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The Gonality of Singular Plane Curves

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1. Introduction

Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree d over the complex number field $\mathbf{C}$. We denote by $\mathbf{C}(C)$ the field of rational functions on C. Let $\tilde{C}$ be the non-singular model of C. Since $\mathbf{C}(\tilde{C}) \cong \mathbf{C}(C)$, a non-constant rational function φ on C induces a non-constant morphism $\varphi : \tilde{C} \to \mathbf{P}^1$. Let $\deg \varphi$ denote the degree of this morphism φ. We remark that $\deg \varphi = [\mathbf{C}(C) : \mathbf{C}(\varphi)] = \deg(\varphi)_0 = \deg(\varphi)_\infty$. The *gonality* of C, denoted by $\mathrm{Gon}(C)$, is defined to be $\min\{\deg \varphi \,|\, \varphi \in \mathbf{C}(C) \setminus \mathbf{C}\}$. So by definition, the gonality of C is nothing but the gonality of $\tilde{C}$. Let ν denote the maximal multiplicity of C. We easily see that $\mathrm{Gon}(C) \leq d - \nu$. We know that the genus of C is equal to $(d-1)(d-2)/2 - \delta$ with $\delta \geq 0$.

THEOREM 1. *Let C be an irreducible plane curve of degree d with $\delta \geq \nu$. Letting $d \equiv i \pmod{\nu}$, define*

$$R(\nu, \delta, i) = \frac{\nu^2 + (\nu - 2)i}{2\nu(\nu - 1)} + \sqrt{\frac{\delta - \nu}{\nu - 1} + \left(\frac{\nu - 2 + i}{2(\nu - 1)}\right)^2}.$$

If $d/\nu > R(\nu, \delta, i)$, then $\mathrm{Gon}(C) = d - \nu$.

REMARK 1. Theorem 1 is a generalization of Theorem 2.1 in Coppens and Kato [1] where they considered the case in which C has only nodes and ordinary cusps. Note that $R(2, \delta, 0) = 1 + \sqrt{\delta - 2}$, $R(2, \delta, 1) = 1 + \sqrt{\delta - 7/4}$. In general, we have the estimation: $R(\nu, \delta, i) < 1 + \sqrt{\delta/(\nu - 1)}$.

We have $\delta < \nu$ if either (i) C is a smooth curve ($\delta = 0$, $\nu = 1$ and $\mathrm{Gon}(C) = d - 1$ for all $d \geq 2$), or (ii) C has one node or one ordinary cusp ($\delta = 1$ and $\nu = 2$ and $\mathrm{Gon}(C) = d - 2$ for all $d \geq 3$). Cf. [1], [3], [5].

DEFINITION. Let $m_1, \cdots, m_n$ denote the multiplicities of all singular points (we include infinitely near singular points) of C. Set $\eta = \sum (m_i/\nu)^2$. Clearly, we have $n \geq \eta \geq 1$.

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THEOREM 2. *Let C be an irreducible plane curve of degree d with $\nu \geq 3$. We have* $\mathrm{Gon}(C) = d - \nu$ *if*

$$d/\nu \begin{cases} > (\eta+1)/2, & \text{for } \eta < a(\nu),\ \eta \geq 5 \\ > 2\sqrt{\eta} - (1 + 1/\nu), & \text{for } a(\nu) \leq \eta < 4\,, \\ \geq 3, & \text{for } 4 \leq \eta < 5\,, \end{cases}$$

where $a(\nu) = (2 - \sqrt{1 - 2/\nu})^2$.

REMARK 2. Note that $a(3) = 2.023\cdots$ and $1 < a(\nu) \leq 1.671\cdots$ for $\nu \geq 4$.

We shall show that if $\eta \geq 2\nu + 5$, then the criterion in Theorem 1 is more effective than that in Theorem 2. We also prove some subtle criterions.

THEOREM 3. *Let C be an irreducible plane curve of degree d with n singular points (infinitely near singular points are also counted). We renumber the multiplicities m_i's as $\nu = m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_n$. We have* $\mathrm{Gon}(C) = d - \nu$ *if either*

(i) $n \leq 2$, *or*

(ii) $n = 3$ *and* $d/\nu > 2$, *or*

(iii) $n \geq 4$, $d \geq m_2 + m_3 + m_4$ *and*

$$d/\nu > \begin{cases} (\eta+1)/2 & \text{if } \nu = 3, 4\,, \\ & \text{if } \nu \geq 5 \text{ and } \eta < b(\nu),\ \eta \geq c(\nu)\,, \\ (1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\} & \text{if } \nu \geq 5 \text{ and } b(\nu) \leq \eta < c(\nu)\,, \end{cases}$$

where $b(\nu) = (3/2 - \sqrt{1/4 - 1/\nu})^2$ *and* $c(\nu) = (3/2 + \sqrt{1/4 - 1/\nu})^2$.

REMARK 3. In view of Theorem 2, the condition (iii) is meaningful only if $a(\nu) \leq \eta < 5$. We remark that $a(\nu) < b(\nu) < c(\nu)$ and $1 < b(\nu) \leq 1.629\cdots$ and $2.970\cdots \leq c(\nu) < 4$ for $\nu \geq 5$.

2. Rational functions on C and on $\mathbf{P}^2$

Let φ be a rational function on C. Set $r = \deg \varphi$. We know that a rational function φ of a plane curve C is a restriction of a rational function $\Phi = g(x, y, z)/h(x, y, z)$ on $\mathbf{P}^2$, where g and h are relatively prime homogeneous polynomials of the same degree, say k. We call k the *degree* of the rational function Φ. A rational function Φ is called a *linear function* if $k = 1$. Classically, one says that φ is cut out by the pencil $\Lambda : t_0 g - t_1 h = 0$ on $\mathbf{P}^2$. Let us consider the rational map

$$\Phi : \mathbf{P}^2 \ni P \mapsto (h(P), g(P)) \in \mathbf{P}^1\,.$$

By a sequence of blowing-ups $\pi : X \to \mathbf{P}^2$, one can resolve the base points of Φ and the singularities of C, so that $\Phi \circ \pi : X \to \mathbf{P}^1$ becomes a morphism and the strict transform $\tilde{C}$

of C is non-singular. Write $\pi = \pi_1 \circ \cdots \circ \pi_s$, where $\pi_i : X_i \to X_{i-1}$ is the blowing-up at a point $P_i \in X_{i-1}$ and $X_0 = \mathbf{P}^2$, $X_s = X$. Let E_i be the total transform on X of the exceptional curve of the blowing-up π_i. We have a relation of divisors: $\tilde{C} = \pi^*C - \sum m_i E_i$, where m_i is the multiplicity of the strict transform of C on X_{i-1} at P_i. Set $H = \pi^* L$, where L is a line on $\mathbf{P}^2$. Then, we have the linear equivalence: $\tilde{C} \sim dH - \sum m_i E_i$. It follows from this and the adjunction formula that $\delta = \sum m_i(m_i - 1)/2$. Any fibre D of the morphism $\Phi \circ \pi$ is linearly equivalent to a divisor $kH - \sum a_i E_i$ with some integers a_i. Since $DE_i \geq 0$ and $D^2 = 0$, we must have the relation:

$$k^2 = \sum a_i^2$$

and also we must have $a_i \geq 0$ for all i. We then obtain the formula:

$$r = dk - \sum a_i m_i \, .$$

If $k = 1$, then we must have $r = d - m_i$ for some i. In particular, there is a rational function φ with $r = d - \nu$. Note that a rational function $g^*/h^* \in \mathbf{C}(\mathbf{P}^2)$ also induces φ if and only if $gh^* - hg^*$ is divisible by the defining polynomial of C. So many different rational functions on $\mathbf{P}^2$ can induce the same rational function φ on C.

LEMMA 1. *We have the inequality:* $r + \delta \geq dk - k^2$.

PROOF. It suffices to show that $k^2 + \delta \geq \sum a_i m_i$. We see that

$$k^2 + \delta - \sum a_i m_i = \sum_{m_i \neq 1} (2a_i - m_i)^2/4 + \sum_{m_i \neq 1} m_i(m_i - 2)/4 + \sum_{m_i = 1} a_i(a_i - 1) \, .$$

If $m_i \geq 2$ or $m_i = 0$, then $m_i(m_i - 2) \geq 0$. Since a_i is an integer, we have $a_i(a_i - 1) \geq 0$. Thus we get the desired inequality. □

Let b denote the number of a_i with $a_i \neq 0$.

LEMMA 2. *If* $r < d - \nu$, *then* $k \geq 2$ *and* $d/\nu < (k\sqrt{b} - 1)/(k - 1)$.

PROOF. If $k = 1$, then we have $r \geq d - \nu$. So assume $k \geq 2$. By Schwarz' inequality, we have $\sum a_i \leq \sqrt{b}k$. We obtain

$$r \geq dk - \left(\sum a_i\right)\nu \geq k(d - \nu\sqrt{b}) = d - \nu + (k-1)\nu\left\{d/\nu - (k\sqrt{b} - 1)/(k - 1)\right\},$$

which implies the assertion. □

LEMMA 3. *If* $r < d - \nu$, *then* $k \geq 2$ *and* $k > d/\nu - 1$. *Furthermore, if* $r = d - \nu + s$ *with* $s \geq 0$ *and* $k \geq 2$, *then* $k \geq d/\nu - s - 1$.

PROOF. In view of the inequality in Lemma 2, it suffices to note that $b \leq k^2$. Suppose $r = d - \nu + s$ with $s \geq 0$. If $k \geq 2$, then we obtain

$$k + s \geq k + s/(k-1) \geq d/\nu - 1 .$$

□

We renumber a_i's so that $a_1 \geq a_2 \geq \cdots \geq a_b \geq 1, a_i = 0$ for $i > b$.

LEMMA 4. *We have $r \geq d - \nu$ either if $b \leq 2$, or if $b = 3$ and $d/\nu \geq 2$.*

PROOF. (i) $b = 1$. We have $r = k(d - m_1) \geq k(d - \nu) \geq d - \nu$. (ii) $b = 2$. By Bezout's theorem applied to the curve C and the line passing through P_1 and P_2, we have the inequality: $d \geq m_1 + m_2$. On the other hand, since $k^2 = a_1^2 + a_2^2$, we must have $a_i < k$ for $i = 1, 2$. Thus we obtain

$$\begin{aligned} r &= d - \nu + (\nu - m_1) + (k-1)d - (a_1 - 1)m_1 - a_2 m_2 \\ &\geq d - \nu + (k - a_1)m_1 + (k - a_2 - 1)m_2 \geq d - \nu . \end{aligned}$$

(iii) $b = 3$. In case $k \geq 4$, by Lemma 2, we have $r > d - \nu$, since $(4\sqrt{3} - 1)/3 = 1.976 \cdots < 2$. In case $k \leq 3$, the equation: $k^2 = a_1^2 + a_2^2 + a_3^2$, $(3 \leq a_1 + a_2 + a_3 \leq 5)$ has only one integer solution: $k = 3, a_1 = a_2 = 2, a_3 = 1$. Under the assumption: $d \geq 2\nu$, we obtain $r = d - \nu + (\nu - m_1) + 2d - (m_1 + 2m_2 + m_3) \geq d - \nu$. □

LEMMA 5. *If $r < d - \nu$ and $k = 2$, then $b = 4$ and $d < m_1 + m_2 + m_3 + m_4 - \nu$.*

PROOF. We have $b = 1$ or $b = 4$. In case $b = 4$, we must have $a_1 = a_2 = a_3 = a_4 = 1$. So we obtain $d - \nu > r = 2d - m_1 - m_2 - m_3 - m_4$, which gives the assertion. □

LEMMA 6. *We have the inequality: $r \geq k(d - \sqrt{\sum m_i^2})$.*

PROOF. By Schwarz' inequality, we have

$$\sum a_i m_i \leq \sqrt{\sum a_i^2} \sqrt{\sum m_i^2} = k \sqrt{\sum m_i^2} ,$$

which gives the assertion. □

3. Proof of Theorem 1

Let C be an irreducible plane curve of degree d.

LEMMA 7 (Cf. Coppens and Kato[1, 2]). *Let φ be a rational function on C with $r = \deg \varphi$. Let l be a positive integer with $l < d$. Suppose $r + \delta < (l + 1)(d - l - 1)$. Then there exists a rational function on $\mathbf{P}^2$ of degree $k \leq l$ which induces φ on C.*

PROOF. Assume to the contrary that there are no rational functions of degree $\leq l$ on $\mathbf{P}^2$ which induces φ on C. Following the arguments in [1, 2], one can prove that there exists a rational function of degree k on $\mathbf{P}^2$ which induces φ on C with $l < k \leq d - 3 - l$. Using

Lemma 1, we have $dk - k^2 \leq r + \delta < (l+1)(d-l-1)$, from which we infer that $(l+1-k)(d-k-l-1) > 0$. This is absurd, because $l+1-k \leq 0$ and $d-k-l-1 \geq 2$. □

PROPOSITION 1. *Assume there is a positive integer l such that $l \leq (d/\nu) - 1$ and $\delta - \nu < l(d-l-2)$. Then we have* $\mathrm{Gon}(C) = d - \nu$.

PROOF. Suppose there exists a rational function φ on C with $r = \deg \varphi < d - \nu$. In this case, we have the inequality:

$$r + \delta \leq d - \nu - 1 + \delta < l(d-l-2) + d - 1 = (l+1)(d-l-1)\,.$$

So by Lemma 7, there exists a rational function Φ of degree $k \leq l$ on $\mathbf{P}^2$ which induces φ on C. But, since $k \leq l \leq (d/\nu) - 1$, by Lemma 3, there cannot exist such a rational function Φ. □

PROPOSITION 2. *If $[d/\nu] \geq 2$ and $([d/\nu]-1)(d-[d/\nu]-1) > \delta - \nu$, then we have* $\mathrm{Gon}(C) = d - \nu$.

REMARK 4. In case $\nu = 2$, this criterion is best possible. See [1], Examples 4,1 and 4,2. We see that the assertion of Proposition 1 is equivalent to that of Proposition 2. Take a positive integer l which satisfies the two assumptions in Proposition 1. We find that $1 \leq l \leq [d/\nu] - 1 \leq (d/\nu) - 1$. The quadratic function $Q(x) = x(d-x-2)$ is a monotone increasing function for the interval $0 \leq x \leq (d/2) - 1$. Hence we infer that $Q(l) \leq Q([d/\nu]-1)$. Thus the integer $[d/\nu] - 1$ also satisfies the two assumptions in Proposition 1.

Using the latter assertion in Lemma 3, we obtain the following

PROPOSITION 3. *Let s be a non-negative integer. Set $l = d/\nu - s - 2$ (if $d \equiv 0 \pmod{\nu}$), $[d/\nu] - s - 1$ (otherwise). If $l \geq 1$ and $\delta - \nu + s + 1 < l(d-l-2)$, then* $\mathrm{Gon}(C) = d - \nu$ *and any rational function φ with $d - \nu \leq \deg \varphi \leq d - \nu + s$ is induced by a linear function on* $\mathbf{P}^2$.

PROOF OF THEOREM 1. We reformulate Proposition 2. Letting $d = [d/\nu]\nu + i$ with $0 \leq i < \nu$, the inequality $\delta - \nu < ([d/\nu]-1)(d-[d/\nu]-1)$ can be written as:

$$\frac{\delta - \nu}{\nu - 1} + \left(\frac{\nu - 2 + i}{2(\nu - 1)}\right)^2 < \left\{\frac{d}{\nu} - \frac{\nu^2 + (\nu-2)i}{2\nu(\nu-1)}\right\}^2.$$

If $\delta - \nu \geq 0$, then the above inequality is equivalent to the inequality $d/\nu > R(\delta, \nu, i)$. Furthermore, we easily see that $R(\delta, \nu, i) \geq 1 + i/\nu$. So it follows from the inequality $d/\nu > R(\delta, \nu, i)$ that $d > \nu + i$, which gives $d \geq 2\nu + i$ if $d \equiv i \pmod{\nu}$ and hence $d/\nu \geq 2$.

REMARK 5. If $\delta - \nu < 0$, then the left hand side of the above inequality is negative. It follows that the above inequality always holds. In case $\delta = 1$, $\nu = 2$, we have $\mathrm{Gon}(C) = d - 2$

for $d \geq 4$. It is well known that $\mathrm{Gon}(C) = 1$ if $d = 3$. In case $\delta = 0, \nu = 1$, we have $\mathrm{Gon}(C) = d - 1$ for $d \geq 2$.

LEMMA 8. *We have the estimation:*

$$R(\nu, \delta, i) < 1 + \sqrt{\delta/(\nu - 1)}\,.$$

PROOF. Since $i \leq \nu - 1$, we have

$$\nu^2 + (\nu - 2)i \leq \nu^2 + (\nu - 2)(\nu - 1) = 2\nu(\nu - 1) - (\nu - 2)$$

and $\nu - 2 + i \leq 2(\nu - 1)$. Thus, we obtain

$$\frac{\nu^2 + (\nu - 2)i}{2\nu(\nu - 1)} \leq 1 \quad \text{and} \quad \frac{\nu}{\nu - 1} - \left(\frac{\nu - 2 + i}{2(\nu - 1)}\right)^2 > 0\,,$$

which gives the desired inequality. □

4. Proof of Theorems 2 and 3

Let C be an irreducible plane curve of degree d. Now let $\pi : X \to \mathbf{P}^2$ be the minimal resolution of the singularities of C. We do not require that the inverse image $\pi^{-1}(C)$ has normal crossings. In this case, $m_i \geq 2$ for all i.

LEMMA 9. *Assume $d/\nu > (\eta + 1)/2$. Let φ be a rational function on C with $r = \deg \varphi < d - \nu$. Then we can find a rational function Φ on $\mathbf{P}^2$ which induces φ on C such that $\Phi \circ \pi : X \to \mathbf{P}^1$ becomes a morphism. Furthermore, the degree k of Φ satisfies the inequality*:

$$k \leq 1 + \frac{\sqrt{\eta} - (1 + 1/\nu)}{d/\nu - \sqrt{\eta}}\,.$$

PROOF. According to Theorem 3.1 in Serrano [5](See also [4]), such a rational function exists if $\tilde{C}^2 > (r + 1)^2$. On X, we have

$$\begin{aligned}\tilde{C}^2 - (r + 1)^2 &\geq d^2 - \sum m_i^2 - (d - \nu)^2 \\ &= 2d\nu - \sum m_i^2 - \nu^2 = 2\nu^2\{d/\nu - (\eta + 1)/2\} > 0\,.\end{aligned}$$

By Lemma 6, we have $d - \nu - 1 \geq r \geq k\nu(d/\nu - \sqrt{\eta})$. Thus we obtain

$$k \leq \frac{d/\nu - (1 + 1/\nu)}{d/\nu - \sqrt{\eta}} = 1 + \frac{\sqrt{\eta} - (1 + 1/\nu)}{d/\nu - \sqrt{\eta}}\,.$$

□

REMARK 6. Under the hypothesis $d/\nu > (\eta + 1)/2$, we see that $d/\nu - \sqrt{\eta} = d/\nu - (\eta + 1)/2 + (\sqrt{\eta} - 1)^2/2 > 0$. Since $k \geq 1$, we must have $\sqrt{\eta} - (1 + 1/\nu) \geq 0$.

In a similar manner to that in the proof of Lemma 9, we can show the following

LEMMA 10. *Let s be a non-negative integer with $s < \nu - 1$. Let φ be a rational function on C with $r = \deg \varphi = d - \nu + s$. If*

$$d/\nu > (\eta + 1)/2 + \frac{s+1}{2(\nu - s - 1)} \left\{ \eta - 1 + \frac{s+1}{\nu} \right\},$$

then we can find a rational function Φ on $\mathbf{P}^2$ which induces φ on C such that $\Phi \circ \pi : X \to \mathbf{P}^1$ becomes a morphism. Furthermore, the degree k of Φ satisfies the inequality:

$$k \leq 1 + \frac{\sqrt{\eta} - 1 + s/\nu}{d/\nu - \sqrt{\eta}}.$$

PROPOSITION 4. *Suppose $d/\nu > (\eta + 1)/2$. We get* $\mathrm{Gon}(C) = d - \nu$ *if either*
(i) $d/\nu > 2\sqrt{\eta} - (1 + 1/\nu)$, *or*
(ii) $\eta \geq 5$, *or*
(iii) $d/\nu \geq 3$ *and* $\eta < 5$, *or*
(iv) $d/\nu > (1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\}$ *and* $d \geq m_2 + m_3 + m_4$ (*if* $n \geq 4$), *where the multiplicities m_i's are renumbered as $m_1 \geq m_2 \geq m_3 \geq \cdots$.*

PROOF. Assume there is a rational function φ on C with $r = \deg \varphi < d - \nu$. By Lemma 9, we can find a rational function Φ on $\mathbf{P}^2$ which induces φ on C such that π has already resolved the base points of Φ. The degree k of Φ must satisfy the inequality in Lemma 9.

(i) If $d/\nu > 2\sqrt{\eta} - (1 + 1/\nu)$, then we infer that $k < 2$. So we get $k = 1$, which is impossible by Lemma 3.

(ii) If $\eta \geq 5$, then we have $d/\nu > 3$. We obtain

$$k < 1 + \frac{\sqrt{\eta} - 1}{(\eta + 1)/2 - \sqrt{\eta}} = 1 + \frac{2}{\sqrt{\eta} - 1} \leq 1 + \frac{2}{\sqrt{5} - 1} = (3 + \sqrt{5})/2 < 3.$$

So $k \leq 2$, which contradicts Lemma 3.

(iii) We have

$$k < 1 + \frac{\sqrt{5} - 1}{3 - \sqrt{5}} = (3 + \sqrt{5})/2 < 3.$$

So $k \leq 2$, which again contradicts Lemma 3.

(iv) In a similar manner to that in the proof of (i), under the assumption on d/ν, we obtain $k < 3$. In case $k = 2$, by Lemma 5, we get a contradiction. □

PROOF OF THEOREM 2. By Proposition 4, (i), we get $\mathrm{Gon}(C) = d - \nu$ if $d/\nu > \max\{2\sqrt{\eta} - (1 + 1/\nu), (\eta + 1)/2\}$. We easily see that $2\sqrt{\eta} - (1 + 1/\nu) \geq (\eta + 1)/2$ if and only if $2 - \sqrt{1 - 2/\nu} \leq \sqrt{\eta} \leq 2 + \sqrt{1 - 2/\nu}$. In case $\nu \geq 3$, we have the relation: $a(\nu) = (2 - \sqrt{1 - 2/\nu})^2 < 5 < (2 + \sqrt{1 - 2/\nu})^2$. Using also Proposition 4, (ii), we get

$\mathrm{Gon}(C) = d - \nu$ if

$$d/\nu > \begin{cases} (\eta + 1)/2\,, & \text{for } \eta < a(\nu),\ \eta \geq 5 \\ 2\sqrt{\eta} - (1 + 1/\nu), & \text{for } a(\nu) \leq \eta < 5\,. \end{cases}$$

On the other hand, by Proposition 4, (iii), for $\eta < 5$, we get $\mathrm{Gon}(C) = d - \nu$ if $d/\nu \geq 3$. Obviously, $2\sqrt{\eta} - (1 + 1/\nu) > 3$ if and only if $\sqrt{\eta} > 2 + 1/(2\nu)$. Thus, for $(2 + 1/(2\nu))^2 < \eta < 5$, the condition $d/\nu \geq 3$ is sharper than the condition $d/\nu > 2\sqrt{\eta} - (1 + 1/\nu)$. Finally, for the interval $4 \leq \eta \leq (2 + 1/(2\nu))^2$, we find that $3 \geq 2\sqrt{\eta} - (1 + 1/\nu) \geq 3 - 1/\nu$. The inequality $d/\nu > 3 - 1/\nu$ implies $d > 3\nu - 1$, hence $d \geq 3\nu$. As a consequence, the conditions $d/\nu \geq 3$ and $d/\nu > 2\sqrt{\eta} - (1 + 1/\nu)$ have the same effect.

REMARK 7. In case $\nu = 2$, we infer from Proposition 4, (i) that if $d/2 > (\eta + 1)/2$, then $\mathrm{Gon}(C) = d - 2$. In this case, $\delta = \eta$. But the criterion in Theorem 1 is sharper than this one.

PROPOSITION 5. *Suppose* $\nu \geq 3$. *If* $\eta \geq 2\nu + 5$, *then the criterion in Theorem* 1 *is sharper than that in Theorem* 2.

PROOF. It suffices to prove the inequality: $(\eta + 1)/2 > R(\nu, \delta, i)$. By definition, we have $\delta < \sum m_i^2/2 = \nu^2\eta/2$. Using Lemma 8, we obtain

$$R(\nu, \delta, i) < R(\nu, \nu^2\eta/2, i) < 1 + \nu\sqrt{\eta/2(\nu - 1)}\,.$$

By an easy manipulation, the inequality: $(\eta + 1)/2 \geq 1 + \nu\sqrt{\eta/2(\nu - 1)}$ can be reduced to the inequality: $\eta \geq t(\nu)$, where

$$t(\nu) = \nu + 2 + \frac{1}{\nu - 1} + \sqrt{\left(\nu + 2 + \frac{1}{\nu - 1}\right)^2 - 1}\,.$$

Clearly, we have $t(\nu) \leq 2\nu + 5$. Thus, if $\eta \geq 2\nu + 5$, then $(\eta + 1)/2 > R(\nu, \delta, i)$. □

PROPOSITION 6. *Assume*

$$d/\nu > (\eta + 1)/2 + \frac{1}{2(\nu - 1)}\left\{\eta - 1 + \frac{1}{\nu}\right\}.$$

If either

(i) $d/\nu > 2\sqrt{\eta} - 1$, *or*
(ii) $\eta > 5$, *or*
(iii) $d/\nu > 3$, $\eta \leq 5$,

then we have $\mathrm{Gon}(C) = d - \nu$ *and any rational function* φ *with* $\deg \varphi = d - \nu$ *is induced by a linear function on* $\mathbf{P}^2$.

PROOF OF THEOREM 3. Suppose $d/\nu > (\eta + 1)/2$. Assume there is a rational function φ on C with $r = \deg \varphi < d - \nu$. We infer from Lemma 9 that there is a rational function

Φ on $\mathbf{P}^2$ which induces φ on C such that π resolves the base points of Φ. It follows that $b \leq n$.

(i), (ii) We first show that $d/\nu > (\eta + 1)/2$ for the case (i). If $n = 1$, then we have $\eta = 1$ and so $d/\nu > 1 = (\eta+1)/2$. If $n = 2$, then, as we have noticed, we have $d \geq m_1+m_2$. It follows that $d/\nu \geq 1+(m_2/\nu) > 1+(1/2)(m_2/\nu)^2 = (\eta+1)/2$. (ii) Since $\eta \leq n = 3$, we have $d/\nu > 2 \geq (\eta + 1)/2$. Thus, we obtain $b \leq n$. By Lemma 4, we derive a contradiction.

(iii) We easily see that $(1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\} \geq (\eta + 1)/2$ if $\nu \leq 4$, or if $\nu \geq 5$ and $b(\nu) \leq \eta \leq c(\nu)$. Thus, under the assumptions in (iii), we have

$$d/\nu > \max\{(1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\},\ (\eta + 1)/2\}\,.$$

By Proposition 4, (iv), we arrive at a contradiction.

5. Examples

EXAMPLE 1. *Let C be an irreducible plane curve of degree $d = km + 1$ defined by the equation*:

$$y\prod_{i=1}^{k}(x - a_i)^m - c\prod_{j=1}^{k}(y - b_j)^m = 0\,,$$

where the a_i's and the b_j's are mutually distinct, respectively, $b_j \neq 0$ for all j and c is a general constant. We have $\mathrm{Gon}(C) = k$.

PROOF. By Eisenstein's criterion applied to the homegenization of the above polynomial, we easily see that the curve C is irreducible. If $m = 1$, then C is a smooth curve with $\mathrm{Gon}(C) = d - 1 = k$. In what follows, we assume that $m \geq 2$. Under the assumption that the constant c is general, the curve C has k^2 ordinary m-fold singular points $P_{ij} = (a_i, b_j)$ for $1 \leq i, j \leq k$. Thus $\nu = m$ and $\eta = k^2$. In this case, $\mathrm{Gon}(C) < d - \nu$. Indeed, the rational function $\Phi = \prod(y - b_j)/\prod(x - a_i)$ of degree k on $\mathbf{P}^2$ induces a rational function φ on C. The function Φ has k^2 base points P_{ij} on C. This proves that $\deg\varphi = (km+1)k - k^2m = k$. Note that $k > d/\nu - 1$.

We now prove that $\mathrm{Gon}(C) = k$. We first see that $\mathbf{C}(C) \cong \mathbf{C}(\varphi, x)$. For simplicity's sake, we also denote by x, y the rational functions on C induced by x, y. Clearly, we have $\mathbf{C}(\varphi, x) \subset \mathbf{C}(C)$. Since $\varphi^m = y/c$, we obtain $y \in \mathbf{C}(\varphi, x)$, which implies $\mathbf{C}(\varphi, x) = \mathbf{C}(C)$. Now $\mathbf{C}(\varphi, x)$ is the rational function field of the curve $C' : \varphi\prod(x-a_i) - c\prod(c\varphi^m - b_j) = 0$. The curve C' is of degree $d' = mk$ and has one singular point with multiplicity sequence $((m - 1)k, k_{m-2}, k - 1)$ on the line at infinity, where by k_{m-2} we mean k's repeated $m - 2$ times. For C', we use the notation d', ν' and η'. We have

$$\eta' = 1 + (m - 2)/(m - 1)^2 + \{(k - 1)/k(m - 1)\}^2 < m/(m - 1)\,.$$

We obtain $2\sqrt{\eta'} - (1 + 1/\nu') < 2\sqrt{m/(m - 1)} - 1 - 1/(m - 1)k$. Hence, we have $d'/\nu' - \{2\sqrt{\eta'} - (1 + 1/\nu')\} > (\sqrt{m/(m - 1)} - 1)^2 + 1/(m - 1)k > 0$. We can show that $\eta' > a(\nu')$.

We therefore conclude from Theorem 2 that $\mathrm{Gon}(C') = d' - \nu' = k$ if $\nu' \geq 3$. In case $\nu' \leq 2$, by Theorem 3, we can easily check that $\mathrm{Gon}(C') = k$. Since C and C' are birational, we get $\mathrm{Gon}(C) = k$. □

EXAMPLE 2. *Let C be an irreducible plane curve of degree d. Suppose C has 9 ordinary triple points. By Theorem 1, we get $\mathrm{Gon}(C) = d - 3$ if $d \geq 14$. Let C be the curve of degree 11 defined by the equation*:

$$y \prod_{i=1}^{3} (x - a_i)^3 (x - a_4) - c \prod_{j=1}^{3} (y - b_j)^3 (y - b_4) = 0\,,$$

where the a_i's and the b_j's are mutually distinct, respectively, $b_j \neq 0$ for all j and c is a general constant. This curve C has 9 ordinary triple points. But we see that $\mathrm{Gon}(C) \leq 6 < 11 - 3$.

PROOF. We consider the rational function $\Phi = \prod_{j=1}^{3} (y - b_j) / \prod_{i=1}^{3} (x - a_i)$ on $\mathbf{P}^2$. Let φ be the rational function on C induced by Φ. It turns out that $\deg \varphi = 6$. □

EXAMPLE 3. *Let C be an irreducible plane curve of degree $d = em$ defined by the equation: $y^m = \prod_{i=1}^{em} (x - a_i)$, where the a_i's are mutually distinct. We have $\mathrm{Gon}(C) = m$ if $e \geq 2$ or $= m - 1$ if $e = 1$.*

PROOF. If $e = 1$ or if $e = 2$ and $m = 1$, then C is smooth. Otherwise, the curve C has one singular point with multiplicity sequence $((e - 1)m, m_{e-1})$ on the line at infinity. We have $\nu = (e - 1)m$, $\eta = e/(e - 1)$ and so $d/\nu = e/(e - 1) = \eta$. In case $\nu \geq 3$, we can apply Theorem 2 and we conclude that $\mathrm{Gon}(C) = d - \nu = m$. In case $\nu = 2$, we see that the genus of C is equal to 1 (if $m = e = 2$) or 0 (if $m = 1$ and $e = 3$). Thus we also get $\mathrm{Gon}(C) = m$. □

EXAMPLE 4. *Let C be the transform of an irreducible plane curve Γ of degree m by a general quadratic transformation. Then C is of degree $2m$ and has three ordinary m-fold singular points other than the singular points of Γ. Since a general line is transformed into a conic, we have a rational function Φ on $\mathbf{P}^2$ of degree two which induces a rational function φ on C with $\deg \varphi \leq m - 1$. In this case, we have $d/\nu = 2$, but $\mathrm{Gon}(C) = \mathrm{Gon}(\Gamma) < d - \nu$. Cf. Lemma 5. As a consequence, we conclude that the condition in Theorem 3, (ii) is sharp.*

EXAMPLE 5. *Let C be the plane curve of degree $2m + 1$ with $m \geq 2$ defined by the equation: $y^{m+1} - (x^m + x^{2m+1}) = 0$. We have $\mathrm{Gon}(C) = m + 1$.*

PROOF. The point $(0, 0)$ is a singular point with mutiplicity sequence (m) and C also has a singular point with multiplicity sequence (m, m) on the line at infinity. We have $d = 2m + 1$, $\nu = m$, $n = 3$ and $\eta = 3$. Thus $d/\nu = 2 + 1/m > 2$. By Theorem 3, (ii), we infer that $\mathrm{Gon}(C) = d - \nu = m + 1$. □

EXAMPLE 6. *Let C be the Fermat curve: $x^m + y^m - 1 = 0$. Take a rational function $\Phi = y/(x - 1)$ on $\mathbf{P}^2$. Let φ be the rational function on C induced by Φ. We know that*

$\mathrm{Gon}(C) = m - 1 = \deg \varphi$. *By the way*, *we have* $\mathbf{C}(C) = \mathbf{C}(x, \varphi) = \mathbf{C}(C')$, *where the curve* C' *is defined by the equation*:

$$\varphi^m (x-1)^{m-1} + (x^m - 1)/(x-1) = 0\,.$$

In this case, the curve C' *has two singular points with multiplicity sequences* (m) *and* $(m-1, m-1)$.

EXAMPLE 7. *Let* C *be the curve of degree* 9 *defined by the equation*:

$$y(x-a_1)^5(x-a_2)^3 - c(y-b_1)^5(y-b_2)^3 = 0\,,$$

where the a_i*'s and the* b_i*'s are mutually distinct*, *respectively and the constant* c *is generally chosen. Then we have* $\mathrm{Gon}(C) = 4$.

PROOF. The curve C has two ordinary singular points of multiplicities 5 and 3, two singular points with multiplicity sequence $(3, 2)$. We have $\nu = 5$ and $\eta = 12/5$. By Theorem 3, (iii), we conclude that $\mathrm{Gon}(C) = 9 - 5 = 4$. In this example, we cannot apply Theorem 2. □

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