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On the Product Formula Approach to a Class of Quasilinear Evolution Systems

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Abstract. We concern ourselves firstly with the abstract evolution equation

$$\begin{cases} \mathbf{x}'(t) = (A+B)\mathbf{x}(t)\,, \quad t > 0\,; \quad \mathbf{x}(0) = \mathbf{x}_0 \in D\,, \\ \varphi(\mathbf{x}(t))' \le g(\varphi(\mathbf{x}(t)))\,, \quad t > 0 \end{cases}$$

in a Banach space X, where A is m-quasidissipative, but B satisfies no global conditions of quasidissipativity. We assume that a secondary topology (and a related notion of limits) can be introduced, through a lower semicontinuous functional φ, such that B is locally Lipschitz continuous with respect to this topology. Under appropriate assumptions, the first product formula

$$\mathbf{x}(t) = (D,\varphi)\text{-}\lim_{h\downarrow 0}\{(I-hA)^{-1}(I+hB)\}^{[t/h]}\mathbf{x}_0\,, \quad t \ge 0$$

can be defined, and we show that this provides unique solutions in a generalized sense to our original equation. Here (D,φ) -lim refers to our new notion of limit. For approximations A_h of A, we also show convergence for the second product formula,

$$\mathbf{x}(t) = (D,\varphi)\text{-}\lim_{h\downarrow 0}\{(I-hA_h)^{-1}(I+hB)\}^{[t/h]}\mathbf{x}_0\,,$$

and use this to generate solutions to a class of advection reaction diffusion systems. As a concrete example, a mathematical model for HIV infection is studied.

Introduction

Generation theorems for nonlinear semigroups associated with evolution equations governed by either quasi-dissipative or quasi-accretive operators are well developed, and have been widely used to treat a variety of mathematical models. There are, however, situations where it is desirable to consider additive combinations of quasi-dissipatve operators and non-quasidissipative perturbations, and this is the basic motivation for the work leading to this paper.

By means of a bornoglogical-type structure, it is possible to treat a broad class of evolution equations that include nolinear perturbations satisfying no quasi-dissipativity conditions in a global sense. For an appropriate lower semicontinuous functional φ on the Banach space $(X, \|\cdot\|)$ in question, φ-bounded sets make up this bornological structure, giving us a system (D,φ) and a natural, related notion of convergence. This secondary structure is important

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in our argument for two reasons. Firstly, fundamental properties of operators appearing in the evolution equations, such as quasidissipativity and continuity, are localized on φ-bounded sets. Secondly, growth of solutions to the evolution equations is restricted with respect to this functional.

Solutions are constructed through the use of the *product formula*, which generates "φ-stable" approximate descrete schemes to an abstract evolution equation in (D, φ). By showing the convergence (in the sense mentioned above) of such schemes in our system we obtain a generation theorem for locally quasi-contractive semigroups providing solutions, perhaps in a generalized sense, to evolution problems of the form

$$\text{(EP)} \qquad \begin{cases} \mathbf{x}'(t) = (A+B)\mathbf{x}(t)\,, \quad t > 0\,; \quad \mathbf{x}(0) = \mathbf{x}_0 \in D\,, \\ \varphi(\mathbf{x}(t))' \leq g(\varphi(\mathbf{x}(t)))\,, \quad t > 0 \end{cases}$$

in the system (D, φ), where the operator $A + B$ is quasi-dissipative on φ-bounded sets, and $g(\cdot)$ is a comparison function on $[0, \infty)$, specifying the growth rate of the solution and ensuring φ-boundedness.

We start by placing assumptions on operators A and B, and the class D of initial data, such that generalized solutions $\mathbf{x}(\cdot)$ to (EP) may be generated by the *first product formula*:

$$\mathbf{x}(t) = (D, \varphi)\text{-}\lim_{h\downarrow 0}\{(I - hA)^{-1}(I + hB)\}^{[t/h]}\mathbf{x}_0\,, \quad t \geq 0\,.$$

Here (D, φ)-lim means the limit in (D, φ), detailed later on. This semi-implicit scheme is designed to extract the relevant "good" properties of the m-quasidissipative operator A and the possibly non-quasidissipative operator B.

We then extend the class of equations which can be treated by decomposing the operator A into a "well-behaved" principal part Λ, and a relatively continuous or non-smooth perturbation Ξ, which may be such that the resolvent $(I - h(\Lambda + \Xi))^{-1}$ does not exist on a sufficiently large domain. In this case we assume that Ξ can be approximated by operators Ξ_h, as h tends to 0, in such a way that a new type of product formula may be employed:

$$\mathbf{x}(t) = (D, \varphi)\text{-}\lim_{h\downarrow 0}\{(I - hA_h)^{-1}(I + hB)\}^{[t/h]}\mathbf{x}_0\,.$$

Here A_h is defined as $\Lambda + \Xi_h$. This second product formula can be used to treat equations containing differential operators with "bad" coefficients, and even certain types of strong coupling (our approach can be applied to the strongly coupled system of equations discussed in [7]).

The first section of this paper contains some preliminary results, definitions and notational conventions which shall be used in the remaining sections, including definitions of the system (D, φ) and related convergence mentioned above. In Section 2, we define the class of evolution equations to be treated, and discuss the abstract theory related to the generation of nonlinear semigroups associated with (EP). Towards the end of Section 2 a more general class of equations and the related second product formula is considered, and it is to this more general class that the semilinear systems of advection-reaction-diffusion equations, treated in

Section 3, belong. The *HIV infection model* is a concrete and significant example of such a system, consisting of three equations with coupled reaction terms, and is treated in Section 4. It is expected to typify the kind of problem to which our approach can be applied.

Coupled reaction terms, such as those contained in the HIV model would ordinarily require rather technical treatment for existence of solutions to be shown by more direct methods. It turns out that the product formula is proven in such a way that these terms may be dealt with in a systematic and comprehensive way, and that as well as existence, we obtain some useful information about the solutions, related to the physical meaning of the parameters.

1. Preliminaries

We begin by giving some definitions and general concepts used in this paper.

Let X be a real Banach space with norm $\|\cdot\|$, and let $F(\cdot)$ denote the duality mapping. An operator $\mathscr{A}$ with domain $D(\mathscr{A})$ in a Banach space X is said to be *quasi-dissipative of type* ω for some $\omega \in \mathbf{R}$, if for all $\mathbf{x}_1, \mathbf{x}_2 \in D(\mathscr{A})$,

$$\langle \mathscr{A}\mathbf{x}_1 - \mathscr{A}\mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2\rangle_i \le \omega\|\mathbf{x}_1 - \mathbf{x}_2\|^2 .$$

Here the *lower semi-inner product* $\langle\cdot,\cdot\rangle_i$ is defined $\langle z_2, z_1\rangle_i = \inf\{\langle z_2, z^*\rangle \mid z \in F(z_1)\}$, $z_1, z_2 \in X$ and the *upper semi-inner product* $\langle\cdot,\cdot\rangle_s$ as the supremum of the same set. $\mathscr{A}$ is *dissipative* on X, or simply *dissipative*, if $\omega = 0$. Recall that an operator $\mathscr{A}$ is *accretive* if and only if $-\mathscr{A}$ is dissipative.

Let $\varphi : X \to [0, \infty]$ be a lower semicontinuous functional with effective domain $D(\varphi) = \{\mathbf{x} \in X : \varphi(\mathbf{x}) < \infty\}$. A "$\varphi$-bounded set" is defined to be a subset $W \subset X$ contained in some level set $D_\alpha = \{\mathbf{x} \in X : \varphi(\mathbf{x}) \le \alpha\}$, $\alpha > 0$, and in this sense, the family $\{D_\alpha : \alpha > 0\}$ defines a secondary structure on X. Denote by (D, φ) the metric space D with this extra structure. Noting that the family $\mathfrak{B}$ of φ-bounded sets satisfies the conditions: (i) $\mathfrak{B}$ is a covering for $D(\varphi) = \bigcup_{\alpha>0} D_\alpha$, (ii) if $E, F \in \mathfrak{B}$, then $E \cup F \in \mathfrak{B}$, and (iii) if a set $E \subset E_1$ for some $E_1 \in \mathfrak{B}$, then $E \in \mathfrak{B}$. Thus $\mathfrak{B}$ is a bornology on the metric space $(D(\varphi), \|\cdot\|)$. We shall say that a generalized sequence $(\mathbf{x}_\gamma)_{\gamma\in\Gamma}$, Γ being a directional set, converges to $\mathbf{x}$ in (D, φ) if $\sup_{\gamma\in\Gamma} \varphi(\mathbf{x}_\gamma) < \infty$ and $\lim_\gamma \mathbf{x}_\gamma = \mathbf{x}$ in X. In this case, we write

$$(D, \varphi)\text{-}\lim_\gamma \mathbf{x}_\gamma = \mathbf{x} \quad \text{or} \quad \mathbf{x}_\gamma \to \mathbf{x} \text{ in } (D, \varphi) .$$

The generalized notions of solution that we shall deal with (*integral solutions* and *strong solutions*) are defined as follows.

1 Definition. Let $\mathscr{A}$ be an operator on a Banach space X, such that (D, φ) is a system of the type discussed above. For a real number ω and a Borel measurable function $g : [0, \infty) \to \mathbf{R}$, we say that $\mathbf{x}(\cdot) : [0, \tau] \to \overline{D(\mathscr{A})}$ is an *integral solution of type* (ω, g) to the evolution problem (EP) if the following hold:

(i) $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(\cdot)$ is strongly continuous on $[0, \tau]$,

(ii) for every $s, t \in [0, \tau]$ with $s < t$ and every $\hat{\mathbf{x}} \in D(\mathscr{A})$,

$$e^{-2\omega t}\|\mathbf{x}(t) - \hat{\mathbf{x}}\|^2 - e^{-2\omega s}\|\mathbf{x}(s) - \hat{\mathbf{x}}\|^2 \le 2\int_s^t e^{-2\omega r}[\mathscr{A}\hat{\mathbf{x}}, \mathbf{x}(r) - \hat{\mathbf{x}}]_s dr,$$

(iii) for $s, t \in [0, \tau]$ with $s < t$,

$$\varphi(\mathbf{x}(t)) \le \varphi(\mathbf{x}(s)) + \int_s^t g[\varphi(\mathbf{x}(r))]dr\,. \tag{1.1}$$

Equivalent to (ii) above is the following, slightly clearer condition:

(ii)′ for every $s, t \in [0, \tau]$ with $s < t$ and every $\hat{\mathbf{x}} \in D(\mathscr{A})$,

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}\|^2 - \|\mathbf{x}(s) - \hat{\mathbf{x}}\|^2 \le 2\int_s^t [\mathscr{A}\hat{\mathbf{x}}, \mathbf{x}(r) - \hat{\mathbf{x}}]_s dr + 2\omega\int_s^t \|\mathbf{x}(r) - \hat{\mathbf{x}}\|^2 dr\,.$$

It is clear that if $\mathbf{x}(t)$ is an integral solution of type ω_1, then $\mathbf{x}(t)$ is an integral solution of type ω_2 to the same problem, whenever $\omega_1 \le \omega_2$.

2 DEFINITION. A function $\mathbf{x}(\cdot)$ shall be called a *strong solution* to equation (EP) above, if $\mathbf{x}(\cdot)$ is strongly absolutely continuous on bounded intervals, the derivative $\mathbf{x}'(t)$ with respect to time exists and is equal to $\mathscr{A}\mathbf{x}(t)$ for almost every $t \ge 0$, and if $\varphi(\mathbf{x}(\cdot))$ satisfies (iii) from the previous definition.

The following fact is known about these generalized solutions.

3 LEMMA. *For $\mathscr{A}$ a quasi-dissipative operator of type ω, a strong solution $\mathbf{x}(t)$ to* (EP) *is always an integral solution of type (ω, g). Conversely, when the following conditions are satisfied by an integral solution $\mathbf{x}(\cdot)$ of type (ω, g) :*

(i) *$\mathbf{x}(\cdot)$ is strongly absolutely continuous on bounded intervals and $\mathbf{x}'(t)$ exists for almost all t in some interval,*

(ii) *$\mathscr{A}$ is a maximal dissipative operator of type ω,*

then $\mathbf{x}(\cdot)$ is in fact a strong solution to (EP).

2. Non-dissipative perturbations of m-quasidissipative operators

In this section we shall prove the existence of solutions, in a generalized sense, to the abstract evolution problem

$$\begin{cases} \mathbf{x}'(t) = (A + B)\mathbf{x}(t)\,, \quad t > 0\,; \quad \mathbf{x}(0) = \mathbf{x}_0 \in D\,, \\ \varphi(\mathbf{x}(t))' \le g(\varphi(\mathbf{x}(t)))\,, \quad t > 0\,, \end{cases} \tag{EP}$$

where X is a Banach space and the class of initial data D, described below, is contained in the domain $D(B)$ of B. The operator A is assumed to be an m-quasi dissipative operator of type ω_0, with domain $D(A)$ dense in X. B denotes a nonlinear perturbing operator, locally continuous in the sense described below, satisfying a certain *growth condition*.

Assume that we may define a lower semicontinuous functional $\varphi : X \to \mathbf{R}^+ \cup \{\infty\}$, with effective domain $D = \{\mathbf{x} \in X \mid \varphi(\mathbf{x}) < \infty\}$. For $r > 0$ let $D_r = \{\mathbf{x} \in X \mid \varphi(\mathbf{x}) \le r\}$, and note that $D = \bigcup_{k=1}^{\infty} D_k$. Of course, in general, the class of admissible initial data may simply be a subset of the effective domain of φ, however by simply restricting φ to an appropriate set we may assume equality with no loss of generality. We further assume that the following hold:

(C1) There exists a Borel measurable function $\psi^* : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\varphi(\mathbf{x} + hB\mathbf{x}) \le \varphi(\mathbf{x}) + h\psi^*(\varphi(\mathbf{x}))$$

holds for all $\mathbf{x} \in D$, and for any real $\eta_0 \ge 0$, the integral inequality

$$\begin{cases} \beta(t) \le \eta_0 + \displaystyle\int_0^t \psi^*(\beta(r))dr \\ \beta(0) = \eta_0 \end{cases}$$

has a bounded and integrable maximal solution $\hat{\beta}(\cdot; \eta_0)$, on any finite interval.

(C2) given an arbitrary $r > 0$, there exists a constant $m(r) > 0$ such that

$$\|B\mathbf{x}_1 - B\mathbf{x}_2\| \le m(r)\|\mathbf{x}_1 - \mathbf{x}_2\|$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D_r$. We may assume further that $m(\cdot)$ is nondecreasing and upper semicontinuous on $\mathbf{R}^+$.

(C3) $\varphi((I - hA)^{-1}\mathbf{x}) \le \varphi(\mathbf{x})$ for all $\mathbf{x} \in D$ and $h > 0$ sufficiently small.

In general, we may wish to consider the case where the resolvent $(I - hA)^{-1}$ satisfies some more general growth condition than (C3), for example by including a factor $(1 - h\alpha)^{-1}$, for some $\alpha > 0$, on the right hand side of the inequality. The arguments presented here can be extended to this case, although even in the rather general application dealt with later on we need only consider the case described by (C3). Condition (C2) is effectively the local Lipschitz continuity of the operator B, although here "locally" means with respect to the secondary topology defined by φ, where as the continuity is with respect to the norm itself. This condition is not strong enough to guarantee, for example, boundedness of the approximate solutions we shall construct later on, (necessitating (C1) or some equivalent) but covers a broad range of operators describing reaction-type terms in the class of equations we shall treat.

We thus have a system (D, φ) of the type mentioned previously, with respect to which both A and B are well behaved. Depending on the choice of φ, the "good properties" of B that are exploited may include, for example, relative continuity with respect to A, or with respect to some family of sets which reflect the structure of $A + B$. It is clear that the choice of φ is heavily related to the explicit form of B, depending on the problem to be treated.

Our approach shall revolve around the *product formula*, used to generate a difference approximation scheme consisting of elements $\mathbf{x}_i^h$. Let the initial value $\mathbf{x}_0$ belong to some D_k and $\tau > 0$ be given. For $h > 0$ and positive integer i such that $ih \le \tau$, define

(PF) $$\mathbf{x}_i^h = [(I - hA)^{-1}(I + hB)]^i \mathbf{x}_0 \, .$$

To start with, we shall need the *stability condition*, namely the φ-boundedness, given in the following lemma.

4 LEMMA. *Given a real number $\tau > 0$ and an element $\mathbf{x}_0 \in D$, there exists $h^0 > 0$ such that the set*

$$\mathscr{S}_{\tau,\mathbf{x}_0} = \{[(I - hA)^{-1}(I + hB)]^{[\frac{t}{h}]}\mathbf{x}_0 \,|\, h < h^0, 0 \le t \le \tau\} \tag{2.1}$$

is uniformly bounded with respect to φ, i.e. there exists $k > 0$ such that $\varphi(\mathbf{x}) \le k$ for all $\mathbf{x} \in \mathscr{S}_{\tau,\mathbf{x}_0}$.

PROOF. For $\mathbf{x}_0 \in D_r$, define $\mathbf{x}_i^h = [(I - hA)^{-1}(I + hB)]^i \mathbf{x}_0$, $ih \le \tau$, $h < h^0$ (sufficiently small, as in C3). Then, by the properties of A and B above, we have

$$\begin{aligned}\varphi(\mathbf{x}_i^h) &= \varphi((I - hA)^{-1}(I + hB)\mathbf{x}_{i-1}^h) \\ &\le \varphi(\mathbf{x}_{i-1}^h + hB\mathbf{x}_{i-1}^h) \le \varphi(\mathbf{x}_{i-1}^h) + h\psi^*(\varphi(\mathbf{x}_{i-1}^h)) \\ &\le \varphi(\mathbf{x}_{i-2}^h) + h\psi^*(\varphi(\mathbf{x}_{i-2}^h)) + h\psi^*(\varphi(\mathbf{x}_{i-1}^h))\,.\end{aligned}$$

Continuing inductively, we eventually obtain

$$\varphi(\mathbf{x}_i^h) \le \varphi(\mathbf{x}_0) + h\sum_{j=0}^{i-1} \psi^*(\varphi(\mathbf{x}_j^h))\,.$$

Thus, defining $\beta_h(t) = \varphi([(I - hA)^{-1}(I + hB)]^{[t/h]}\mathbf{x}_0)$, it is clear that

$$\beta_h(t) \le \beta_h(0) + \int_0^t \psi^*(\beta_h(r))dr\,, \quad 0 \le t \le \tau\,, \tag{2.2}$$

for $h \in (0, h^0]$. Here $\beta_h(0) = \varphi(\mathbf{x}_0)$ and so by (C1) there exists a maximal solution $\hat{\beta}(\cdot; \varphi(\mathbf{x}_0))$, with

$$\varphi(\mathbf{x}_i^h) \le \sup_{t\in[0,\tau]} \hat{\beta}(t; \varphi(\mathbf{x}_0)) = \hat{\beta}(\tau; \varphi(\mathbf{x}_0)) \quad \text{for all } \mathbf{x} \in \mathscr{S}_{\tau,\mathbf{x}_0}\,. \tag{2.3}$$

Note that $\hat{\beta}(\cdot, \eta)$ is non-decreasing. □

We are now in a position to treat a discrete scheme of the form (PF) and prove its convergence. Let $\mathbf{x}_0 \in D$, and define $\mathbf{x}_i^h$ as in (PF), for $h \in (0, h^0)$, $i = 1, \ldots, N_h$, $hN_h \le \tau$. Define $\mathbf{f}_i^h$ by

$$\mathbf{f}_i^h = \frac{\mathbf{x}_i^h - \mathbf{x}_{i-1}^h}{h} - (A + B)\mathbf{x}_i^h\,. \tag{2.4}$$

We shall show a localized version of the *consistency condition* as used in [2], i.e. that the error terms $\mathbf{f}_i^h$ satisfy

$$\lim_{h\downarrow 0} h \sum_{i=1}^{N_h} \|\mathbf{f}_i^h\| = 0\,, \tag{2.5}$$

and finally estimate terms of the form $\|\mathbf{x}_i^\lambda - \mathbf{x}_j^\mu\|$, for $\lambda, \mu \in (0, h^0)$.

5 LEMMA. *Let $\hat{\mathbf{x}} \in D(A)$ and let k be the φ-bound on $\mathscr{S}_{\tau,\mathbf{x}_0}$ from Lemma 4. Then,*

$$\|\mathbf{f}_i^h\| \le m(k)\,\mathrm{e}^{\tau(m(k)+\omega_h)}\cdot\{2\|\hat{\mathbf{x}} - \mathbf{x}_0\| + h(\|B\mathbf{x}_0\| + \|A\hat{\mathbf{x}}\|)\} \tag{2.6}$$

for all $i = 1, \dots, N_h$, where $\omega_h \equiv (1 - h\omega_0)^{-1}\omega_0$.

PROOF. Firstly, noting that $(I - hA)\mathbf{x}_i^h = (I + hB)\mathbf{x}_{i-1}^h$, we obtain that for $\hat{\mathbf{x}} \in D(A)$,

$$\begin{aligned} \|\mathbf{x}_1^h - \mathbf{x}_0\| &\le \|\mathbf{x}_1^h - \hat{\mathbf{x}}\| + \|\hat{\mathbf{x}} - \mathbf{x}_0\| \\ &\le (1 - h\omega_0)^{-1}\|(I + hB)\mathbf{x}_0 - (I - hA)\hat{\mathbf{x}}\| + \|\hat{\mathbf{x}} - \mathbf{x}_0\| \\ &\le (1 + (1 - h\omega_0)^{-1})\|\mathbf{x}_0 - \hat{\mathbf{x}}\| + h(1 - h\omega_0)^{-1}\|B\mathbf{x}_0 + A\hat{\mathbf{x}}\|\,. \end{aligned} \tag{2.7}$$

Secondly, using the stability condition,

$$\begin{aligned} \|\mathbf{x}_i^h - \mathbf{x}_{i-1}^h\| &\le (1 - h\omega_0)^{-1}\|(I - hA)\mathbf{x}_i^h - (I - hA)\mathbf{x}_{i-1}^h\| \\ &= (1 - h\omega_0)^{-1}\|\mathbf{x}_{i-1}^h - \mathbf{x}_{i-2}^h + h(B\mathbf{x}_{i-1}^h - B\mathbf{x}_{i-2}^h)\| \\ &\le (1 - h\omega_0)^{-1}(1 + hm(k))\|\mathbf{x}_{i-1}^h - \mathbf{x}_{i-2}^h\|, \end{aligned}$$

and inductively,

$$\|\mathbf{x}_i^h - \mathbf{x}_{i-1}^h\| \le (1 - h\omega_0)^{-(i-1)}(1 + hm(k))^{i-1}\|\mathbf{x}_1^h - \mathbf{x}_0\|\,. \tag{2.8}$$

Now, the error terms may be estimated, since

$$\begin{aligned} \|\mathbf{f}_i^h\| &= h^{-1}\|(I - hA)\mathbf{x}_i^h - \mathbf{x}_{i-1}^h - hB\mathbf{x}_i^h\| \\ &= h^{-1}\|(I + hB)\mathbf{x}_{i-1}^h - \mathbf{x}_{i-1}^h - hB\mathbf{x}_i^h\| = \|B\mathbf{x}_i^h - B\mathbf{x}_{i-1}^h\|\,. \end{aligned} \tag{2.9}$$

Using again the Lipscitz continuity of B on the set $\mathscr{S}_{\tau,\mathbf{x}_0}$ the statement follows by combining (2.7), (2.8) and (2.9), and noting that $(1 - h\omega_0)^{-1} \le \mathrm{e}^{h\omega_h}$. □

Using the above it is seen that the error terms $\mathbf{f}_i^h$ satisfy (2.5), in the following way: Given an $\varepsilon > 0$, we may choose some $\hat{\mathbf{x}} \in D(A)$ sufficiently close to $\mathbf{x}_0$ that $2\tau m(k)\,\mathrm{e}^{\tau(m(k)+\omega_0)}\|\hat{\mathbf{x}} - \mathbf{x}_0\| \le \varepsilon$. Then, letting h tend to zero from above, we see that

$$\begin{aligned} \lim_{h\downarrow 0} h \sum_{i=1}^{N_h} \|\mathbf{f}_i^h\| &\le \lim_{h\downarrow 0} \tau m(k)\,\mathrm{e}^{\tau(m(k)+\omega_h)}\cdot\{2\|\hat{\mathbf{x}} - \mathbf{x}_0\| + h(\|B\mathbf{x}_0\| + \|A\hat{\mathbf{x}}\|)\} \\ &= 2\tau m(k)\,\mathrm{e}^{\tau(m(k)+\omega_0)}\|\hat{\mathbf{x}} - \mathbf{x}_0\| \le \varepsilon\,. \end{aligned} \tag{2.10}$$

Since we may choose ε arbitrarily small, we conclude that $\lim_{h\downarrow 0} h \sum_{i=1}^{N_h} \|\mathbf{f}_i^h\| = 0$.

Let $\mathbf{x}, \hat{\mathbf{x}} \in D(A) \cap D_k$. There exists some functional $f \in X^*$ in the image of the duality mapping of $\mathbf{x} - \hat{\mathbf{x}}$ such that $\langle A\mathbf{x} - A\hat{\mathbf{x}}, f\rangle \le \omega_0 \|\mathbf{x} - \hat{\mathbf{x}}\|^2$. Considering the full operator $A + B$, we have

$$\begin{aligned}\langle (A+B)\mathbf{x} - (A+B)\hat{\mathbf{x}}, f\rangle &= \langle A\mathbf{x} - A\hat{\mathbf{x}}, f\rangle + \langle B\mathbf{x} - B\hat{\mathbf{x}}, f\rangle \\ &\le \omega_0\|\mathbf{x} - \hat{\mathbf{x}}\|^2 + \|B\mathbf{x} - B\hat{\mathbf{x}}\| \cdot \|\mathbf{x} - \hat{\mathbf{x}}\| \\ &\le (\omega_0 + m(k))\|\mathbf{x} - \hat{\mathbf{x}}\|^2 ,\end{aligned}$$

whereby it is seen that the operator $A + B$ is quasi-dissipative, of type $\omega_k = \omega_0 + m(k)$ on the set $D(A) \cap D_k$. Therefore, we have (see e.g. [1]), for $\mathbf{x}, \hat{\mathbf{x}} \in D(A) \cap D_k$,

$$\|(I - \lambda(A+B))\mathbf{x} - (I - \lambda(A+B))\hat{\mathbf{x}}\| \ge (1 - \lambda\omega_k)\|\mathbf{x} - \hat{\mathbf{x}}\| , \quad \text{and}$$

$$(\lambda + \mu - \lambda\mu\omega_k)\|\mathbf{x} - \hat{\mathbf{x}}\| \le \mu\|\mathbf{x} - \lambda(A+B)\mathbf{x} - \hat{\mathbf{x}}\| + \lambda\|\hat{\mathbf{x}} - \mu(A+B)\hat{\mathbf{x}} - \mathbf{x}\| .$$

It is then possible to show local convergence through the following Lemma. For details of the proof, the reader is refered to the work of Y. Kobayashi, and K. Kobayasi, Y. Kobayashi and S. Oharu, who treated a similar localized, and in fact time-dependent, case. Their arguments hold locally with little adjustment.

6 LEMMA. *Given* $\mathbf{x}_0 \in D$ *and* $\lambda, \mu > 0$, *let the elements* $\mathbf{x}_i^\lambda$ *and* $\mathbf{x}_j^\mu$ *be generated by the product formula* (PF). *Let* h^0 *be the appropriate constant from Lemma* 4, *and* $k = \hat{\beta}(\tau; \varphi(\mathbf{x}_0))$ *be the* φ*-bound on* $\mathscr{S}_{\tau,\mathbf{x}_0}$. *Then, given an element* $\hat{\mathbf{x}}$ *in* $D(A) \cap D$,

$$\begin{aligned}(1-\lambda\omega_k)^i(1-\mu\omega_k)^j\|\mathbf{x}_i^\lambda - \mathbf{x}_j^\mu\| &\le \|\mathbf{x}_0^\lambda - \hat{\mathbf{x}}\| + \|\mathbf{x}_0^\mu - \hat{\mathbf{x}}\| \\ &\quad + \{(\lambda i - \mu j)^2 + \lambda^2 i + \mu^2 j\}^{\frac{1}{2}}\|(A+B)\hat{\mathbf{x}}\| + \sum_{k=1}^{i}\lambda\|\mathbf{f}_k^\lambda\| + \sum_{k=1}^{j}\mu\|\mathbf{f}_k^\mu\| .\end{aligned} \tag{2.11}$$

This estimate is used to prove that the scheme converges to some limit as $h \downarrow 0$, and that the limit itself is a unique integral solution of type ω_k to the problem (EP) on the interval $[0, \tau]$. The proof of Lemma 4 shows that $k = \hat{\beta}(\tau; \varphi(\mathbf{x}_0))$. Let $\mathbf{x}_\tau(\cdot)$ represent the corresponding solution on the interval $[0, \tau]$. If $\tau' < \tau$ then, since $\mathbf{x}_\tau(\cdot)$ restricted to the interval $[0, \tau']$ is still an integral solution of type $\omega_{\hat{\beta}(\tau;\varphi(\mathbf{x}_0))}$, and $\mathbf{x}_{\tau'}(\cdot)$ is an integral solution of type $\omega_{\hat{\beta}(\tau';\varphi(\mathbf{x}_0))} \le \omega_{\hat{\beta}(\tau;\varphi(\mathbf{x}_0))}$, uniqueness implies that

$$\mathbf{x}_\tau(t) = \mathbf{x}_{\tau'}(t) \quad \text{for all } \ t \in [0, \tau'] . \tag{2.12}$$

Thereby we can continue taking arbitrarily large $\tau > 0$ and define a solution $\mathbf{x}(\cdot)$ on the whole interval $[0, \infty)$. Also, on $[0, \tau']$, the solution $\mathbf{x}_\tau(\cdot)$ satisfies

$$\|\mathbf{x}_\tau(t)\| \le \mathrm{e}^{tm(k')}\|\mathbf{x}_0\| , \tag{2.13}$$

where $k' = \hat{\beta}(\tau'; \varphi(\mathbf{x}_0))$. These facts allow us to show the following theorem.

7 THEOREM (First product formula). *Given any* $\mathbf{x}_0 \in D$ *there exists a unique* $\mathbf{x}(t) : [0,\infty) \to D$, *defined by*

$$\mathbf{x}(t) = (D,\varphi)\text{-}\lim_{h\downarrow 0}\{(I-hA)^{-1}(I+hB)\}^{[t/h]}\mathbf{x}_0\,, \quad t \geq 0\,, \tag{2.14}$$

where, for any $\tau > 0$, *convergence is uniform and* $\mathbf{x}(\cdot)$ *is an integral solution of type* $(m(\hat{\beta}(\tau;\varphi(\mathbf{x}_0))),\psi^*)$ *to the problem* (EP) *on* $[0,\tau]$. *Moreover, for* $0 \leq t$, $\mathbf{x}(\cdot)$ *satisfies*

$$\varphi(\mathbf{x}(t)) \leq \hat{\beta}(t) \equiv \hat{\beta}(t;\varphi(\mathbf{x}_0)) \tag{2.15}$$

and

$$\|\mathbf{x}(t)\| \leq \exp\left\{t\omega_0 + \int_0^t m(\hat{\beta}(r))dr\right\}\left(\|\mathbf{x}_0\|\left(1+\int_0^t m(\hat{\beta}(r))dr\right) + t\|B\mathbf{x}_0\|\right). \tag{2.16}$$

PROOF. The only part of the statement whose proof is not complete or immediate from the above, is the last estimate. However this follows readily since we know that $\|\mathbf{x}_{i+1}^h\| \leq (1-h\omega_0)^{-1}\|(I+hB)\mathbf{x}_i^h\|$ and $\varphi(\mathbf{x}_i^h) \leq \hat{\beta}(hi)$. Hence,

$$\|(I+hB)\mathbf{x}_i^h\| \leq (1+hm(\hat{\beta}(hi)))\|\mathbf{x}_i^h\| + h(m(\hat{\beta}(hi))\|\mathbf{x}_0\| + \|B\mathbf{x}_0\|)\,.$$

It then follows that

$$\|\mathbf{x}_i^h\| \leq \exp\left\{ih\omega_h + h\sum_{j=0}^{i-1} m(\hat{\beta}(hj))\right\}\left(\|\mathbf{x}_0\| + ih\|B\mathbf{x}_0\| + h\sum_{j=1}^{i} m(\hat{\beta}(hj))\|\mathbf{x}_0\|\right)$$

(ω_h as is Lemma 5). Letting h tend to 0 and ih to t, we obtain the desired result. □

It is not difficult to imagine a situation in which the operator A, although being quasi-dissipative in the appropriate sense, does not have a resolvent $(I-hA)^{-1}$ on all of X, as was assumed for the scheme discussed above. Examples include differential operators with coupled terms, and with non-smooth coefficients (as discussed in the following sections). It turns out that these operators can often be dealt with effectively through the use of approximate operators A_h, satisfying some convergence condition.

From now on, we shall assume that A is quasi-dissipative of type ω_0, with domain $D(A)$ dense in X. As before, assume that we have some class D of initial data, on which we define the functional φ. Let B satisfy conditions (C1) and (C2). Assume that we may write the operator A in the form $A = \Lambda + \Xi$, where Λ is m-quasi-dissipative, $D(\Xi) \supset D(\Lambda)$, and where we may approximate Ξ by a sequence $\{\Xi_h\}_{0<h<h^0}$ of operators such that $D(\Xi_h) \supset D(\Lambda)$.

(i) $\lim_{h\downarrow 0}\|\Xi_h\mathbf{x} - \Xi\mathbf{x}\| = 0$ for all $\mathbf{x} \in D(\Lambda)$;

(ii) the operators $A_h \equiv \Lambda + \Xi_h$ are uniformly quasidissipative of type ω_0.

(iii) the resolvents $(I-hA_h)^{-1}$ are defined over the whole of D and preserve φ-boundedness, as was previously assumed for $(I-hA)^{-1}$.

As before, let $\tau > 0$ and $\mathbf{x}_0 \in D$ be given. Under the conditions above we may generate, for each h with $0 < h < h^0$, a sequence $\{\hat{\mathbf{x}}_i^h\}_{ih\leq\tau}$ defined

$$\begin{cases} \hat{\mathbf{x}}_0^h = \mathbf{x}_0 \\ \hat{\mathbf{x}}_i^h = (I - hA_h)^{-1}(I + hB)\hat{\mathbf{x}}_{i-1}^h \quad \text{for } i \geq 1, ih \leq \tau\,. \end{cases} \tag{2.17}$$

Note that since D is invariant under $(I - hA_h)^{-1}$ and we assume condition (C1), $(I + hB)$ maps D into itself, and all elements $\hat{\mathbf{x}}_i^h$ are well-defined.

8 THEOREM (Second product formula). *Under the above assumptions, and the added condition that*

$$\lim_{h\downarrow 0} \|\Xi_h \hat{\mathbf{x}}_i^h - \Xi \hat{\mathbf{x}}_i^h\| = 0 \tag{2.18}$$

uniformly over i, *with* $ih \leq \tau$, *there exists a unique integral solution* $\mathbf{x}(\cdot) : [0, \tau) \to D$ *to* (EP). *Furthermore, the solution is given by*

$$\mathbf{x}(t) = (D, \varphi)\text{-}\lim_{h\downarrow 0}\{(I - hA_h)^{-1}(I + hB)\}^{[t/h]}\mathbf{x}_0\,, \tag{2.19}$$

9 REMARK. As an example of a case in which this convergence condition is satisfied, let Λ be an elliptic operator generating an analytic semigroup, and let Ξ be a differential operator of lower order. If $\|A_h \hat{\mathbf{x}}_i^h\|$ is bounded, then the terms $(-\Lambda)^\gamma \hat{\mathbf{x}}_i^h$ are also bounded, where $(-\Lambda)^\gamma$ denotes a fractional power of the generator $-\Lambda$, and this fact can be used to show convergence of $\Xi_h \hat{\mathbf{x}}_i^h$ to $\Xi \hat{\mathbf{x}}_i^h$ in the appropriate way. The reader is refered to [7] for a concrete example in which this condition also holds for operators coupled strongly in differential terms.

PROOF. Firstly, note that we obtain the φ-bound of Lemma 4 for the set $\widehat{\mathscr{S}}_{\tau,\mathbf{x}_0} = \{\hat{\mathbf{x}}_i^h : 0 < h < h_0,\ ih \leq \tau\}$. Re-examining the proof of lemma 5, it is seen that we may replace A by A_h in (2.7) to obtain

$$\|\hat{\mathbf{x}}_1^h - \mathbf{x}_0\| \leq (1 + (1 - h\omega_0)^{-1})\|\mathbf{x}_0 - \hat{\mathbf{x}}\| + h(1 - h\omega_0)^{-1}\|B\mathbf{x}_0 + A_h\hat{\mathbf{x}}\| \tag{2.20}$$

for $\hat{\mathbf{x}} \in D(A_h)$. In the next step we may proceed similarly, to obtain

$$\|\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h\| \leq (1 - h\omega_0)^{-(i-1)}(1 + hm(k))^{i-1}\|\hat{\mathbf{x}}_1^h - \mathbf{x}_0\| \tag{2.21}$$

from (2.8), and finally we evaluate the error terms $\hat{\mathbf{f}}_i^h$, by proceeding as follows:

$$\begin{aligned} \|\hat{\mathbf{f}}_i^h\| &= h^{-1}\|(I - hA)\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h - hB\hat{\mathbf{x}}_i^h\| \\ &\leq h^{-1}\|\{(I - hA_h) - (I - hA)\}\hat{\mathbf{x}}_i^h\| \\ &\quad + h^{-1}\|(I - hA_h)\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h - hB\hat{\mathbf{x}}_i^h\| \\ &= \|A_h\hat{\mathbf{x}}_i^h - A\hat{\mathbf{x}}_i^h\| + \|B\hat{\mathbf{x}}_i^h - B\hat{\mathbf{x}}_{i-1}^h\| \\ &= \|\Xi_h\hat{\mathbf{x}}_i^h - \Xi\hat{\mathbf{x}}_i^h\| + \|B\hat{\mathbf{x}}_i^h - B\hat{\mathbf{x}}_{i-1}^h\| \end{aligned} \tag{2.22}$$

by (2.17). Combining these estimates as before, it is seen that

$$\|\mathbf{f}_i^h\| \le \|\Xi_h \hat{\mathbf{x}}_i^h - \Xi \hat{\mathbf{x}}_i^h\| + m(k)\, e^{\tau(m(k)+\omega_h)}\{2\|\hat{\mathbf{x}} - \mathbf{x}_0\| + h(\|B\mathbf{x}_0\| + \|A\hat{\mathbf{x}}\|)\}\,. \tag{2.23}$$

By an argument similar to that given after the proof of Lemma 5 we may show the consistency condition for our new approximate scheme. The dissipativity of A can now be used in the same way as before, and we have hence shown the existence of a solution of the form described in Theorem 7. The estimate (2.16) for the growth of the solution $\mathbf{x}(\cdot)$ also holds. We note here that the uniform quasidissipativity of A_h is equivalent to the case in which the constants of quasidissipativity have bounded lim sup, as h converges to zero. □

3. Semilinear Parabolic Systems

We shall treat a general class of systems of semilinear equations containing reaction, diffusion and advection terms, by applying the results above concerning the product formula approach. Initially, we require some preliminary definitions.

Let Ω be a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary, and fix a positive integer m. Define the operators Λ_j and Ξ_j by

$$\Lambda_j = d_j \Delta \quad \text{and} \quad \Xi_j = \mathbf{b}_j \cdot \nabla\,, \quad j = 1, \cdots, m \tag{3.1}$$

for positive constants d_j, and variable coefficients $\mathbf{b}_j \in L^\infty(\Omega)^n$. We assume that the Laplace operator Δ is defined under homogeneous Neumann boundary conditions. Let functions Ψ_j describe coupled *reaction terms*, so that we have a system of equations over the domain Ω, given by

$$\text{(RDS)} \qquad \begin{cases} (u_1)_t = \Lambda_1 u_1 + \Xi_1 u_1 + \Psi_1(u_1, \cdots, u_m)\,, & u_1(0,\cdot) = u_1^0(\cdot) \\ \quad \vdots \qquad \vdots & \qquad \vdots \\ (u_m)_t = \Lambda_m u_m + \Xi_m u_m + \Psi_m(u_1, \cdots, u_m)\,, & u_m(0,\cdot) = u_m^0(\cdot)\,, \end{cases}$$

under 0-Neumann boundary conditions. We assume that $u_j \equiv u_j(\cdot, x)$, $j = 1, \cdots, m$, are all elements of $L^p \subset L^1$, for some $p > n$, and define our Banach space X to be $L^p(\Omega)^m$, with norm $\|\cdot\|_X$ given by

$$\|(z_1, \cdots, z_m)\|_X = (\|z_1\|_p^p + \cdots + \|z_m\|_p^p)^{\frac{1}{p}}\,.$$

Let $A = \Lambda + \Xi$, where

$$\Lambda \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} \Lambda_1 z_1 \\ \vdots \\ \Lambda_m z_m \end{bmatrix} \quad \text{and} \quad \Xi \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} \Xi_1 z_1 \\ \vdots \\ \Xi_m z_m \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in D(A)\,. \tag{3.2}$$

Here, the domain $D(A)$ is defined

$$D(A) = \left\{ z \in L^p \;\middle|\; z \in W^{2,p}(\Omega),\ \frac{\partial}{\partial \nu} z \bigg|_{\partial\Omega} = 0 \right\}^m, \tag{3.3}$$

where ν represents the outward normal vector on the boundary $\partial\Omega$.

As in the previous section, we define the set of permissible initial data D by

$$D = \{(z_1, \cdots, z_m) \in X \mid \|z_j\|_\infty < \infty \; j = 1, \cdots, m\}\,. \tag{3.4}$$

Let the functional $\varphi : D \to \mathbf{R}^+$ be given by

$$\varphi((z_1, \cdots, z_m)) = \max\{\|z_1\|_\infty, \cdots, \|z_m\|_\infty\}\,. \tag{3.5}$$

We assume that the functions Ψ_j are Lipschitz continuous on φ-bounded sets, in such a way that the operator B, defined

$$B \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} \Psi_1(z_1, \cdots, z_m) \\ \vdots \\ \Psi_m(z_1, \cdots, z_m) \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in D\,, \tag{3.6}$$

satisfies conditions (C1) and (C2).

To apply Theorem 7 directly, we would need to construct the resolvents of the operators $\Lambda_j + \Xi_j$, which, given the fact that the $\mathbf{b}_j$s are possibly discontinuous, is not straightforward. We therefore attempt to apply Theorem 8, by constructing resolvents to approximate operators, and showing convergence in the appropriate sense, as well as the remaining conditions as set out in the previous section.

10 DEFINITION. The operators A_h, used to approximate A, shall be defined as follows: Firstly, for $j = 1, \cdots, m$, if $\mathbf{b}_j = (b_{j,1}, \cdots, b_{j,n}) \in (L^\infty)^n$, let $\{b_{j,k}^{(h)}\}_{h>0}$ in $C^\infty(\bar{\Omega})$ converge to $b_{j,k}$ in L^p as $h \downarrow 0$, with $\|b_{j,k}^{(h)}\|_\infty \le \|b_{j,k}\|_\infty$ for $k = 1, \cdots, n$. Then

$$\mathbf{b}_j^h \equiv (b_{j,1}^{(h)}, \cdots, b_{j,n}^{(h)}) \to \mathbf{b}_j \quad \text{as} \quad h \downarrow 0\,, \tag{3.7}$$

in $(L^1)^n$. Define

$$\Xi_j^h = \mathbf{b}_j^h \cdot \nabla \quad \text{for} \quad j = 1, \cdots, m \tag{3.8}$$

and $A_h = \Lambda + \Xi_h$, where

$$\Xi_h \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} \Xi_1^h z_1 \\ \vdots \\ \Xi_m^h z_m \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in D(A)\,. \tag{3.9}$$

Note that the resolvents $(I - hA_h)^{-1}$, of the operators A_h, are defined by

$$(I - hA_h)^{-1} \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} (I - h(\Lambda_1 + \Xi_1^h))^{-1} z_1 \\ \vdots \\ (I - h(\Lambda_m + \Xi_m^h))^{-1} z_m \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in X\,. \tag{3.10}$$

The following well-known elliptic estimates shall be useful at a number of points throughout this section.

11 LEMMA (Elliptic Estimates). *Let* $z \in W^{2,p}$ *and let* $\mathscr{A} = d\Delta + \mathbf{b}\cdot\nabla$ *for positive constant* d *and* $\mathbf{b} = [b_j]_{j=1}^n \in C^1(\Omega;\mathbf{R}^n)$. *Then there exists some constant* C *such that*

$$\|\nabla z\|_\infty \le C(\|\mathscr{A}z\|_p + \|z\|_p)\,. \tag{3.11}$$

Here $C \equiv C(n,\Omega,d,K)$, *and* $\|b_j\|_\infty < K$.

PROOF. This follows simply from well known estimates. Since

$$\|z\|_{W^{2,p}} \le C'(\|\mathscr{A}z\|_p + \|z\|_p)\,,$$

we know that $\nabla z \in W^{1,p}$, and given the embedding $W^{1,p} \hookrightarrow C^{1-n/p}(\bar{\Omega})$, the statement is obtained through the estimates

$$\|\nabla z\|_\infty + [\nabla z]_{1-n/p} \le C''\|\nabla z\|_{W^{1,p}} \le C'''\|z\|_{W^{2,p}}\,.$$

Details may be found in [6]. □

12 PROPOSITION. *Let* $1 < p < \infty$, *and* $\mathbf{F} : L^p \to (L^p)^*$ *be the duality mapping from* L^p *into its dual. Let* $\mathscr{A} = d\Delta + \mathbf{b}\cdot\nabla$ *for positive constant* d *and* $\mathbf{b} \in C^1(\Omega;\mathbf{R}^n)$. *Then, the following holds for* $z \in D(\mathscr{A})$:

$$\langle \mathscr{A}z, \mathbf{F}(z)\rangle \le \left(\frac{\|\mathbf{b}\|_\infty^2}{4d(p-1)}\right)\|z\|_p^2\,. \tag{3.12}$$

PROOF. Let $\mathscr{A}$ be written $d\Delta + \mathbf{b}\cdot\nabla$. We begin by recalling that $(L^p)^* \equiv L^q$, for p and q such that $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, and that $\mathbf{F}(z)(x) = \|z\|_p^{2-p}|z(x)|^{p-2}z(x)$ for a.e. $x \in \Omega$ where $z \ne 0$ in L^p. Let $z \in D(\mathscr{A})$, and $z \ne 0$. Then $z \in C^1(\bar{\Omega})$, and

$$\langle \mathscr{A}z, \mathbf{F}(z)\rangle = \int_\Omega \mathscr{A}z \cdot \mathbf{F}(z)dx\,. \tag{3.13}$$

Write $\langle \mathscr{A}z, \mathbf{F}(z)\rangle = \langle d\Delta z, \mathbf{F}(z)\rangle + \langle \mathbf{b}\cdot\nabla z, \mathbf{F}(z)\rangle$. The contraction semigroup $\mathrm{e}^{t\Delta}$ is analytic and positivity preserving on L^p for general p with $1 \le p < \infty$, so that for $\eta > 0$, $\mathrm{e}^{\eta\Delta} z \in D(\Delta) \cap C^\infty$, and in particular $\mathrm{e}^{\eta\Delta} z$ satisfies the 0-Neumann boundary conditions for any $z \in L^p$. Thus $\mathbf{F}(z)\nabla\,\mathrm{e}^{\eta\Delta} z \in C^1$, and hence we may apply the divergence theorem to obtain

$$\begin{aligned}\int_\Omega \Delta\,\mathrm{e}^{\eta\Delta} z \cdot \mathbf{F}(z)dx &= \int_\Omega \operatorname{div}\left(\mathbf{F}(z)\nabla\,\mathrm{e}^{\eta\Delta} z\right)dx - \int_\Omega \nabla\,\mathrm{e}^{\eta\Delta} z\cdot\nabla\mathbf{F}(z)dx \\ &= \int_{\partial\Omega} \mathbf{F}(z)\left(\nabla\,\mathrm{e}^{\eta\Delta} z\cdot\nu\right)dS - \int_\Omega \nabla\,\mathrm{e}^{\eta\Delta} z\cdot\nabla\mathbf{F}(z)dx \\ &= -\int_\Omega \nabla\,\mathrm{e}^{\eta\Delta} z\cdot\nabla\mathbf{F}(z)dx\,,\end{aligned}$$

where $\int_{\partial\Omega} dS$ represents the integral over the surface of Ω, and as usual ν is the outward normal to $\partial\Omega$. Let η tend to 0. Then $\Delta\,\mathrm{e}^{\eta\Delta} z$ tends to Δz, and by the well-known elliptic estimate, $\|\nabla\hat{z}\|_\infty \le C(\|\Delta\hat{z}\|_p + \|\hat{z}\|_p)$, $\hat{z} \in D(\mathscr{A})$, we see that $\nabla\, e^{\eta\Delta} z$ tends to ∇z in L^p. Hence, we have

$$\langle \Delta z, \mathbf{F}(z)\rangle = -\int_\Omega \nabla z \cdot \nabla \mathbf{F}(z) dx\,.$$

Noting that $\nabla\mathbf{F}(z) = (p-1)\|z\|_p^{2-p}|z|^{p-2}\nabla z$ we obtain

$$\begin{aligned}\langle \mathscr{A}z, \mathbf{F}(z)\rangle &= -d(p-1)\|z\|_p^{2-p}\int_\Omega |z(x)|^{p-2}|\nabla z(x)|^2 dx\\ &\quad + \|z\|_p^{2-p}\int_\Omega z(x)|z(x)|^{p-2}\mathbf{b}\cdot\nabla z(x) dx\\ &\le -d(p-1)\|z\|_p^{2-p}\int_\Omega |z(x)|^{p-2}|\nabla z(x)|^2 dx\\ &\quad + \|z\|_p^{2-p}\int_\Omega |z(x)|^{p-2}|\mathbf{b}|\cdot|\nabla z(x)|\cdot|z(x)| dx\,,\end{aligned}$$

and since

$$\begin{aligned}-|\nabla z(x)|^2 + (d(p-1))^{-1}|\mathbf{b}|\cdot|\nabla z(x)|\cdot|z(x)| &= -\{|\nabla z(x)| - (2d(p-1))^{-1}|\mathbf{b}|\cdot|z(x)|\}^2\\ &\quad + (2d(p-1))^{-2}|\mathbf{b}|^2\cdot|z(x)|^2\end{aligned}$$

it is finally seen that

$$\begin{aligned}\langle \mathscr{A}z, \mathbf{F}(z)\rangle &\le -d(p-1)\|z\|_p^{2-p}\int_\Omega |z(x)|^{p-2}\left\{|\nabla z(x)| - \frac{|\mathbf{b}|}{2d(p-1)}|z(x)|\right\}^2 dx\\ &\quad + (4d(p-1))^{-1}\|\mathbf{b}\|_\infty^2\|z\|_p^{2-p}\int_\Omega |z(x)|^p dx\\ &\le \|\mathbf{b}\|_\infty^2(4d(p-1))^{-1}\|z\|_p^2\,.\end{aligned}$$

□

13 REMARK. Since $\|\mathbf{b}_j^h\|_\infty \le \|\mathbf{b}_j\|_\infty$, there exists some uniform constant of quasidissipativity, ω_0, for the operators A_h and A. Furthermore, the operators $(I - hA_h)^{-1}$ are non-expansive with respect to φ. This is shown through the use of the maximum principle for elliptic equations (for details see [7] and [5]).

14 DEFINITION. Let $\mathbf{x}_0 = (u_1^0, \cdots, u_m^0) \in D$ and let $\tau > 0$ be a constant. We now define the scheme as set out in (2.17), namely

$$\begin{cases}\hat{\mathbf{x}}_0^h = \mathbf{x}_0\\ \hat{\mathbf{x}}_i^h = (I - hA_h)^{-1}\,(I + hB)\,\hat{\mathbf{x}}_{i-1}^h \quad \text{for } i \ge 1\,,\ ih \le \tau\,.\end{cases}$$

Denote the uniform φ-bound of the scheme by k, as previously. The convergence of $\Xi_h \mathbf{x}$ to $\Xi \mathbf{x}$, for $\mathbf{x} \in D(\Lambda)$ is clear. We set about showing the appropriate uniform convergence, starting with the following lemma.

15 LEMMA. *Let* $\mathbf{x}_0 \in D \cap D(A)$ *and let elements* $\hat{\mathbf{x}}_i^h$ *be generated as above for* $0 < h < h_0$ *and* $ih \le \tau$. *Then the set* $\{A_h \hat{\mathbf{x}}_i^h\}$ *is uniformly* $\|\cdot\|$*-bounded for sufficiently small* h.

PROOF. Performing the argument of equation (2.8) we obtain

$$\|\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h\| \le (1 - h\omega_0)^{-(i-1)}(1 + hm(k))^{i-1}\|\hat{\mathbf{x}}_1^h - \mathbf{x}_0\| .$$

Next, we see that

$$\|\hat{\mathbf{x}}_1^h - \mathbf{x}_0\| \le (1 - h\omega_0)^{-1}\|(I + hB)\mathbf{x}_0 - \mathbf{x}_0 + hA_h\mathbf{x}_0\| ,$$

and thus

$$\|\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h\| \le h\, \mathrm{e}^{\tau(\omega_h + m(k))}\, \|B\mathbf{x}_0 + A_h\mathbf{x}_0\| . \tag{3.14}$$

As before we define $\omega_h = \omega_0(1 - h\omega_0)^{-1}$. It is now clear that

$$\begin{aligned} \|h^{-1}(\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h) - (A_h + B)\hat{\mathbf{x}}_i^h\| &= h^{-1}\|(I - hA_h)\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h - B\hat{\mathbf{x}}_i^h\| \\ &\le h^{-1}\|(I + hB)\hat{\mathbf{x}}_{i-1}^h - \hat{\mathbf{x}}_{i-1}^h - B\hat{\mathbf{x}}_i^h\| \\ &\le hm(k)\, \mathrm{e}^{\tau(\omega_h + m(k))}\, \|B\mathbf{x}_0 + A_h\mathbf{x}_0\| . \end{aligned}$$

Thus, it is immediately seen that $h^{-1}\|\hat{\mathbf{x}}_i^h - \hat{\mathbf{x}}_{i-1}^h\|$ is uniformly bounded, and it is clear by the Lipschitz continuity of B on D_k that $\|B\hat{\mathbf{x}}_i^h\|$ is. Hence it follows that $A_h\hat{\mathbf{x}}_i^h$ must also be uniformly bounded. □

16 THEOREM. *Given* $\mathbf{x}_0 \in D \cap D(A)$*, there exists a unique solution* $\mathbf{x}(\cdot) : \mathbf{R}^+ \to D$ *of the type described in Theorem* 7 *to the parabolic system* (RDS)*, given by the product formula*

$$\mathbf{x}(t) = (D, \varphi)\text{-}\lim_{h\downarrow 0}\{(I - hA_h)^{-1}(I + hB)\}^{[\frac{t}{h}]}\mathbf{x}_0 , \quad t \ge 0 . \tag{3.15}$$

PROOF. It remains only to show the uniform convergence as described in the remark after Theorem 7. We let the elements $\hat{\mathbf{x}}_i^h$ be written $(u_{1,i}^h, \cdots, u_{m,i}^h)$ and proceed as follows. We have

$$\|\Xi_h\hat{\mathbf{x}}_i^h - \Xi\hat{\mathbf{x}}_i^h\| = (\|(\mathbf{b}_1^h - \mathbf{b}_1)\cdot\nabla u_{1,i}^h\|_p^p + \cdots + \|(\mathbf{b}_m^h - \mathbf{b}_m)\cdot\nabla u_{m,i}^h\|_p^p)^{\frac{1}{p}} .$$

The elliptic estimates then give us

$$\|\nabla u_{j,i}^h\|_\infty \le C(\|(\Lambda_j + \Xi_j^h)u_{j,i}^h\|_p + \|u_{j,i}^h\|_p) \le C(\|A_h\hat{\mathbf{x}}_i^h\| + \|\hat{\mathbf{x}}_i^h\|) ,$$

where the constant C may be chosen uniformly over h, since $\|\mathbf{b}_j^h\|_\infty$ is uniformly bounded. Thus, since $\mathbf{b}_j^h$ converges to $\mathbf{b}_j$ in L^p, uniform convergence is seen to hold. □

Since convergence in L^p implies convergence in L^1, (3.15) gives solutions to (RDS) in this larger space.

For equations of the type discussed here, we can extract further information about the generated solutions. In preparation for this, the following lemma is necessary.

17 LEMMA. *Let the sequence* $\{\mathbf{x}_h\}_{h>0} \subset X$ *converge to some* $\mathbf{x} \in X$ *as* $h \downarrow 0$. *Then,*

$$(I - hA_h)^{-1}\mathbf{x}_h \to \mathbf{x} \quad as \quad h \downarrow 0\,. \tag{3.16}$$

PROOF. Firstly,

$$\begin{aligned}\|(I - hA_h)^{-1}\mathbf{x}_h - \mathbf{x}\| &\leq \|(I - hA_h)^{-1}\mathbf{x}_h - (I - hA_h)^{-1}\mathbf{x}\| \\ &\quad + \|(I - hA_h)^{-1}\mathbf{x} - \mathbf{x}\| \\ &\leq (1 - h\omega_0)^{-1}\|\mathbf{x}_h - \mathbf{x}\| + \|(I - hA_h)^{-1}\mathbf{x} - \mathbf{x}\|\,.\end{aligned}$$

Given some $\hat{\mathbf{x}} \in D(A)$ with $\|\hat{\mathbf{x}} - \mathbf{x}\| \leq \varepsilon$,

$$\begin{aligned}\|(I - hA_h)^{-1}\mathbf{x} - \mathbf{x}\| &\leq \|(I - hA_h)^{-1}\mathbf{x} - (I - hA_h)^{-1}\hat{\mathbf{x}}\| \\ &\quad + \|(I - hA_h)^{-1}\hat{\mathbf{x}} - \hat{\mathbf{x}}\| + \|\hat{\mathbf{x}} - \mathbf{x}\| \\ &\leq (1 + (1 - h\omega_0)^{-1})\varepsilon + h\|(I - hA_h)^{-1}A_h\hat{\mathbf{x}}\|\,.\end{aligned}$$

In particular, letting h tend to 0 we see that in the limit the difference above is bounded by 2ε. Since ε may be chosen arbitrarily we obtain the desired result. □

Note that the equivalent result for weak convergence also holds.

18 PROPOSITION. *The solution* $\mathbf{x}(\cdot)$*, constructed in Theorem* 16, *satisfies*

$$\mathbf{x}(t) = \mathbf{x}_0 + (\Lambda + \Xi)\int_0^t \mathbf{x}(s)ds + \int_0^t B\mathbf{x}(s)ds \tag{3.17}$$

for all $t \geq 0$.

PROOF. Fix some $t > 0$ and let $\hat{\mathbf{x}}_i^h$ be generated as in (2.17). Initially, note that

$$\begin{aligned}\hat{\mathbf{x}}_i^h - \mathbf{x}_0 &= \sum_{k=1}^{i} \hat{\mathbf{x}}_k^h - \hat{\mathbf{x}}_{k-1}^h \\ &= \sum_{k=1}^{i} (I - hA_h)^{-1}(I + hB)\hat{\mathbf{x}}_{k-1}^h - \hat{\mathbf{x}}_{k-1}^h \\ &= h\sum_{k=1}^{i} A_h(I - hA_h)^{-1}\hat{\mathbf{x}}_{k-1}^h + h\sum_{k=1}^{i}(I - hA_h)^{-1}B\hat{\mathbf{x}}_{k-1}^h \\ &= (I - hA_h)^{-1}\Lambda\int_0^{ih} \hat{\mathbf{x}}_h(s)ds + (I - hA_h)^{-1}\int_0^{ih} \Xi_h\hat{\mathbf{x}}_h(s)ds\end{aligned}$$

$$+ (I - hA_h)^{-1} \int_0^{ih} B\hat{\mathbf{x}}_h(s)ds\,,$$

where the functions $\mathbf{x}_h(\cdot)$ are defined $\mathbf{x}_h(s) = \hat{\mathbf{x}}_i^h$, $i = \left[\frac{s}{h}\right]$, for $s \geq 0$. On the interval $[0, t]$, $\mathbf{x}_h(\cdot)$ and $B\mathbf{x}_h(\cdot)$ converge uniformly to $\mathbf{x}(\cdot)$ and $B\mathbf{x}(\cdot)$, and so letting h converge to 0 and ih to t we see that $\int_0^{ih} \mathbf{x}_h(s)ds$ converges to $\int_0^t \mathbf{x}(s)ds$, and by Lemma 17, the final term converges to $\int_0^t B\mathbf{x}(s)ds$. We already know that $\|\nabla\mathbf{x}_i^h\|_\infty$ is uniformly bounded, and thus the (vector valued) integral $\int_0^{ih} \nabla\mathbf{x}_h(s)ds$ is also uniformly bounded (we permit ourselves use of this notation to represent the vector in the appropriate product Banach space, with components formed by the appropriate integral). Since X is a reflexive Banach space, there exists some null sequence $\{h_j\}_{j=1}^\infty$ and some $\mathbf{w}(t) \in X^n$ such that

$$ih_j \to t \quad \text{and} \quad \int_0^{ih_j} \nabla\mathbf{x}_{h_j}(s)ds \rightharpoonup \mathbf{w}(t)\,.$$

The weak convergence of $(I - hA_{h_j})^{-1}\Lambda \int_0^{ih_j} \mathbf{x}_{h_j}(s)ds$ then also follows (since all other terms converge), and hence the boundedness of $\Lambda \int_0^{ih_j} \mathbf{x}_{h_j}(s)ds$ and the demi-closedness of the operator Δ imply that

$$\mathbf{x}(t) - \mathbf{x}_0 = \Lambda \int_0^t \mathbf{x}(s)ds + \mathbf{b}\cdot\mathbf{w}(t) + \int_0^t B\mathbf{x}(s)ds\,. \tag{3.18}$$

Now, let $\phi \in C_0^\infty(\bar{\Omega})$, and denote $\mathbf{w}(t)$ by $(w_1, \cdots, w_n)$. Then,

$$\begin{aligned}
\int_\Omega \phi(x)w_k(x)dx &= \lim_{j\to\infty} \int_\Omega \phi(x)\left[\int_0^{ih_j} \frac{\partial}{\partial x_k}\mathbf{x}_{h_j}(s)ds\right](x)dx \\
&= -\lim_{j\to\infty} \int_\Omega \phi_{x_k}(x)\left[\int_0^{ih_j} \mathbf{x}_{h_j}(s)ds\right](x)dx \\
&= -\int_\Omega \phi_{x_k}\left[\int_0^t \mathbf{x}(s)ds\right](x)dx\,.
\end{aligned}$$

Here, the first and last equalities hold by the weak convergence of the appropriate terms. Hence $\mathbf{w}(t) = \nabla\{\int_0^t \mathbf{x}(s)ds\}$, in the sense of distributions, and the statement holds. □

19 REMARK. As mentioned in the proof of Lemma 15, the estimate in (3.14) implies that for the solution $\mathbf{x}(\cdot)$ above, we have

$$\|\mathbf{x}(t) - \mathbf{x}(s)\| \leq e^{(\omega_0+m(k))(t+s)}\,\|(A+B)\,\mathbf{x}_0\|\cdot|t-s| \quad \text{for } t, s \in [0,\tau]\,. \tag{3.19}$$

This also follows by examining the estimate (2.11) in the limit as $h \downarrow 0$ and $ih \to t$, $jh \to s$. Recall that $m(k)$ depends on the φ-bound k of the solution in the interval $[0, \tau]$, which in turn depends on τ. However, we can say that the solution is Lipschitz continuous on any bounded interval, with Lipschitz constant

$$\text{Lip}\left(\mathbf{x}(\cdot)\big|_{[0,\tau]}\right) \leq \mathrm{e}^{2(\omega_0+m(k))\tau} \,\|(A+B)\,\mathbf{x}_0\|\,. \tag{3.20}$$

It follows then that $\mathbf{x}(\cdot)$ is strongly differentiable at almost every point on the real line. Let t be one such point. Calculating the derivative at t, we have

$$\begin{aligned}\mathbf{x}'(t) = \lim_{\delta\downarrow 0}\frac{\mathbf{x}(t+\delta)-\mathbf{x}(t)}{\delta} &= \lim_{\delta\downarrow 0}(\Lambda+\Xi)\frac{1}{\delta}\int_t^{t+\delta}\mathbf{x}(s)ds + \lim_{\delta\downarrow 0}\frac{1}{\delta}\int_t^{t+\delta}B\mathbf{x}(s)ds\\ &= \lim_{\delta\downarrow 0}(\Lambda+\Xi)\frac{1}{\delta}\int_t^{t+\delta}\mathbf{x}(s)ds + B\mathbf{x}(t)\,.\end{aligned}$$

It is clear that $\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_t^{t+\delta}\mathbf{x}(s)ds = \mathbf{x}(t)$, and also that $\|\nabla\frac{1}{\delta}\int_t^{t+\delta}\mathbf{x}(s)ds\|$ is bounded for δ sufficiently small, using an argument similar to that used in the proof of Proposition 18. Thus, as was done previously, we may construct a null-sequence $\{\delta_j\}_{j=1}^{\infty}$ such that

$$\underset{j\to\infty}{\text{w-lim}}\,\nabla\frac{1}{\delta_j}\int_t^{t+\delta_j}\mathbf{x}(s)ds = \mathbf{w}(t)$$

for some $\mathbf{w}(t)\in X^n$. Again we obtain the weak convergence of $\Lambda\frac{1}{\delta_j}\int_t^{t+\delta_j}\mathbf{x}(s)ds$, which by the demi-closedness of Λ gives

$$\mathbf{x}'(t) = \Lambda\mathbf{x}(t) + \mathbf{b}\cdot\mathbf{w}(t) + B\mathbf{x}(t)\,.$$

We proceed in the same way as in Proposition 18 to show that $\mathbf{w}(t)$ is in fact equal to $\nabla\mathbf{x}(t)$ in the sense of distributions, and that $\nabla\mathbf{x}(t)$ makes sense in X^n. Hence we obtain the following theorem.

20 THEOREM. *The solution* $\mathbf{x}(\cdot)$ *to the system* (RDS) *of equations generated by the second product formula for* $\mathbf{x}_0\in D\cap D(A)$ *is a unique strong solution.*

21 REMARK. For $\mathbf{x}_0\in\overline{D_\alpha\cap D(A)} = D_\alpha$ for some $\alpha>0$, we still know that all elements $\mathbf{x}_i^h$ of the scheme belong to some D_k for any bounded time interval, where k naturally depends on this interval. The estimate in (2.11) implies continuous dependence of solutions on initial data. Thus we can construct solutions to (RDS) as the pointwise limit of strong solutions for any $\mathbf{x}_0\in D_\alpha$ by considering a sequence $\{h_0^{(i)}\}$ of elements in $D_\alpha\cap D(A)$ converging to $\mathbf{x}_0$. In fact, we require only that

$$(D,\varphi)\text{-}\lim_{i\to\infty}\mathbf{x}_0^{(i)} = \mathbf{x}_0$$

for convergence to hold. Thus we have generalized solutions to (RDS) for initial data on all of D.

4. Application to HIV model

The system of equations (HIV) below describes the processes of HIV infection in the human body, represented by a bounded, open domain Ω in $\mathbf{R}^3$ with sufficiently smooth boundary $\partial\Omega$. Let $u(t,x)$ represent the spatial distribution of uninfected cells at time t, $v(t,x)$ that of infected cells, and $w(t,x)$ the concentration of the virus itself. We shall assume that $u(t,\cdot)$, $v(t,\cdot)$ and $w(t,\cdot)$ are all elements of $L^p \subset L^1$ $(t \geq 0)$ for some $p > n$, and we therefore define the Banach space X to be $L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega)$, with norm $\|\cdot\|$ given by

$$\|(u,v,w)\|_X = (\|u\|_p^p + \|v\|_p^p + \|w\|_p^p)^{\frac{1}{p}} .$$

Since the values of u, v and w shall represent quantities such as density and concentration, it would seem more natural to use the space L^1. However, in order to obtain estimates necessary for convergence to be shown we begin by using the space L^p and later use the fact that Ω is bounded, which implies the appropriate convergence and boundedness in L^1. The equations themselves take the form

$$\text{(HIV)} \qquad \begin{cases} u_t = d_1\Delta u + \mathbf{b}_1 \cdot \nabla u + S(w) - \alpha u + (p(w) - \gamma)wu\,, & u(0) = u_0 \\ v_t = d_2\Delta v + \mathbf{b}_2 \cdot \nabla v + \gamma wu - \beta v - q(w)wv\,, & v(0) = v_0 \\ w_t = d_3\Delta w + \mathbf{b}_3 \cdot \nabla w + r(w)vw - \delta uw + g(w)\,, & w(0) = w_0 \end{cases}$$

where $(u_0, v_0, w_0) \in D$, the set of permissible initial data detailed later on. The homogeneous Neumann boundary condition is imposed on the unknown functions u, v and w. This system of equations is based on a model given in [4], where a detailed description of the specific reaction terms can also be found. We give a brief outline here and refer the reader to [4] for more specific information.

The constants α, β, γ and δ represent, repectively, mortality rates of u and v, the rate of infection, and the rate of decrease of HIV virus through immune response. The functions p, q and r are given by

$$(4.1) \qquad p(w) = \frac{p^*}{c_p + |w|}\,, \quad q(w) = \frac{q^*}{c_q + |w|}\,, \quad r(w) = \frac{r^*}{c_r + |w|}$$

and correspond to production of uninfected cells and virus cells, and loss of infected cells due to presence of the virus. The constants p^*, q^*, r^*, and c_p, c_q, c_r are the primary and secondary constants of saturation. Finally, $g(\cdot)$ and $S(\cdot)$ correspond to the supply of the virus and uninfected cells, and are assumed here to be uniformly bounded, Lipschitz continuous functions in $C^\infty(\bar{\Omega} \times \mathbf{R}; \mathbf{R})$.

Diffusion of the individual substances and advection effects along the vector fields $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, are described by the terms $d_j\Delta$ and $\mathbf{b}_j \cdot \nabla$. Here we shall assume each $\mathbf{b}_j$ to be a function in the space $(L^\infty(\Omega))^3$, expecting the vector field to describe the paths of veins and other such channels.

22 DEFINITION. We formulate the problem in the manner described in the previous section, defining D, φ, A and A_h as set out there. The operator B is defined

$$B\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} S(w) - \alpha u + (p(w) - \gamma)wu \\ \gamma wu - \beta v - q(w)wv \\ r(w)wv - \delta uw + g(w) \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in D\,. \tag{4.2}$$

It therefore follows that, explicitly, the approximate scheme takes the form:

$$\begin{cases} u_{i+1}^h = (I - h(\Lambda_1 + \Xi_1^h))^{-1}\{u_i^h + h(S(w_i^h) - \alpha u_i^h + (p(w_i^h) - \gamma)w_i^h u_i^h)\} \\ v_{i+1}^h = (I - h(\Lambda_2 + \Xi_2^h))^{-1}\{v_i^h + h(\gamma w_i^h u_i^h - \beta v_i^h - q(w_i^h)w_i^h v_i^h)\} \\ w_{i+1}^h = (I - h(\Lambda_3 + \Xi_3^h))^{-1}\{w_i^h + h(r(w_i^h)w_i^h v_i^h - \delta u_i^h w_i^h + g(w_i^h))\}\,. \end{cases} \tag{4.3}$$

The operator B is locally Lipschitz continuous in the sense described in (C4), and in fact, since its terms are all rather well-behaved, we can see that it should be possible to find a function $\psi^*(\cdot)$ to bound the growth with respect to time of our descrete scheme. However, we shall show the φ-boundedness in a slightly more technical way, and at the same time extract some useful information about the elements $\{\hat{\mathbf{x}}_i^h\}$, related to the physical meaning of the quantities u, v and w.

23 PROPOSITION. *Let $(u_0, v_0, w_0) \in D$ satisfy $u_0(x)$, $v_0(x)$ and $w_0(x) \geq 0$ for almost every $x \in \Omega$. Then there exists some $h^0 > 0$ such that the components u_i^h, v_i^h and w_i^h of $\hat{\mathbf{x}}_i^h$, are all non-negative in the same sense and are uniformly L^∞-bounded for $ih \leq \tau$ and $0 < h < h^0$.*

PROOF. To begin, note that for $w_i^h \geq 0$,

$$p(w_i^h)w_i^h = \frac{p^* w_i^h}{c_p + w_i^h} \leq p^*\,, \tag{4.4}$$

and similarly, $q(w_i^h)w_i^h \leq q^*$ and $r(w_i^h)w_i^h \leq r^*$. Let $g^* = \|g\|_\infty$ and $S^* = \|S\|_\infty$. To simplify notation we define the following constants. The uniform bound C^u on u_i^h shall be given by

$$C^u = \mathrm{e}^{\tau p^*}\left\{\|u_0\| + \frac{S_0}{p^*}\right\}, \tag{4.5}$$

and constants C_1 and C_2, used next in the uniform bound $C^{\mathbf{x}}$ for $\varphi(\mathbf{x}_i^h)$, defined

$$\begin{aligned} C_1 &= \max\{\alpha + p^* + \gamma C^u, \beta + q^* + \gamma C^u, r^* + \delta C^u\} \\ C_2 &= \max\{S^*, g^*\}\,. \end{aligned} \tag{4.6}$$

Define $C^{\mathbf{x}} = \mathrm{e}^{\tau C_1}\{\varphi(\mathbf{x}_0) + \frac{C_2}{C_1}\}$ and finally

$$h^0(\tau) = \min\left\{\frac{1}{\beta + q^*}, \frac{1}{\alpha + \gamma C^{\mathbf{x}}}, \frac{1}{\delta C^u}\right\}. \tag{4.7}$$

The statement shall be shown by proving the following:

CLAIM. For all h such that $0 \leq h \leq h^0$, and all $i = 0, 1, \cdots, N_h$ such that $hN_h \leq \tau$, we have

(i) $\|u_i\|_\infty \leq (1 + hp^*)^i \, \|u_0\|_\infty + h \sum_{j=0}^{i-1} (1 + hp^*)^j \, S^*$

(ii) $\varphi(\mathbf{x}_i^h) \leq (1 + hC_1)^i \, \varphi(\mathbf{x}_0) + h \sum_{j=0}^{i-1} (1 + hC_1)^j \, C_2$

(iii) $u_i^h, v_i^h, w_i^h \geq 0$,

where the sums in (i) and (ii) are naturally taken to be zero for $i = 0$. Note that under (i) and (ii) the uniform bounds $\|u_i\|_\infty \leq C^u$ and $\varphi(\mathbf{x}_i^h) \leq C^{\mathbf{x}}$ follow simply from the definitions.

PROOF OF CLAIM. Choose some appropriate h. We shall use an induction argument to show the result. Firstly, it is easily verified that (i),(ii) and (iii) are satisfied for $i = 0$. We therefore assume that (i) through (iii) hold for some i, and show the result for $i + 1$. For notational convenience, let $\hat{u}_{i+1} = (I - h(\Lambda_1 + \Xi_1^h))u_{i+1}$ and define $\hat{v}_{i+1}$ and $\hat{w}_{i+1}$ similarly. Also, we shall drop the superscript h and write u_i etc for u_i^h where no confusion will be caused.

Consider the terms $\varphi(\mathbf{x}_i)$. The nature of our operators A_h is such that $\varphi(\mathbf{x}_{i+1}) \leq \varphi((I - hA_h)\mathbf{x}_{i+1}) = \varphi((\hat{u}_{i+1}, \hat{v}_{i+1}, \hat{w}_{i+1}))$. By (4.3),

$$\begin{aligned}\|\hat{u}_{i+1}\|_\infty &\leq \|u_i\|_\infty + h\left(S^* + (\alpha + p^*)\|u_i\|_\infty + \gamma C^u \|w_i\|_\infty\right) \\ &\leq \left\{1 + h\left(\alpha + p^* + \gamma C^u\right)\right\} \varphi(\mathbf{x}_i) + hS^* .\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\hat{v}_{i+1}\|_\infty &\leq \left\{1 + h\left(\gamma C^u + \beta + q^*\right)\right\} \varphi(\mathbf{x}_i) \quad \text{and} \\ \|\hat{w}_{i+1}\|_\infty &\leq \left\{1 + h\left(r^* + \delta C^u\right)\right\} \varphi(\mathbf{x}_i) + hg^* .\end{aligned}$$

It is clear that

$$\begin{aligned}\varphi(\mathbf{x}_{i+1}) &\leq (1 + hC_1)\,\varphi(\mathbf{x}_i) + hC_2 \\ &\leq (1 + hC_1)^{i+1}\varphi(\mathbf{x}_0) + h\sum_{j=0}^{i}(1 + hC_1)^j C_2 ,\end{aligned}$$

showing that (ii) holds for $\mathbf{x}_{i+1}$.

By the definition of h^0, the following estimates hold

$$\begin{aligned}&\hat{u}_{i+1} \geq (1 - h(\alpha + \gamma w_i))u_i \geq (1 - h(\alpha + \gamma C^{\mathbf{x}}))u_i \geq 0 \\ &\hat{v}_{i+1} \geq (1 - h(\beta + q^*))v_i \geq 0 \\ &\hat{w}_{i+1} \geq (1 - h\delta u_i)w_i \geq 0 .\end{aligned}$$

By hypothesis, u_i, v_i and w_i are all non-negative, so that by (4.3) and (4.4) we see that

$$\begin{aligned}\hat{u}_{i+1} &\le (1 + hp^*)u_i + hS^* \\ &\le (1 + hp^*)\|u_i\|_\infty + hS^* \\ &\le (1 + hp^*)^{i+1}\|u_0\|_\infty + h\sum_{j=0}^{i}(1 + hp^*)^j S^* .\end{aligned}$$

Statements (i) and (iii) then follow by the L^∞-norm and order preserving properties of the operator $(I - hA_h)^{-1}$ and its components. This completes the proof of the claim, and hence that of the proposition. □

The reader will note the dependence of h^0 on both the time-bound τ and the φ-bound of initial data. In other words we have had to take full advantage of the localization used in the convergence theorem. The following statement can now be made:

24 THEOREM. *Unique generalized solutions* $\mathbf{x}(\cdot) : \mathbf{R}^+ \to D$ *exist to the system* (HIV) *for all initial data* $\mathbf{x}_0 \in D$. *When* $\mathbf{x}_0 \in D \cap D(A)$, *these solutions are strong solutions and are given by the product formula*

$$\mathbf{x}(t) = \lim_{h\downarrow 0}\{(I - hA_h)^{-1}(I + hB)\}^{[t/h]}\mathbf{x}_0 , \quad t \ge 0 . \tag{4.8}$$

Furthermore, for initial data $\mathbf{x}_0$ *consisting of component functions* u_0, v_0 *and* w_0, *all non-negative almost everywhere, the generated solution* $\mathbf{x}(\cdot)$ *retains this property, and is therefore physically reasonable in this sense.*

25 REMARK. Note that when $\mathbf{x}(\cdot)$ is a strong solution $\mathbf{x}(\cdot) \in D(\Lambda)$, and hence by definition the 0-Neumann boundary conditions are satisfied in a strict sense.

5. Concluding remarks

The reader will note that in Section 3, the assumption that initial data $\mathbf{x}_0$ belongs not only to some φ-bounded set, but also to the domain of A, is essential to the proof of convergence of $\Xi_h\hat{\mathbf{x}}_i^h - \Xi\hat{\mathbf{x}}_i^h$ to zero uniformly over i, and thus the convegence of the product formula. We then use the fact that the semigroup generating solutions depends on initial data in a continuous way to show that solutions exist for all $\mathbf{x}_0 \in D$. Note that such an extension is possible for any operator A satisfying $\overline{D(A) \cap D_\alpha} = D_\alpha$ for all $\alpha > 0$, using the local quasidissipativity of $A + B$ as was done in Section 3. This means that the space (D, φ) is complete with respect to the topology introduced in Section 1.

In the inductive argument used in the proof of Lemma 23 we firstly require a bound on terms $\|u_i^h\|_\infty$, namely the first component of the elements of the scheme, in order to find the uniform bound for $\varphi(\mathbf{x}_i^h)$. Expecting the operator B used here to be typical of the type of perturbations encountered in other applications, we note that this proof can be generalized to the case of multiple functionals with bounds dependent on each other. In the case of the HIV model above we could have defined a second functional $\varphi_1 : X \to (R^+)^2$ by

$$\varphi_1[(u, v, w)] = \|u\|_\infty , \quad (u, v, w) \in X$$

and it is easy to think of a situation in which a larger number of relevant inter-related functionals can be defined to obtain similar results concerning boundedness and non-negativity for reaction operators of other convective reaction-diffusion systems.

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