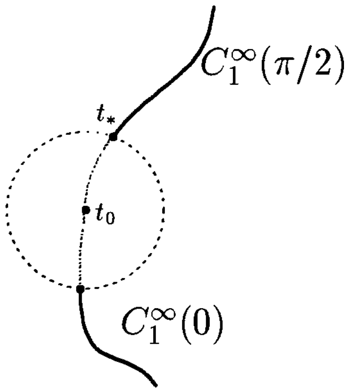

FIGURE 7.

with $N = [M/2]$. Furthermore put $w = t_0^4 v^{-1}$ with $w \sim -1$ as $x \to \infty$. By this change of variable, (2.10) becomes

$$(it_0 v^{-\frac{1}{2}} e^{h(t_0)})^{-1} I_1 = \sqrt{\pi} \sum_{m=0}^{N} \sum_{k=0}^{2m} \frac{(-)^{m-k}(-k)_{2m-k}\,(1/2)_{m+k}}{(1)_{2m-k} 4^{2m-k}} w^k v^{-m} + O(x^{-(M+1)}) .$$

Thus we obtain the right-hand side of (2.2).

2.3.3. Estimate of the Remaining Part of the Integral z_1. We estimate the remaining part $\int_{C_1, |t-t_0|>|x|^{\varepsilon}} f dt$. There are two components of C_1 outside the circle $|t - t_0| = |y|^{\varepsilon}$. Let $C_1^{\infty}(\pi/2)$ be the part tending to $t = \infty$ ($\arg t = \pi/2$), and $C_1^{\infty}(0)$ be the other part. By t_* ($t_* \neq \infty$) we denote the terminal point of it (cf. Fig. 7). For $t \in C_1^{\infty}(\pi/2)$, we have

$$|dt| = \left| \frac{1}{h'(t)} \frac{dh}{ds} \right| ds = O(x^{-1-\varepsilon}) ds ,$$

where s denotes the length of the part of the image $h(C_1^{\infty}(\pi/2))$ from $h(t_*)$ to $h(t)$. Using this and property (iii), we get

$$\begin{aligned} \int_{C_1^{\infty}(\pi/2)} f dt &= O\left(\int_{C_1^{\infty}(\pi/2)} |\exp h(t)| |dt| \right) \\ &= O\left(\int_{C_1^{\infty}(\pi/2)} \exp(\mathrm{Re}\, h(t)) |dt| \right) \\ &= O\left(\int_{C_1^{\infty}(\pi/2)} \exp(h(t_0) - |x|^{\varepsilon}/2) x^{-1-\varepsilon} ds \right) \end{aligned}$$

$$= O\left(it_0 v^{-\frac{1}{2}} e^{h(t_0)} x^{-\frac{1}{2}-\varepsilon} \exp(-|x|^{\varepsilon}/2)\right).$$

The other part $\int_{C_1^\infty(0)} f\,dt$ is evaluated in a similar way. Thus the theorem is proved. □

2.4. Proof of Theorem 2.3. Consider the function

$$h_\infty(t) = -\frac{t^4}{4} + \frac{xt^2}{2}.$$

The saddle points of $h_\infty(t)$ are $\pm x^{1/2}$ and 0. By Proposition 2.1, they are approximate saddle points of $h(t)$. We calculate an asymptotic expansion of z_1 by the saddle point method, using the point $x^{1/2}$ instead of t_0.

2.4.1. Modification of the Path C_1. In the calculation, we need to construct the path C_1 with the properties:

(a) C_1 consists of three curves $\Gamma_-(0)$, Γ_+, $\Gamma_-(\pi/2)$.

(a.1) Γ_+ is an arc passing through $t = x^{1/2}$ and lying inside the circle $|t - x^{1/2}| = 2|x|^{\varepsilon}$. Both ends a_0, $a_{\pi/2}$ of Γ_+ are located on $|t - x^{1/2}| = 2|x|^{\varepsilon}$.

(a.2) $\Gamma_-(0)$ is a curve starting from $t = \infty$ ($\arg t = 0$) and ending at a_0.

(a.3) $\Gamma_-(\pi/2)$ is a curve starting from $a_{\pi/2}$ and tending to $t = \infty$ ($\arg t = \pi/2$).

(b) C_1 lies outside the circles $|t - t_j| = |x|^{\varepsilon}$ ($j = 1, 2$), and $\Gamma_-(0)$ and $\Gamma_-(\pi/2)$ lies outside the circle $|t - t_0| = 2|x|^{\varepsilon}$.

(c) $\mathrm{Re}(h_\infty(t) - x^2/4) \le 0$, $\mathrm{Im}(h_\infty(t) - x^2/4) = 0$ for $t \in \Gamma_+$.

(d) $(d/d\rho)\mathrm{Re}(h(t)) \le -c$ for $t \in \Gamma_-(0)$ (or $t \in \Gamma_-(\pi/2)$), where c is a positive constant and ρ denotes the length of the part of the image curve $h(\Gamma_-(0))$ (or $h(\Gamma_-(\pi/2))$) from $h(a_0)$ (or $h(a_{\pi/2})$) to $\tau = h(t)$.

First we consider the case where $y = 0$ and $\arg x = \pi/2$. The curve Γ_1 in the proof of Theorem 2.2 satisfies the conditions (a), (b), (c) and (d). We divide Γ_1 into the three parts $\Gamma_-(0)$, Γ_+ and $\Gamma_-(\pi/2)$, and denote these curves by $\Gamma_-^0(0)$, Γ_+^0 and $\Gamma_-^0(\pi/2)$, respectively. Let T_-^0 be the image $h(\Gamma_-^0(0))$ and $h(\Gamma_-^0(\pi/2))$ in the τ-plane.

In the case where x and y are in general position, we can get Γ_+ by a continuous modification of Γ_+^0 preserving properties (a) and (c). Note that, when t is on the circle $|t - t_j| = 2|x|^{\varepsilon}$, the function

$$h(t) - h(x^{\frac{1}{2}}) = y(t - x^{\frac{1}{2}}) - x(t - x^{\frac{1}{2}})^2 - x^{\frac{1}{2}}(t - x^{\frac{1}{2}})^3 - \frac{1}{4}(t - x^{\frac{1}{2}})^4 \tag{2.11}$$

satisfies $|h(t) - h(x^{1/2})| \sim |x|^{1+2\varepsilon}$ as $x \to \infty$. We draw a curve $T_-(0)$ (or $T_-(\pi/2)$) in the τ-plane with the properties below (cf. Fig. 8):

(A) $T_-(0)$ (or $T_-(\pi/2)$) is a curve starting from $\tau = \infty$ ($\arg\tau = \pi$) and ending at $h(a_0)$ (or starting from $h(a_{\pi/2})$ and tending to $\tau = \infty$ ($\arg\tau = \pi$)).

(B) $T_-(0)$ (or $T_-(\pi/2)$) lies outside the circles $|\tau - h(t_j)| = 2|x|^{1+2\varepsilon}$ ($j = 0, 1, 2$)

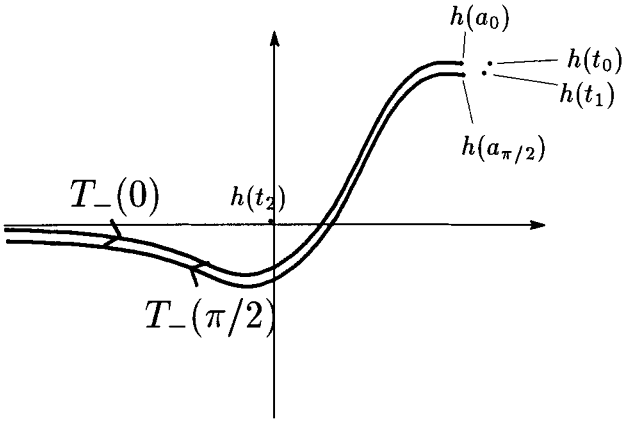

FIGURE 8.

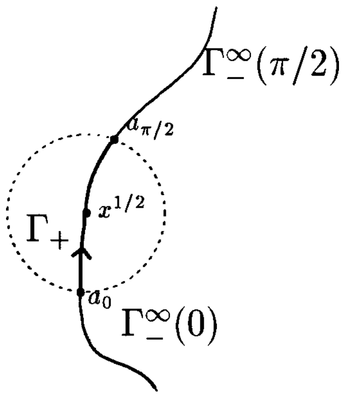

FIGURE 9.

(C) $(d/d\rho)\mathrm{Re}\tau \le -c$ for $\tau \in T_-(0)$ (or $\tau \in T_-(\pi/2)$), where c is a positive constant and ρ denotes the length of the part of $T_-(0)$ (or $T_-(\pi/2)$) from $h(a_0)$ (or $h(a_{\pi/2})$) to τ.

(D) $T_-(0)$ (or $T_-(\pi/2)$) is obtained by a continuous modification of T_-^0 preserving properties (A) through (C).

As long as $|\arg x - \pi/2| < 3\pi/4 - \delta$, such a drawing of $T_-(0)$ and $T_-(\pi/2)$ is possible. Note that the mapping $\tau = h(t)$ is biholomorphic at each point in $\mathbf{C} - \{t_0, t_1, t_2\}$. We take the curve $\Gamma_-(0)$ (or $\Gamma_-(\pi/2)$) to be the inverse image of $T_-(0)$ (or $T_-(\pi/2)$) tending to $t = \infty$ ($\arg t = 0$) (or $t = \infty$ ($\arg t = \pi/2$)). Combine the three curves $\Gamma_-(0)$, Γ_+ and $\Gamma_-(\pi/2)$ obtained above (Fig. 9). This is the desired path C_1 with properties (a) through (d).

2.4.2. Calculation of the Principal Part of the Integral z_1. We calculate the principal part $I_1' = \int_{\Gamma_+} f\,dt$. We put $t = x^{1/2}(1+\sigma)$ in the integral. In the case where $y = 0$ and $\arg x = \pi/2$, if t moves in the direction of $\arg t = 3\pi/4$ near $t = x^{1/2}$, then σ in the direction of $\arg\sigma = \pi/2$ near $\sigma = 0$. We get

$$\begin{aligned} I_1' &= \int_{|\sigma|\le\theta(x)} \exp(h(x^{\frac12}(1+\sigma)))d\sigma \\ &= x^{\frac12}\exp(h(x^{\frac12}))\int_{|\sigma|\le\theta(x)} \exp(x^{\frac12}y\sigma - x^2\sigma^2)\exp\left(-x^2\sigma^3\left(1+\frac{\sigma}{4}\right)\right)d\sigma \end{aligned}$$

with $\theta(x) = |x|^{-1+\varepsilon'}$. Furthermore the change of variables $\sigma = -2^{-1/2}x^{-1}\tau$ yields

$$\begin{aligned} I_1' = -x^{\frac12}\exp(h(x^{\frac12}))2^{-\frac12}x^{-1}\int_{|\tau|\le|x|^{\varepsilon'}} \exp\left(-2^{-\frac12}x^{-\frac12}y\tau - \frac{\tau^2}{2}\right) \\ \times\exp(2^{-\frac32}x^{-1}\tau^3(1-2^{-\frac52}x^{-1}\tau))d\tau\,, \end{aligned}$$

where τ moves on the real axis. For $|\sigma| \le \theta(x)$, we have

$$\begin{aligned} &\exp(2^{-\frac23}x^{-1}\tau^3(1-2^{-\frac52}x^{-1}\tau)) \\ &\quad= \sum_{k=0}^{M}\frac{1}{(1)_k}2^{-\frac23 k}x^{-k}\tau^{3k}(1-2^{-\frac52}x^{-1}\tau)^k + O(x^{-(M+1)}\tau^{3(M+1)}) \\ &\quad= \sum_{k=0}^{M}\sum_{l=0}^{M}\frac{(-k)_l}{(1)_k(1)_l}2^{-\frac{3k+5l}{2}}x^{-(k+l)}\tau^{3k+l} + O(x^{-(M+1)}\tau^{3(M+1)})\,. \end{aligned}$$

Substituting this into I_1', we get

$$\begin{aligned} I_1' &= -2^{-\frac12}x^{-\frac12}\exp(h(x^{\frac12})) \\ &\quad\times\left\{\sum_{k=0}^{M}\sum_{l=0}^{M}\frac{(-k)_l}{(1)_k(1)_l}2^{-\frac{3k+5l}{2}}x^{-(k+l)}J_{k,l}(x,y) + O(x^{-(M+1)})\right\} \end{aligned} \tag{2.12}$$

with

$$J_{k,l}(x,y) = \int_{|\tau|\le|x|^{\varepsilon'}}\tau^{3k+l}\exp\left(-2^{-\frac12}x^{-\frac12}y\tau - \frac{\tau^2}{2}\right)d\tau\,.$$

Let $D_\nu(x)$ be the parabolic cylinder function

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)}\int_0^{+\infty} t^{-\nu-1}\exp\left(-zt - \frac{t^2}{2}\right)dt \quad (\mathrm{Re}\nu < 0)\,.$$

The integral above is expressed in terms of the Hermite polynomial (cf. [1] vol. II pp. 117–119). In fact

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \tau^{3k+l} \exp\left(-2^{-\frac{1}{2}} x^{-\frac{1}{2}} y\tau - \frac{\tau^2}{2}\right) d\tau \\
&= \Gamma(3k+l+1) e^{\frac{1}{8}x^{-1}y^2} \\
&\quad \times (D_{-(3k+l+1)}(2^{-\frac{1}{2}} x^{-\frac{1}{2}} y) - (-)^{3k+l+1} D_{-(3k+l+1)}(-2^{-\frac{1}{2}} x^{-\frac{1}{2}} y)) \\
&= (-)^{k+l} (2\pi)^{\frac{1}{2}} e^{\frac{1}{8}x^{-1}y^2} e^{-\frac{3k+l}{2}\pi i} D_{3k+l}(2^{-\frac{1}{2}} i x^{-\frac{1}{2}} y) \\
&= (-)^{k+l} (2\pi)^{\frac{1}{2}} e^{\frac{1}{4}x^{-1}y^2} e^{-\frac{3k+l}{2}\pi i} 2^{-\frac{3k+l}{2}} H_{3k+l}\left(\frac{i}{2} x^{-\frac{1}{2}} y \right),
\end{aligned}$$

which differs from $J_{k,l}(x, y)$ by $O(-|x|^{\varepsilon})$. Hence (2.12) becomes

$$\begin{aligned}
I_1' &= -\sqrt{\pi} x^{-\frac{1}{2}} \exp\left(h(x^{\frac{1}{2}}) + \frac{1}{4} x^{-1} y^2 \right) \left\{ \sum_{k=0}^{M} \sum_{l=0}^{M} \frac{(-)^{k+l}(-k)_l}{(1)_k (1)_l 2^{3(k+l)}} \right. \\
&\qquad \left. \times\, e^{-\frac{3k+l}{2}\pi i} H_{3k+l}\left(\frac{i}{2} x^{-\frac{1}{2}} y \right) x^{-(k+l)} + O(x^{-(M+1)}) \right\} \\
&= -\sqrt{\pi} x^{-\frac{1}{2}} \exp\left(h(x^{\frac{1}{2}}) + \frac{1}{4} x^{-1} y^2 \right) \left\{ \sum_{m=0}^{M} \sum_{k=0}^{M} \frac{(-)^{m+k}(-k)_{m-k}}{(1)_k (1)_{m-k} 2^{3m}} \right. \\
&\qquad \left. \times\, e^{-\frac{m}{2}\pi i} H_{2k+m}\left(\frac{i}{2} x^{-\frac{1}{2}} y \right) x^{-m} + O(x^{-(M+1)}) \right\} .
\end{aligned}$$

Thus we get the desired asymptotic expansion.

2.4.3. Estimate of the Remaining Part of the Integral z_1**.** We estimate the remaining part $\int_{\Gamma_-(0)\cup\Gamma_-(\pi/2)} f\,dt$. For $t \in \Gamma_-(0)$, we have

$$|dt| = \left| \frac{1}{h'(t)} \frac{dh}{ds} \right| ds = O(x^{-1-\varepsilon}) ds ,$$

where s denotes the length of the part of the image $h(\Gamma_-(0))$ from $h(a_0)$ to $h(t)$. Noting that the image $h(\Gamma_-(0))$ possesses the properties (B) and (C), we get

$$\begin{aligned}
\int_{\Gamma_-(0)} f\,dt &= O\left(\int_{\Gamma_-(0)} |\exp h(t)|\, |dt| \right) = O\left(\int_{\Gamma_-(0)} \exp(\operatorname{Re} h(t)) |dt| \right) \\
&= O\left(\int_{\Gamma_-(0)} \exp(h(x^{\frac{1}{2}}) - |x|^{\varepsilon}) |dt| \right) \\
&= O\left(\int_{\Gamma_-(0)} \exp(h(x^{\frac{1}{2}}) - |x|^{\varepsilon}) x^{-1-\varepsilon} ds \right)
\end{aligned}$$

$$= O\left(x^{-\frac{1}{2}} \exp\left(h(x^{\frac{1}{2}}) + \frac{1}{4}x^{-1}y^2\right)x^{-\frac{1}{2}-\varepsilon} \exp(-|x|^{\varepsilon})\right).$$

The other part $\int_{\Gamma_-(\pi/2)} f dt$ is evaluated in a similar way. Thus the theorem is proved. □

2.5. Proof of Theorem 2.5. We give an asymptotic expansion of the integral z_R. Let $h_0(t)$ be the function given by

$$h_0(t) = \frac{xt^2}{2} + yt\,.$$

The saddle point of $h_0(t)$ is $-y/x$ which is an approximate saddle point t_2 and is close to the origin O as x tends to ∞.

2.5.1. Modification of the Path C_R. In this calculation, we wish to modify the path C_R in such a way that it fulfills the following conditions (cf. Fig. 10).

(p) C_R consists of three curves $\Gamma'_-(\pi)$, Γ'_+, $\Gamma'_-(0)$, where

(p.1) Γ'_+ is a curve passing through the origin O and lying inside the circle $|t| = 2|x|^{\varepsilon}$. Both ends b_π, b_0 of Γ'_+ are located on $|t| = 2|x|^{\varepsilon}$.

(p.2) $\Gamma'_-(\pi)$ is a curve starting from $t = \infty$ ($\arg t = \pi$) and ending at b_π.

(p.3) $\Gamma'_-(0)$ is a curve starting from b_0 and tending to $t = \infty$ ($\arg t = 0$).

(q) C_R lies outside the circles $|t - t_j| = |x|^{\varepsilon}$ $(j = 0, 1)$, and $\Gamma'_-(\pi)$ and $\Gamma'_-(0)$ lies outside the circle $|t - t_2| = 2|x|^{\varepsilon}$.

(r) $\mathrm{Re}\, h_0(t) \leq 0$, $\mathrm{Im}\, h_0(t) = 0$ for $t \in \Gamma'_+$.

(s) $(d/d\rho)\mathrm{Re}(h(t)) \leq -c$ for $t \in \Gamma'_-(\pi)$ (or $t \in \Gamma'_-(0)$), where c is a positive constant and ρ denotes the length of the part of the image curve $h(\Gamma'_-(\pi))$ (or $h(\Gamma'_-(0))$) from $h(b_\pi)$ (or $h(b_0)$) to $\tau = h(t)$.

In the case $y = 0$ and $\arg x = \pi$, take the path C_R to be the real axis. Of course this path is a suitable path with the properties (p), (q), (r) and (s). We divide this path into

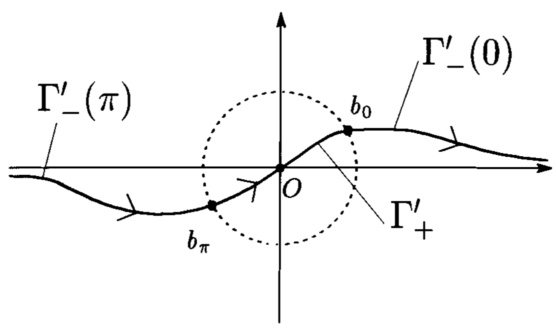

FIGURE 10.

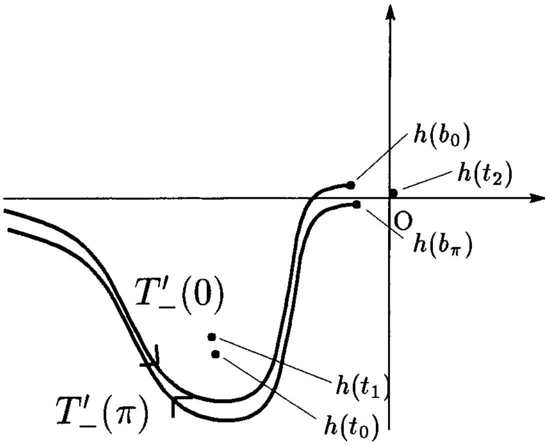

FIGURE 11.

the three parts $\Gamma'_-(0)$, Γ'_+ and $\Gamma'_-(\pi)$, and denote these curves by $\Gamma'^0_-(0)$, Γ'^0_+ and $\Gamma'^0_-(\pi)$, respectively. Consider the case where x and y are in general position. We can obtain Γ'_+ by a continuous modification of Γ'^0_+, preserving properties (p) and (r). Note that, when t is on the circle $|t| = 2|x|^\varepsilon$, $|h(t)| \sim |x|^{1+2\varepsilon}$ as $x \to \infty$. As long as $|\arg x - \pi| < \frac{3}{4}\pi - \delta'$, we can draw a curve $T'_-(0)$ (or $T'_-(\pi)$) in the τ-plane with the properties bellow (cf. Fig. 11), where δ is a sufficiently small positive constant.

(P) $T'_-(0)$ (or $T'_-(\pi)$) is a curve starting from $\tau = \infty$ ($\arg \tau = \pi$) and ending at $h(b_0)$ (or starting from $h(b_\pi)$ and tending to $\tau = \infty$ ($\arg \tau = \pi$)).

(Q) $T'_-(0)$ (or $T'_-(\pi)$) lies outside the circles $|\tau - h(t_j)| = 2|x|^{1+2\varepsilon}$ $(j = 0, 1, 2)$

(R) $(d/d\rho)\mathrm{Re}\tau \leq -c$ for $\tau \in T'_-(0)$ (or $\tau \in T'_-(\pi)$), where c is a positive constant and ρ denotes the length of the part of $T'_-(0)$ (or $T'_-(\pi)$) from $h(b_0)$ (or $h(b_\pi)$) to τ.

(S) $T'_-(0)$ (or $T'_-(\pi)$) is obtained by a continuous modification of the image $h(\Gamma'^0_-(0))$ (or $h(\Gamma'^0_-(\pi))$) preserving properties (P) through (R).

We take the inverse image $\Gamma'_-(0) = h^{-1}(T'_-(0))$ (or $\Gamma'_-(\pi) = h^{-1}(T'_-(\pi))$) tending to $t = \infty$ ($\arg t = 0$) (or $t = \infty$ ($\arg t = \pi$)). It is easy to see that the path $C_R = \Gamma'_-(0) \cup \Gamma'_+ \cup \Gamma'_-(\pi)$ is a continuous modification of the real axis and fulfills the conditions (p) through (s).

2.5.2. Calculation of the Principal Part of the Integral z_R. We calculate the principal part $I_R = \int_{\Gamma'_+} f\,dt$. Putting $t = ix^{-1/2}\tau$ in the integral, we get

$$\begin{aligned} I_R &= \int_{C_R, |t| \leq \theta(x)} e^{-\frac{t^4}{4}} \exp\left(yt + \frac{xt^2}{2}\right) dt \\ &= ix^{-\frac{1}{2}} \int_{|\tau| \leq |x|^{\varepsilon'}} e^{-\frac{x^{-2}\tau^4}{4}} \exp\left(ix^{-\frac{1}{2}} y\tau - \frac{\tau^2}{2}\right) d\tau\,, \end{aligned}$$

where τ moves on the real axis, and $\theta(x) = |x|^{-\frac{1}{2}+\varepsilon'}$. For $|t| \le \theta(x)$, we have

$$I_R = ix^{-\frac{1}{2}} \left\{ \sum_{k=0}^{M} \frac{(-)^k}{(1)_k 4^k} x^{-2k} J_k(x, y) + O(x^{-2(M+1)}) \right\} ,$$

with

$$J_k(x, y) = \int_{|\tau| \le |x|^{\varepsilon'}} \tau^{4k} \exp\left(ix^{-\frac{1}{2}} y\tau - \frac{\tau^2}{2} \right) d\tau ,$$

where M is an arbitrary positive integer. We express the integral $J_k(x, y)$ by using the parabolic cylinder function $D_\nu(x)$ and the Hermite polynomial $H_\nu(z)$. By [1] (vol.II pp.117–119),

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \tau^{4k} \exp\left(ix^{-\frac{1}{2}} y\tau - \frac{\tau^2}{2} \right) d\tau \\
&= \Gamma(4k+1) e^{-\frac{1}{4}x^{-1}y^2} (D_{-(4k+1)}(ix^{-\frac{1}{2}}y) + (-)^{4k} D_{-(4k+1)}(-ix^{-\frac{1}{2}}y)) \\
&= e^{-\frac{1}{4}x^{-1}y^2} (2\pi)^{\frac{1}{2}} D_{4k}(x^{-\frac{1}{2}}y) \\
&= (2\pi)^{\frac{1}{2}} 2^{-2k} e^{-\frac{1}{2}x^{-1}y^2} H_{4k}(2^{-\frac{1}{2}} x^{-\frac{1}{2}} y) ,
\end{aligned}$$

which differs from $J_k(x, y)$ by $O(-|x|^{\varepsilon})$. Thus we obtain the desired asymptotic expansion.

2.5.3. Estimate of the Remaining Part of the Integral z_R. We estimate the remaining part $\int_{\Gamma'_-(0) \cup \Gamma'_-(\pi)} f dt$. For $t \in \Gamma'_-(0)$, we have

$$|dt| = \left| \frac{1}{h'(t)} \frac{dh}{ds} \right| ds = O(x^{-1-\varepsilon}) ds ,$$

where s denotes the length of the part of the image $h(\Gamma'_-(0))$ from $h(b_0)$ to $h(t)$. Using this and property (Q) and (R), we get

$$\begin{aligned}
\int_{\Gamma'_-(0)} f dt &= O\bigg(\int_{\Gamma'_-(0)} \exp(-|x|^{\varepsilon}) x^{-1-\varepsilon} ds \bigg) \\
&= O\bigg(x^{-\frac{1}{2}} \exp\bigg(-\frac{1}{2} x^{-1} y^2 \bigg) x^{-\frac{1}{2}-\varepsilon} \exp(-|x|^{\varepsilon}) \bigg) .
\end{aligned}$$

The other part $\int_{\Gamma_-(\pi/2)} f dt$ is evaluated in a similar way. Thus the theorem is proved. □

3. Stokes Multipliers near $x = \infty$

By Theorems 2.3, 2.5 and Corollary 2.4, for $|y| < r'$,

$$z_1 \simeq Z_1 , \quad \text{in} \quad |\arg x - \pi/2| < 3\pi/4 - \delta , \tag{3.1}$$

$$z_4 \simeq Z_4 , \quad \text{in} \quad |\arg x - 3\pi/2| < 3\pi/4 - \delta , \tag{3.2}$$

$$z_R \simeq Z_R\,, \quad \text{in} \quad |\arg x - \pi| < 3\pi/4 - \delta\,, \tag{3.3}$$

as $y \to \infty$ through each sector. The Stokes multipliers near $x = \infty$ are given by the following.

THEOREM 3.1. (1) *In the sector* $|\arg x + \pi/4| < \pi/2 - \delta$,

$$-z_4 \simeq Z_1\,, \quad -z_1 - z_R \simeq Z_4\,, \quad z_1 + z_4 \simeq Z_R\,.$$

(2) *In the sector* $|\arg x - \pi/4| < \pi/2 - \delta$,

$$z_1 \simeq Z_1\,, \quad z_4 - z_R \simeq Z_4\,, \quad z_1 + z_4 \simeq Z_R\,.$$

(3) *In the sector* $|\arg x - 3\pi/4| < \pi/2 - \delta$,

$$z_1 \simeq Z_1\,, \quad z_4 - z_R \simeq Z_4\,, \quad z_R \simeq Z_R\,.$$

(4) *In the sector* $|\arg x - 5\pi/4| < \pi/2 - \delta$,

$$z_1 + z_R \simeq Z_1\,, \quad z_4 \simeq Z_4\,, \quad z_R \simeq Z_R\,.$$

In addition to (3.1), (3.2) and (3.3), we need the following lemma.

LEMMA 3.2. *We have*

$$-z_1 - z_R \simeq Z_4\,, \quad \textit{in} \quad |\arg x + \pi/2| < 3\pi/4 - \delta\,, \tag{3.4}$$
$$z_1 + z_4 \simeq Z_R\,, \quad \textit{in} \quad |\arg x| < 3\pi/4 - \delta\,, \tag{3.5}$$
$$-z_4 \simeq Z_1\,, \quad \textit{in} \quad |\arg x + \pi/2| < 3\pi/4 - \delta\,, \tag{3.6}$$
$$z_1 + z_R \simeq Z_1\,, \quad \textit{in} \quad |\arg x - 3\pi/2| < 3\pi/4 - \delta\,, \tag{3.7}$$
$$z_4 - z_R \simeq Z_4\,, \quad \textit{in} \quad |\arg x - \pi/2| < 3\pi/4 - \delta\,. \tag{3.8}$$

PROOF. Suppose that $|\arg(e^{\pi i}x) - \pi/2| = |\arg x + \pi/2| < (3/4)\pi - \delta$. By Theorem 2.3 and Corollary 2.4,

$$z_1(e^{\pi i}x, e^{\frac{1}{2}\pi i}y) \simeq Z_1(e^{\pi i}x, e^{\frac{1}{2}\pi i}y) = -iZ_4(x, y)\,. \tag{3.9}$$

On the other hand, by Propositions 1.2 and 1.1,

$$z_1(e^{\pi i}x, e^{\frac{1}{2}\pi i}y) = -iz_2 = -i(z_4 + z_1)\,.$$

This relation combined with (3.9) yield (3.4). In the sector $|\arg(e^{\pi i}x) - \pi| = |\arg x| < (3/4)\pi - \delta$, by Theorem 2.5,

$$z_R(e^{\pi i}x, e^{\frac{1}{2}\pi i}y) \simeq Z_R(e^{\pi i}x, e^{\frac{1}{2}\pi i}y) = -iZ_R(x, y)\,. \tag{3.10}$$

By Propositions 1.1 and 1.2,

$$z_R(e^{\pi i}x, e^{\frac{1}{2}\pi i}y) = -iz_I = -i(z_4 + z_1)\,.$$

Using this and (3.10), we get (3.5). The other formulas are also obtained in a similar way. □

PROOF OF THEOREM 3.1. Noting that (3.4), (3.5) and (3.6) are valid in the sector $|\arg x + \pi/2| < (3/4)\pi - \delta$, we obtain (1). Assertion (2) is derived from (3.1), (3.5) and (3.8). We have (3) and (4) by similar methods. □

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