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Minimally Knotted Spatial Graphs are Totally Knotted

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Abstract. Applying Jaco's Handle Addition Lemma, we give a condition for a 3-manifold to have an incompressible boundary. As an application, we show that the boundary of the exterior of a minimally knotted planar graph is incompressible.

1. Introduction

A 3-manifold M is said to be *∂-irreducible* if ∂M is *incompressible* in M, namely, for any disk D properly embedded in M, ∂D bounds a disk in ∂M. Otherwise M is *∂-reducible*. See [4] for the basic terminology in three-dimensional topology which is not stated here. In [2], Haken constructed an algorithm to detect if an irreducible 3-manifold is ∂-irreducible or not. See also Jaco-Oertel [6] for a survey. The algorithm is valid for all irreducible 3-manifolds with handle decompositions, but it is not adapted for an execution by hand. In this paper, we give a sufficient condition for a certain 3-manifold with non-empty connected boundary to be ∂-irreducible, and consider some properties of minimally knotted spatial graphs in S^3. Indeed our sufficient condition is adaptable for not all irreducible 3-manifolds, but is much easier to check than Haken's algorithm.

In §2, we introduce some concept for curves in the boundary of a 3-manifold to state a sufficient condition to be ∂-irreducible as follows. (See §2 for definitions and notation.)

THEOREM 1.1. *Let M be a 3-manifold with non-empty connected boundary ∂M. If there exists a disjoint union of simple closed curves J in ∂M such that (M, J) is almost trivial, then M is irreducible and ∂-irreducible.*

In fact, Theorem 1.1 has various applications to spatial graphs as will be described in §3. A spatial graph means an embedded 1-dimensional graph in S^3. A graph G is said to be *good* if the degree of each vertex of G is greater than one. In this paper, we deal with good planar graphs, and our result obtained here can be generalized for more general good graphs. Let Γ be a spatial graph of a planar graph G embedded in S^3. We say that Γ is *minimally knotted* if any proper subgraph Γ' is contained in a sphere in S^3, and Γ itself is not. A spatial

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graph Γ is said to be *totally knotted* if the exterior $E(\Gamma)$ is irreducible and ∂-irreducible. By using some tangles with the *Brunnian property*, we can show that every planar graph has a spatial embedding which is minimally knotted and totally knotted. Inaba and Soma [3, Theorem 2], Kawauchi [8, Theorem 2.1] and Wu [19] showed that every planar graph has minimally knotted spatial embeddings with some additional conditions. On the other hand, it is easy to construct totally knotted spatial embeddings of every graph which are not minimally knotted by Myers' technique [13] or Kawauchi's [8, Theorem 1.1]. Together with a result of Scharlemann and Thompson [15, Theorem 7.5], the following is obtained by our result. The total knottedness is available under some weaker condition, as will be considered in §3.

THEOREM 1.2. *Minimally knotted connected planar spatial graphs are totally knotted.*

Scharlemann and Thompson [15, Theorem 7.5] showed similar results, and gave an algorithm to detect the triviality of embedded planar graphs, via the extended Haken's algorithm [6], and Wu [20] reproved it and gave a necessary and sufficient condition for a planar graph in general 3-manifold to be minimally knotted in terms of "cycle-triviality".

We say that a 3-manifold with non-empty boundary is *acylindrical* if it is irreducible, ∂-irreducible and does not contain essential tori nor annuli. By Thurston's hyperbolization result ([11], [17]), such a 3-manifold admits a complete hyperbolic structure with totally geodesic boundary. For example, see [7] and [18] for algorithms decomposing 3-manifolds into acylindrical 3-manifolds which are based on normal surface theory.

It is noticed that Theorem 1.2 gives a sufficient condition for a spatial graph Γ to be totally knotted, namely $E(\Gamma)$ is irreducible and ∂-irreducible. Now it is natural to ask the following.

PROBLEM 1.3. Give a sufficient condition for spatial graphs to be acylindrical.

In §3, several examples of minimally knotted spatial graphs are given. The spatial graphs in Figures 3-(A) and -(C) are acylindrical (cf. [17] and [12, Proposition 4.4]), but the exterior of the graph shown in Figure 4-(A) contains essential annuli.

2. Proof of Theorem 1.1

In this paper, we use the following notation:

- $\mathrm{cl}(\cdot)$: the closure,
- $N(Y, X)$: a regular neighborhood of Y in X where $Y \subset X$,
- $\partial N(Y, X)$: the frontier of $N(Y, X)$ in X, and
- $E(Y, X)$: an exterior of Y in X, namely $E(Y, X) = \mathrm{cl}(X - N(Y, X))$.

The total space X is not indicated if it is well-understood.

Let M be a compact, orientable 3-manifold with non-empty boundary ∂M. For a disjoint union J of simple closed curves in the boundary ∂M, the manifold obtained by attaching 2-handles $D^2 \times I$'s along J is denoted by $M(J)$. Let $J = J_1 \cup \cdots \cup J_n$ be a disjoint union of simple closed curves, possibly empty (i.e. $n = 0$), in ∂M.

We say that (M, J) is *trivial* (otherwise it is *non-trivial*) if:

(T.1) There are mutually disjoint essential disks $D_1, \cdots, D_n$ in M transverse to J such that $|\partial D_i \cap J_j| = \delta^i_j$, and

(T.2) $M(J)$ is a 3-ball.

For our convenience, we define the quasi-triviality for (M, J) inductively as follows. We say that (M, J) is *n-quasi-trivial* provided that:

(Q.1) For some i, there is an essential disk D_i in M transverse to J with $|\partial D_i \cap J_j| = \delta^i_j$,

(Q.2) for i in (Q.1), the pair $(M(J_i), J - J_i)$ is $(n-1)$-quasi-trivial, and

(Q.3) if $n = 0$, then M is a 3-ball.

It is noticed that if (M, J) is trivial, then it is $|J|$-quasi-trivial and the genus of ∂M coincides with the number of the components of J. If (M, J) is n-quasi-trivial, then $n = |J|$ and we say that (M, J) is *quasi-trivial*.

We say that (M, J) is *almost trivial* if:

(A.1) For each $J_i \subset J$, $(M(J_i), J - J_i)$ is trivial,

(A.2) (M, J) is not trivial. (By Lemma 2.3, we can replace this with that (M, J) is not quasi-trivial.)

We will prove Theorem 1.1 by applying Jaco's Handle Addition Lemma [5]. The following result is known as the Handle Addition Lemma.

THEOREM 2.1 ([5]). *Let M be an irreducible 3-manifold with compressible boundary, and J a simple closed curve in ∂M. If $\partial M - J$ is incompressible, then $\partial M(J)$ is incompressible.*

Theorem 2.1 was generalized in several ways. (See [10], [14]). The following is needed later.

LEMMA 2.2 ([10, Lemma 1.6], [14, Lemma 2.3]). *Suppose $\partial M - (J_1 \cup \cdots \cup J_n)$ is incompressible in M and $\partial M - (J - J_i)$ is compressible in M. Then, $\partial M(J_i) - (J - J_i)$ is incompressible in $M(J_i)$.*

In order to prove Theorem 1.1, we describe some properties of quasi-trivial pairs.

LEMMA 2.3. *Assume (M, J) is quasi-trivial. Then (M, J) is trivial and M is a handlebody.*

PROOF. We prove this by induction on $|J| = g(\partial M)$. In the case where $n = 0$, we are done by condition (Q.3). Suppose $n > 0$. By condition (Q.2) and by the assumption of the induction, $(M(J_i), J - J_i)$ is trivial for some i. In particular, $M(J_i)$ is a handlebody. It is noticed that M is viewed as the exterior of a properly embedded arc τ in the handlebody $M(J_i)$, where τ corresponds to a cocore of the attached 2-handle. By condition (Q.1), there is a disk D properly embedded in $E(\tau, M(J_i))$ such that $D \cap \partial N(\tau)$ is a single arc in the annulus $\partial N(\tau)$ and $D \cap (J - J_i) = \emptyset$. Let D' be the union of mutually disjoint properly embedded disks in $M(J_i)$ corresponding to the disks of condition (T.1) for $(M(J_i), J - J_i)$. We may assume that D' and τ are in general position and $N(\tau) \cap D'$ consists of meridian disks of $N(\tau)$. By using an innermost disk argument, we can isotope D so that $D \cap D'$ consists of arcs.

Let Δ be an outermost disk of D with respect to $D \cap D'$. Put $\alpha = \Delta \cap \partial D$ and $\beta = \mathrm{cl}(\partial\Delta - \alpha)$. Notice that there are three possibilities for Δ as follows: (A) $\alpha \subset \partial N(\tau)$, (B) α consists of two connected arcs $\alpha \cap \partial N(\tau)$ and $\alpha \cap \partial M(J_i)$, and (C) $\alpha \subset \partial M(J_i)$. In the case of (A), by sliding τ along Δ, we can isotope τ so that $D \cap D'$ is reduced. This isotopy preserves $\partial M(J_i)$. In the case of (B), we can also isotope τ so that $D \cap D'$ is reduced by sliding along Δ. Though this isotopy does not preserve $\partial\tau$, by condition (Q.1), $\alpha \cap \partial M(J_i)$ does not meet $J - J_i$. In the case of (C), we can replace D' by another union of disks as follows. Let D'' be the component of D' containing β. Let D_1 and D_2 be components of $D'' - \beta$. By condition (T.1), we may assume that D_1 does not meet $J - J_i$. By removing D_1 from D'', pasting Δ, and pushing it slightly, we obtain the new system D^* of disks satisfying condition (T.1) and $|D^* \cap D| < |D' \cap D|$. Thus we may assume $D' \cap D = \emptyset$. Now we see that $D' \cup D$ satisfies condition (T.1), and $M = E(\tau, M(J_i))$ is a handlebody. □

Note that by Lemma 2.3, any quasi-trivial pair (M, J) is a handlebody of genus $|J|$ and J is a *primitive set* (in Gordon's notion [1]) of simple closed curves in ∂M as illustrated in Figure 1.

LEMMA 2.4. *Assume (M, J) is almost trivial. Then $M(J_i)$ is a handlebody for each i, and either $\partial M(J_i) - (J - J_i)$ is compressible in $M(J_i)$, or $M(J_i)$ is a solid torus.*

PROOF. By condition (A.1), the pair $(M(J_i), J - J_i)$ is trivial, and thus quasi-trivial. By Lemma 2.3, $M(J_i)$ is a handlebody. By conditions (A.1) and (T.1), for some j, there is a disk $D \subset M(J_i)$ such that $D \cap J = D \cap J_j$ is a transverse point. If $M(J_i)$ is not a solid torus, the frontier $\partial N(D \cup J_j)$ is actually a compressing disk of $\partial M(J_i) - (J - J_i)$. □

LEMMA 2.5. *For an almost trivial pair (M, J), M is irreducible. If M is not a handlebody, then $\partial M - J$ is incompressible in M.*

PROOF. By condition (A.1), $(M(J_i), J - J_i)$ is trivial. Thus, $M(J)$ is a 3-ball by condition (T.2) and we see that M is embeddable in S^3. Since ∂M is connected, M is irreducible.

Now M is viewed as the exterior of properly embedded arcs $\tau_1, \cdots, \tau_n$ in a 3-ball B such that each meridian of τ_i corresponds to J_i. Thus, any compressing disk D for $\partial M - J$ is isotoped so that $\partial D \subset \partial B$. Hence, D separates B into two 3-balls B_1 and B_2. Let M_i denote the submanifold of M corresponding to B_i.

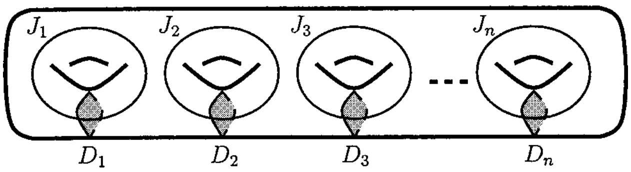


FIGURE 1.

Without loss of generality, we may assume $\tau_1, \cdots, \tau_m$ $(m < n)$ is contained in B_1, and the rest in B_2, after reordering if necessary. By condition (A.1), $M(J_1)$ is trivial. Thus by Lemma 2.3, $M(J_1)$ is a handlebody. Since M_2 is a component of the result of cutting the handlebody $M(J_1)$ along D, M_2 is a handlebody. Similarly, M_1 is a handlebody. Thus $M = M_1 \cup_D M_2$ is a handlebody. Hence we can conclude that if M is not a handlebody, then $\partial M - J$ is incompressible. □

LEMMA 2.6. *Assume (M, J) is almost trivial. Then M is irreducible and either M is ∂-irreducible or M is a handlebody.*

PROOF. Suppose that M is ∂-reducible and not a handlebody. First, we consider the case where $g(\partial M) > 2$. Then, there is a number $1 \leq h \leq n$ such that $\partial M - (J - \bigcup_{k=1}^{h} J_k)$ is compressible and $\partial M - (J - (\bigcup_{k=1}^{h} J_k - J_h))$ is incompressible in M. By Lemma 2.5, $\partial M - J$ is incompressible. Thus, it follows from Lemma 2.2 that $\partial M(J_h) - (J - \bigcup_{k=1}^{h} J_k)$ is incompressible. On the other hand, $\partial M(J_h) - (J - J_h)$ is compressible in $M(J_h)$ by Lemma 2.4, since $M(J_h)$ is not a solid torus for $g(\partial M) > 2$. Hence we see that $h > 1$. It is noticed that $\partial M(J_h) - (J - J_h)$ is a subsurface of $\partial M(J_h) - (J - J_1 \cup \cdots \cup J_{h-1} \cup J_h)$ and each of loops $J_1, \cdots, J_{h-1}$ is non-separating in $\partial M(J_h)$. Thus a compression of $\partial M(J_h) - (J - J_h)$ gives a compression of $\partial M(J_h) - (J - J_1 \cup \cdots \cup J_h)$ in $M(J_h)$. This is a contradiction. This shows that ∂M is incompressible in M if M is not a handlebody.

Suppose $g(\partial M) = 2$ and $J = J_1 \cup J_2$. Since (M, J) is almost trivial, and $g(\partial M(J_i)) = 1$, the manifold $M(J_i)$ is a solid torus. By Lemma 2.5, $\partial M - J$ is incompressible in M. Suppose $\partial M - J_1$ is incompressible in M. It follows from Theorem 2.1 that $\partial M(J_1)$ is incompressible in $M(J_1)$. This contradicts that $M(J_1)$ is a solid torus. Hence $\partial M - J_1$ is compressible. Namely a compressing disk D of ∂M can be chosen so that $\partial D \cap J_1 = \emptyset$. Similarly, $\partial M - J_2$ is compressible and we let E be a compressing disk of $\partial M - J_2$, possibly $E \cap J_1 \neq \emptyset$. Now M is viewed as the exterior of an arc τ_1 properly embedded in a solid torus V and J_2 is considered to be a longitude of V. Thus, D is isotoped so that $\partial D \subset \partial V$ since $\partial D \cap J_1 = \emptyset$. If D separates V, then V is separated into a 3-ball V_1 and a solid torus V_2 such that V_1 contains τ_1, and a meridian disk of V_2 is a compressing disk of $\partial M - J_1$. Hence we may assume D is non-separating and D is a meridian disk of V. If $E(D, M)$ is a solid torus, then M is a handlebody and we are done.

By the same reason as above, E can be assumed to be non-separating in M so that E is a meridian disk of the solid torus $M(J_2)$. Thus, we may assume that the *algebraic intersection number* $\partial E \cdot J_1 = 1$. Let us consider the intersection $D \cap E$. By using an innermost argument, we can remove all circles of $D \cap E$. Let Δ be an outermost disk in D. Now E is ∂-compressed by Δ to two disks E_1 and E_2, possibly $\partial E_i \cap J_2 \neq \emptyset$. Without loss of generality, we may assume $\partial E_1 \cdot J_1$ is odd since $\partial E_1 \cdot J_1 + \partial E_2 \cdot J_1 = \partial E \cdot J_1$ is odd. Repeating such ∂-compressions, finally we get a properly embedded disk E' in $M' = E(D, M)$ with $\partial E' \cdot J_1$ odd. This means that E' is a non-separating compressing disk of $\partial M'$ in M'. Since M is irreducible, M' is also irreducible. Thus, the sphere obtained by compressing $\partial M'$ along E'

bounds a 3-ball in M' on the side not containing E', and we see that M' is a solid torus. Hence, $M = M' \cup N(D)$ is a handlebody of genus two and the conclusion follows in the case $g(\partial M) = 2$.

In the case where $g(\partial M) = 1$, it is easy to see that if M is not a solid torus, then it is a non-trivial knot exterior in S^3 and it is ∂-irreducible. □

LEMMA 2.7. *If (M, J) is almost trivial, then M is not a handlebody.*

PROOF. Suppose M is a handlebody. By condition (A.1) and Lemma 2.3, we see that $M(J')$ is a handlebody for any subsystem J' of J. Now the conclusion follows directly form [1, Theorem 1]. □

PROOF OF THEOREM 1.1. By Lemmas 2.6 and 2.7, M is irreducible and ∂-irreducible. □

Now the following is available. (Compare with [1, Theorem 1].)

THEOREM 2.8. *Let (M, J) be such that for any $J_i \subset J$, $(M(J_i), J - J_i)$ is trivial. Then either*

- *If M is ∂-reducible, then M is a handlebody and (M, J) is trivial, or*
- *M is ∂-irreducible.* □

3. Proof of Theorem 1.2 and examples

Let Γ be a spatial graph in S^3 of a connected graph G. For edges $\mathcal{E} = \{e_1, \cdots, e_n\}$ of Γ, we denote the simple closed curve in $\partial E(\Gamma)$ corresponding to a meridian of e_i by e_i^*, and put $\mathcal{E}^* = \{e_1^*, \cdots, e_n^*\}$. Then by the notion of 2-handle addition, we have $E(\Gamma - \mathcal{E}) = E(\Gamma)(\mathcal{E}^*)$. (cf. Figure 2) We use the same letters for the edges of Γ corresponding to edges of G. For a graph or a spatial graph, we denote the set of vertices, edges by $\mathfrak{V}(\cdot)$ and $\mathfrak{E}(\cdot)$ respectively.

A set of edges $\mathcal{E}$ of G is called a *base edge system of G* if $G - \mathcal{E}$ is connected and simply connected, and a set of edges $\mathcal{E}$ of Γ is called a *base edge system of Γ* if $\Gamma - \mathcal{E}$ is connected and simply connected, equivalently $E(\Gamma - \mathcal{E}) = E(\Gamma)(\mathcal{E}^*)$ is a 3-ball.

LEMMA 3.1. *Let Γ be a spatial graph in a sphere F in S^3. For any base edge system $\mathcal{E} = \{e_1, \cdots, e_n\}$ of Γ, the pair $(E(\Gamma), \mathcal{E}^*)$ is trivial.*

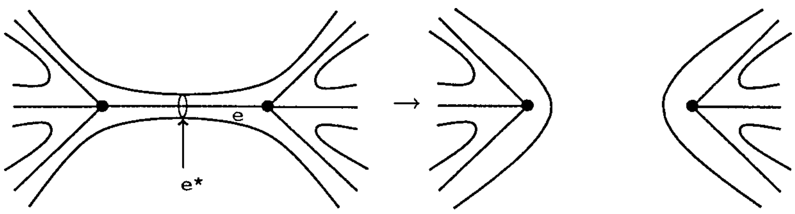


FIGURE 2.

PROOF. By Lemma 2.3, it is sufficient to prove that $(E(\Gamma), \mathcal{E}^*)$ is quasi-trivial.

Let $\Gamma_{\mathcal{E}}^*$ be the subgraph of the dual graph of Γ in F whose vertices are dual of all faces of $F - \Gamma$ and edges consist of the dual of $\mathcal{E}$. Since $\mathcal{E}$ is a base edge system, $\Gamma - \mathcal{E}$ is simply connected.

First we claim that $\Gamma_{\mathcal{E}}^*$ contains a vertex of valence 1. Put $v = |\mathfrak{V}(\Gamma)|$, $e = |\mathfrak{E}(\Gamma)|$, $f = |F - \Gamma|$, and put $v_{\mathcal{E}}^* = |\mathfrak{V}(\Gamma_{\mathcal{E}}^*)|$, $e_{\mathcal{E}}^* = |\mathfrak{E}(\Gamma_{\mathcal{E}}^*)|$. Since Γ is embedded in the sphere F, we have $v - e + f = 2$. Put $g = 1 + e - v$. Notice that g is equal to the genus of the handlebody $N(\Gamma)$. Clearly, $e_{\mathcal{E}}^* = g$, $v_{\mathcal{E}}^* = f$. Hence we have $v_{\mathcal{E}}^* = e_{\mathcal{E}}^* + 1$. On the other hand, if we assume that all valences are greater than or equal to 2, then we have $e_{\mathcal{E}}^* \geq 2v_{\mathcal{E}}^*/2 = v_{\mathcal{E}}^* = e_{\mathcal{E}}^* + 1$. This is a contradiction.

Since $E(\Gamma - \mathcal{E})$ is homeomorphic to a 3-ball, the subgraph $\Gamma - \mathcal{E}$ does not contain any cycles. Hence, each face of $F - \Gamma$ meets $\mathcal{E}$. Thus, each vertex of $\Gamma_{\mathcal{E}}^*$ has non-zero valence. Thus, the exterior $E(\Gamma)$ contains a non-separating disk D coming from a face of $F - \Gamma$ corresponding to a vertex with valence 1 of $\Gamma_{\mathcal{E}}^*$ such that $|\partial D \cap \mathcal{E}^*| = 1$. Now it is easy to check that $(E(\Gamma), \mathcal{E}^*)$ is quasi-trivial by induction on $|\mathcal{E}|$. □

Now we are ready to prove the following.

LEMMA 3.2. *If Γ is a minimally knotted planar spatial graph, then for any base edge system $\mathcal{E} = \{e_1, \cdots, e_n\}$ of Γ, the pair $(E(\Gamma), \mathcal{E}^*)$ is almost trivial.*

PROOF. Since Γ is minimally knotted, $\Gamma - e_i$ is in a sphere in S^3. By Lemma 3.1, we have $(E(\Gamma - e_i), \mathcal{E}^* - e_i^*)$ is trivial. Hence condition $(A.1)$ holds for $(E(\Gamma), \mathcal{E}^*)$. By [15, Theorem 7.5], if the exterior $E(\Gamma)$ is a handlebody ($\pi_1(E(\Gamma))$ is free), then Γ is trivial, namely, Γ is embedded in a sphere in S^3, since $\pi_1(E(\Gamma - e))$ is free for each non-separating edge e by the minimal knottedness. Thus, $E(\Gamma)$ is not a handlebody. Hence $(E(\Gamma), \mathcal{E}^*)$ is non-trivial by Lemma 2.3. Thus, it is an almost trivial pair. □

The converse is true in the following sense.

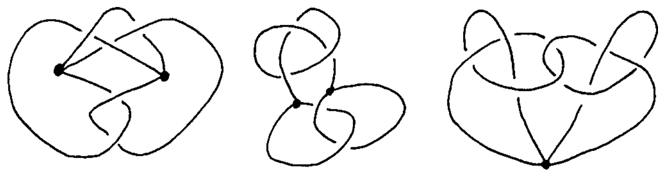
PROPOSITION 3.3. *For any almost trivial pair (M, J), there exists a spatial graph Γ such that $(E(\Gamma), \mathcal{E}^*) = (M, J)$ for some base edge system $\mathcal{E}$ of Γ. In fact, Γ can be chosen to be a bouquet with n loops.*

PROOF. Since $M(J)$ is a 3-ball, M is the exterior of some properly embedded arcs $\tau_1, \cdots, \tau_n$ in a 3-ball B. Embedding (B, τ) in S^3 and shrinking $S^3 - B$ into a point, we obtain a bouquet Γ embedded in S^3 such that $E(\Gamma) = M$. This completes the proof. □

THEOREM 3.4. *Let Γ be a spatial graph. If Γ has a base edge system $\mathcal{E}$ such that $(E(\Gamma), \mathcal{E}^*)$ is almost trivial, then Γ is totally knotted.*

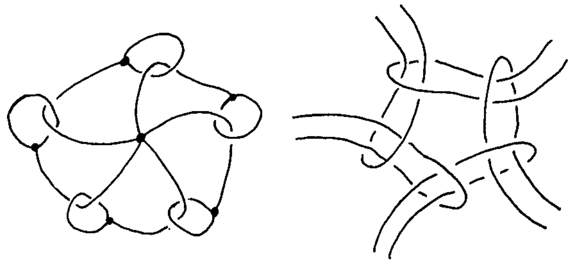
PROOF. By Theorem 1.1, $E(\Gamma)$ is irreducible and ∂-irreducible. Thus Γ is totally knotted. □

PROOF OF THEOREM 1.2. Theorem 1.2 follows directly from Lemma 3.2 and Theorem 3.4. □



(A) (B) (C)

FIGURE 3.



(A) (B)

FIGURE 4.

Here we describe some examples of spatial graphs which are totally knotted. The θ-curve Γ_1 illustrated in Figure 3-(A) is known to be non-trivial ([9]), but is minimally knotted. The handcuff graph Γ_2 embedded as shown in Figure 3-(B) is not minimally knotted for the two loops e_1, e_2 have the linking number one. However it is not hard to see that the exterior M contains an incompressible torus, thus M is not a handlebody. On the other hand, taking meridians of e_1, e_2 as J, we see that (M, J) is almost trivial. Hence by Theorem 3.4, Γ_2 is totally knotted. The graph Γ_3 illustrated in Figure 3-(C) is not minimally knotted, in fact, each subgraph is a trefoil knot and we cannot adapt Theorem 3.4, but it is totally knotted since $E(\Gamma_3)$ is homeomorphic to the tangle space of the "true lover's tangle", which was proved by Myers [12, Proposition 4.1] to be *atoroidal*.

In [16], Taniyama gave a useful method to confirm the non-triviality of certain spatial graphs, and the graph illustrated in Figure 4-(A) is shown to be *irreducible* (see [16] for definition), thus it is non-trivial. Now it is easy to see that it is minimally knotted. Hence by Theorem 1.2, it is totally knotted. It is remarked that the exterior is homeomorphic to the tangle space of the tangle illustrated in Figure 4-(B), and the tangle is non-trivial.

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