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On the Parametric Decomposition of Powers of Parameter Ideals in a Noetherian Local Ring

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Abstract. There is given a characterization of Noetherian local rings A with $d = \dim A \geq 2$, in which the equality $(a_i \mid 1 \leq i \leq d)^n = \bigcap_{\alpha}(a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots, a_d^{\alpha_d})$ holds true for all systems $a_1, a_2, \cdots, a_d$ of parameters and integers $n \geq 1$, where the suffix α runs over $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbf{Z}^d$ such that $\alpha_i \geq 1$ for all $1 \leq i \leq d$ and $\sum_{i=1}^{d} \alpha_i = d + n - 1$.

1. Introduction

Let A be a commutative ring and let $\underline{a} = a_1, a_2, \cdots, a_d$ $(d \geq 1)$ be a sequence of elements in A. We denote by $(\underline{a}) = (a_1, a_2, \cdots, a_d)$ the ideal in A generated by $a_1, a_2, \cdots, a_d$. For each integer $n \geq 1$ let

$$\Lambda_{d,n} = \left\{ (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbf{Z}^d \;\middle|\; \alpha_i \geq 1 \text{ for all } 1 \leq i \leq d \text{ and } \sum_{i=1}^{d} \alpha_i = d + n - 1 \right\}.$$

Let $(\underline{a}; \alpha) = (a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots, a_d^{\alpha_d})$ for each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \Lambda_{d,n}$. In this paper we are interested in the question of when the equality $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ $(n \geq 1)$ holds true for a given system $\underline{a} = a_1, a_2, \cdots, a_d$ $(d = \dim A)$ of parameters in a Noetherian local ring A, and our main result partially answers the question in the following way.

THEOREM 1.1. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 2$. Let $\mathrm{H}^0_{\mathfrak{m}}(A) = \bigcup_{n \geq 1}[(0) :_A \mathfrak{m}^n]$ denote the 0^{th} local cohomology module of A. Then the following two conditions are equivalent.*

(1) *$A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is a Cohen-Macaulay ring and $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(A) = (0)$.*

(2) *The equality*

$$(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$$

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holds true for all systems $\underline{a} = a_1, a_2, \cdots, a_d$ *of parameters and integers* $n \geq 1$.

When this is the case, A is a very special kind of Buchsbaum ring, so that every system of parameters in A forms a d-sequence.

We note here that condition (2) in Theorem 1.1 is always satisfied if $d = 1$, which shows the assumption that $d = \dim A \geq 2$ is crucial in Theorem 1.1.

In general, one has the inclusion $(\underline{a})^n \subseteq \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ for all integers $n \geq 1$, and W. Heinzer, L. J. Ratliff Jr., and K. Shah [HRS, Theorem 2.4] proved that the equality

$$(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$$

holds true for all $n \geq 1$, if the sequence $\underline{a} = a_1, a_2, \cdots, a_d$ is A-regular. The converse is also true, if A is a Noetherian local ring, $(\underline{a}) \subsetneq A$, and each a_i is a non-zerodivisor in A ([GS, Theorem (1.1)]). A Noetherian local ring A with $d = \dim A \geq 1$ is, therefore, necessarily a Cohen-Macaulay ring, if $\dim A/\mathfrak{p} = d$ for all $\mathfrak{p} \in \operatorname{Ass} A$ and if A contains a system $\underline{a} = a_1, a_2, \cdots, a_d$ of parameters for which the equality $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ holds true for all integers $n \geq 1$ ([GS, Corollary 3.7]). Our Theorem 1.1 gives an answer also to the question of whether the converse of [HRS, Theorem 2.4] holds true, showing that the assumption in [GS, Corollary 3.7] that $\dim A/\mathfrak{p} = d$ for all $\mathfrak{p} \in \operatorname{Ass} A$ and the one in [GS,Theorem (1.1)] that each a_i is a non-zerodivisor are not superfluous.

We now briefly explain how this paper is organized. The proof of Theorem 1.1 will be given in Section 3. A Noetherian local ring with $d = \dim A \geq 1$ is not necessarily Cohen-Macaulay, even if depth $A \geq d - 1$ and A contains a system $\underline{a} = a_1, a_2, \cdots, a_d$ of parameters such that $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ for all $n \geq 1$. We will give an example in Section 2 (Corollary (2.3)). In Section 4 we shall explore generalized Cohen-Macaulay local rings A with $\dim A = 2$ in order to show that the ring $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is Cohen-Macaulay, once A contains a standard system a_1, a_2 of parameters such that $(a_1, a_2)^n = \bigcap_{\alpha \in \Lambda_{2,n}} (a_1, a_2; \alpha)$ for *some* $n \geq 2$. The authors do not know whether the assertion is true or not also for higher-dimensional generalized Cohen-Macaulay local rings. A Noetherian local ring A with $d = \dim A \geq 1$ may contain two systems $\underline{a} = a_1, a_2, \cdots, a_d$ and $\underline{b} = b_1, b_2, \cdots, b_d$ of parameters such that $Q = (\underline{a}) = (\underline{b})$ in A, $Q^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ for all $n \geq 1$, but $Q^n \neq \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{b}; \alpha)$ for *any* $n \geq 2$. Thus the *parametric* decomposition $Q^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ of powers of the parameter ideal $Q = (\underline{a})$ heavily depends on the choice of systems $\underline{a} = a_1, a_2, \cdots, a_d$ of generators. We will explore in Section 5 such an example of dimension two.

Throughout this paper, let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Let $\mathrm{H}^i_{\mathfrak{m}}(*)$ $(i \in \mathbf{Z})$ be the local cohomology functors of A with respect to the maximal ideal $\mathfrak{m}$. Let $\ell_A(*)$ and $\mu_A(*)$ denote, respectively, the length and the number of generators. For a given system $\underline{a} = a_1, a_2, \cdots, a_d$ of parameters in A let $\mathrm{e}^0_{(\underline{a})}(A)$ denote the multiplicity of A with respect to the ideal $(\underline{a})$.

2. Non-Cohen-Macaulay local rings with high depth containing parameter ideals whose powers all possess parametric decompositions

In this section we shall show that for a given integer $d \geq 2$, there exists a Noetherian local ring A with $d = \dim A$ and depth $A = d-1$, containing at least one system $\underline{a} = a_1, a_2, \cdots, a_d$ of parameters such that $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ for all $n \geq 1$.

Let us begin with the following, which is a direct consequence of [HRS, Theorem 2.4] via the principle of idealization. We shall note a brief proof for the sake of completeness.

LEMMA 2.1. *Let R be a commutative ring and let M be an R-module. Let $\underline{a} = a_1, a_2, \cdots, a_d$ ($d \geq 1$) be a sequence of elements in R and assume that $\underline{a}$ is M-regular. Then*

$$(\underline{a})^n M = \bigcap_{\alpha \in \Lambda_{d,n}} [(\underline{a}; \alpha) M]$$

for all $n \geq 1$.

PROOF. We may assume that the sequence $\underline{a}$ is also R-regular (replace R by the polynomial ring $\mathbf{Z}[X_1, X_2, \cdots, X_d]$ and $\underline{a}$ by $\underline{X} = X_1, X_2, \cdots, X_d$). Let $S = R \ltimes M$ be the idealization of M over R. Hence $S = R \oplus M$ as additive groups and the multiplication in S is defined by $(a, x) \cdot (b, y) = (ab, ay + bx)$. We put $f_i = (a_i, 0)$ $(1 \leq i \leq d)$. Then, since the sequence $\underline{f} = f_1, f_2, \cdots, f_d$ is S-regular, for all $n \geq 1$ we have by [HRS, Theorem 2.4] that

$$(\underline{f})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{f}; \alpha)$$

in the ring S. Hence $(\underline{a})^n M = \bigcap_{\alpha \in \Lambda_{d,n}} [(\underline{a}; \alpha) M]$, because $(\underline{f})^n = (\underline{a})^n \times (\underline{a})^n M$ and $(\underline{f}; \alpha) = (\underline{a}; \alpha) \times [(\underline{a}; \alpha) M]$ for all $n \geq 1$ and $\alpha \in \Lambda_{d,n}$. □

The local rings A cited in the following are exactly *approximately Cohen-Macaulay* rings in the sense of [G, Theorem (1.1)].

PROPOSITION 2.2. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 2$. Let I ($(0) \neq I \subsetneq A$) be an ideal in A, and assume that A/I is a Cohen-Macaulay ring with $\dim A/I = d$ and that I is a Cohen-Macaulay A-module with $\dim_A I = d - 1$. Let $\underline{a} = a_1, a_2, \cdots, a_d$ be a system of parameters in A such that $a_1 I = (0)$. Then*

$$(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$$

for all $n \geq 1$.

PROOF. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \Lambda_{d,n}$. Then

$$(\underline{a}; \alpha) \cap I = (\underline{a}; \alpha) I = (a_2^{\alpha_2}, \cdots, a_d^{\alpha_d}) I,$$

since $a_1^{\alpha_1} I = (0)$ and the sequence $a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots, a_d^{\alpha_d}$ is A/I-regular. Let $x \in \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ and let $\overline{*}$ denote the reduction mod I. Then

$$\overline{x} \in \bigcap_{\alpha \in \Lambda_{d,n}} (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}; \alpha) = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d})^n$$

by [HRS, Theorem 2.4], because the sequence $\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}$ is A/I-regular. Hence $x \in (\underline{a})^n + I$ so that we have

$$\begin{aligned} \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha) &= (\underline{a})^n + \left[\left(\bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha) \right) \cap I \right] \\ &= (\underline{a})^n + \bigcap_{\alpha \in \Lambda_{d,n}} [(\underline{a}; \alpha) \cap I] \\ &= (\underline{a})^n + \left[\bigcap_{(\alpha_1, \alpha_2, \cdots, \alpha_d) \in \Lambda_{d,n}} (a_2^{\alpha_2}, \cdots, a_d^{\alpha_d}) I \right]. \end{aligned}$$

Notice that the sequence $a_2, \cdots, a_d$ is I-regular, because $a_2, \cdots, a_d$ is a system of parameters for the Cohen-Macaulay A-module I. By Lemma 2.1 we then have

$$\begin{aligned} \bigcap_{(\alpha_1, \alpha_2, \cdots, \alpha_d) \in \Lambda_{d,n}} (a_2^{\alpha_2}, \cdots, a_d^{\alpha_d}) I &\subseteq \bigcap_{(\beta_2, \cdots, \beta_d) \in \Lambda_{d-1,n}} [(a_2^{\beta_2}, \cdots, a_d^{\beta_d}) I] \\ &= (a_2, \cdots, a_d)^n I \\ &\subseteq (\underline{a})^n, \end{aligned}$$

since $(1, \beta_2, \cdots, \beta_d) \in \Lambda_{d,n}$ for every $\beta = (\beta_2, \cdots, \beta_d) \in \Lambda_{d-1,n}$. Thus

$$(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$$

as is claimed. $\square$

The reader may consult [G] for characterizations and examples of approximately Cohen-Macaulay rings. Here let us note the simplest one for which, as an immediate consequence of Proposition 2.2, we have the following.

EXAMPLE 2.3 ([G, Example (3.5) (5)]). Let R be Cohen-Macaulay local ring with $d = \dim R \geq 2$ and let M be a Cohen-Macaulay R-module with $\dim_R M = d-1$. Let $A = R \ltimes M$. Then $\dim A = d$, depth $A = d-1$, and A contains a system $a_1, a_2, \cdots, a_d$ of parameters such that $a_2, a_3, \cdots, a_d$ forms an A-regular sequence and $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ for all $n \geq 1$.

3. Proof of Theorem 1.1

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. We then have the following.

LEMMA 3.1. *The following conditions are equivalent.*

(1) *A/W is a Cohen-Macaulay ring.*

(2) *There exists an integer $\ell \gg 0$ such that for every system $\underline{a} = a_1, a_2, \cdots, a_d$ of parameters contained in $\mathfrak{m}^\ell$, the equality $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ holds true for all $n \geq 1$.*

PROOF. Choose an integer $N \gg 0$ so that $W \cap \mathfrak{m}^N = (0)$.

(1) $\Rightarrow$ (2). Let $\underline{a} = a_1, a_2, \cdots, a_d$ be any system of parameters in A. Then, since the sequence $a_1, a_2, \cdots, a_d$ is A/W-regular, it follows for the same reason as in the proof of Proposition (2.2) that

$$\begin{aligned}\bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha) &= (\underline{a})^n + \left[\left(\bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)\right) \cap W\right] \\ &= (\underline{a})^n + \bigcap_{\alpha \in \Lambda_{d,n}} [(\underline{a}; \alpha)W].\end{aligned}$$

Thus $(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$ for all $n \geq 1$, if $(\underline{a}) \subseteq \mathfrak{m}^N$, or more generally, if $(\underline{a})W = (0)$.

(2) $\Rightarrow$ (1). Choose a system $\underline{a} = a_1, a_2, \cdots, a_d$ of parameters in $\mathfrak{m}^{\ell+N}$ so that each a_i is A/W-regular. We denote by $\overline{*}$ the reduction mod W. Let $\varphi \in \bigcap_{\alpha \in \Lambda_{d,n}} (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}; \alpha)$ and write $\varphi = \overline{x}$ with $x \in (\underline{a})$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \Lambda_{d,n}$. Then

$$x \in [(\underline{a}; \alpha) + W] \cap (\underline{a}) = (\underline{a}; \alpha) + [W \cap (\underline{a})] = (\underline{a}; \alpha),$$

since $W \cap (\underline{a}) \subseteq W \cap \mathfrak{m}^N = (0)$. Therefore $x \in \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha) = (\underline{a})^n$, because $(\underline{a}) \subseteq \mathfrak{m}^\ell$. Hence $\varphi = \bar{x} \in (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d})^n$. Thus $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}; \alpha)$ for all $n \geq 1$, whence by [GS, Theorem (1.1)] A/W is a Cohen-Macaulay ring. □

Thanks to Lemma 3.1 and the proof of the implication (1) $\Rightarrow$ (2), we have the following.

COROLLARY 3.2. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Then $A/\mathrm{H}^0_\mathfrak{m}(A)$ is a Cohen-Macaulay ring, if the equality*

$$(\underline{a})^n = \bigcap_{\alpha \in \Lambda_{d,n}} (\underline{a}; \alpha)$$

holds true for all systems $\underline{a} = a_1, a_2, \cdots, a_d$ of parameter in A and $n \geq 1$. The converse is also true, when $\mathfrak{m}\mathrm{H}^0_\mathfrak{m}(A) = (0)$.

We are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1 We put $W = \mathrm{H}^0_\mathfrak{m}(A)$. By Corollary 3.2 it suffices to show that $\mathfrak{m}W = (0)$, when condition (2) is satisfied. Let $a \in \mathfrak{m}$ such that $\dim A/aA = d - 1$ and extend it to a system $a, x_2, \cdots, x_d$ of parameter in A such that $(x_2, \cdots, x_d)W = (0)$. Let $n \geq 1$ be an integer and put $a_i = a + x_i^n$ for $2 \leq i \leq d$. We look at the system

$a = a_1, a_2, \cdots, a_d$ of parameters in A. Then $a_i W = aW$ for all $1 \leq i \leq d$, so that

$$aW \subseteq (a) \cap (a_2, \cdots, a_d) \subseteq \bigcap_{\alpha \in \Lambda_{d,2}} (\underline{a}; \alpha) = (\underline{a})^2 .$$

Hence $aW \subseteq (a, x_2^n, \cdots, x_d^n)^2 \subseteq (a^2) + (x_2^n, \cdots, x_d^n) \subseteq (a^2) + \mathfrak{m}^n$ for all $n \geq 1$. Thus $aW \subseteq (a^2)$. Let $x \in W$ and write $ax = a^2 y$ with $y \in A$. Then $x - ay \in (0) : a$. Hence $W \subseteq (a) + [(0) : a]$, so that $W = aW + [(0) : a]$, because $(a) \cap W = aW$ and $(0) : a \subseteq W$ (recall that a is A/W-regular, since A/W is by Corollary 3.2 a Cohen-Macaulay ring). Thus $W = (0) : a$, whence $\mathfrak{m}W = (0)$, because the maximal ideal $\mathfrak{m}$ is generated by the elements a with $\dim A/aA = d - 1$. □

4. Generalized Cohen-Macaulay local rings with $\dim A = 2$

In this section we shall explore standard systems of parameters in generalized Cohen-Macaulay local rings with $\dim A = 2$.

To begin with let R be a commutative ring and $a, b \in R$. Let M be an R-module. Then we have the following.

LEMMA 4.1 (K. Nishida). *Let $n \geq 2$. Then there exists an exact sequence*

$$0 \to M \Big/ \bigcap_{i=1}^{n} (a^{n+1-i}, b^i)M \xrightarrow{\varphi} \bigoplus_{i=1}^{n} M/(a^{n+1-i}, b^i)M \xrightarrow{\psi} \bigoplus_{i=1}^{n-1} M/(a^{n-i}, b^i)M \to 0$$

of R-modules, where the homomorphism φ and ψ are defined by

$$\varphi\left(x \bmod \bigcap_{i=1}^{n} (a^{n+1-i}, b^i)M\right) = \{x \bmod (a^{n+1-i}, b^i)M\}_{1 \leq i \leq n} \text{ and}$$

$$\psi(\{x_i \bmod (a^{n+1-i}, b^i)M\}_{1 \leq i \leq n}) = \{x_i - x_{i+1} \bmod (a^{n-i}, b^i)M\}_{1 \leq i \leq n-1} .$$

PROOF (K. Kurano). We certainly have φ is a monomorphism and $\psi\varphi = 0$. It is standard to show that ψ is an epimorphism. Let us check that $\operatorname{Ker}\psi \subseteq \operatorname{Im}\varphi$. Let $\alpha \in \operatorname{Ker}\psi$ and write

$$\alpha = \{x_i \bmod (a^{n+1-i}, b^i)M\}_{1 \leq i \leq n}$$

with $x_i \in M$. Then $x_i - x_{i+1} \in (a^{n-i}, b^i)M$ for all $1 \leq i \leq n - 1$. We write

$$x_i = x_{i+1} + a^{n-i} f_i + b^i g_i \tag{4.2}$$

with $f_i, g_i \in M$. Let $x = x_1 - \sum_{i=1}^{n-1} b^i g_i$. We then have the following.

CLAIM. $x = x_i + \sum_{j=1}^{i-1} a^{n-j} f_j - \sum_{j=i}^{n-1} b^j g_j$ *for all* $1 \leq i \leq n$.

PROOF OF CLAIM. We may assume that $1 \leq i < n$ and our equality holds true for i. Then

$$\begin{aligned} x &= (x_i - b^i g_i) + \sum_{j=1}^{i-1} a^{n-j} f_j - \sum_{j=i+1}^{n-1} b^j g_j \\ &= (x_{i+1} + a^{n-i} f_i) + \sum_{j=1}^{i-1} a^{n-j} f_j - \sum_{j=i+1}^{n-1} b^j g_j \quad \text{(by (4.2))} \\ &= x_{i+1} + \sum_{j=1}^{i} a^{n-j} f_j - \sum_{j=i+1}^{n-1} b^j g_j \end{aligned}$$

as is claimed. □

Consequently, $x - x_i \in (a^{n+1-i}, b^i)M$ for all $1 \leq i \leq n$, so that

$$\alpha = \varphi(\{x \bmod (a^{n+1-i}, b^i)M\}_{1 \leq i \leq n}) .$$

Thus $\alpha \in \operatorname{Im} \varphi$. □

The following is an immediate consequence of Lemma 4.1.

PROPOSITION 4.3. *Let A be a Noetherian local ring with* $\dim A = 2$ *and let* $\underline{a} = a_1, a_2$ *be a system of parameters in A. Then*

$$\ell_A\left(A \Big/ \bigcap_{\alpha \in \Lambda_{2,n}} (\underline{a}; \alpha)\right) = \sum_{i=1}^{n} \ell_A(A/(a_1^{n+1-i}, a_2^i)) - \sum_{i=1}^{n-1} \ell_A(A/(a_1^{n-i}, a_2^i))$$

for all $n \geq 2$.

Now let A be a two-dimensional generalized Cohen-Macaulay local ring with the Stückrad-Vogel invariant $\mathrm{I}(A)$. Hence the A-module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated and $\mathrm{I}(A) = h^0(A) + h^1(A)$, where $h^i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ (cf. [SV, Appendix, Theorem and Definition 17]). Let a_1, a_2 be a standard system of parameters in A, that is a_1, a_2 is a system of parameters in A and the equalities

$$\begin{aligned} \mathrm{I}(A) &= \ell_A(A/(a_1, a_2)) - \mathrm{e}^0_{(a_1,a_2)}(A) \\ &= \ell_A(A/(a_1^m, a_2^n)) - \mathrm{e}^0_{(a_1^m,a_2^n)}(A) \\ &= \ell_A(A/(a_1^m, a_2^n)) - mn\mathrm{e}^0_{(a_1,a_2)}(A) \end{aligned} \tag{4.4}$$

hold true for all integers $m, n \geq 1$. (We note here that there exists an integer $\ell \gg 0$ such that every system of parameters contained in $\mathfrak{m}^\ell$ is standard.) Let $Q = (a_1, a_2)$. Then there exist integers $\mathrm{e}^i_Q(A)$ $(i = 0, 1, 2)$ such that

$$\ell_A(A/Q^{n+1}) = \mathrm{e}^0_Q(A)\binom{n+2}{2} - \mathrm{e}^1_Q(A)\binom{n+1}{1} + \mathrm{e}^2_Q(A) \tag{4.5}$$

for all $n \geq 0$ ([S]). We furthermore have that $\mathrm{e}_Q^1(A) = -h^1(A)$ and $\mathrm{e}_Q^2(A) = h^0(A)$, whence $\mathrm{I}(A) = \mathrm{e}_Q^2(A) - \mathrm{e}_Q^1(A)$ ([S]).

With this notation we have the following.

COROLLARY 4.6. *Let $n \geq 1$ be an integer. Then*

(1) $\ell_A(A/\bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)) = \binom{n+1}{2}\mathrm{e}_Q^0(A) + \mathrm{I}(A)$ *and*

(2) $\ell_A([\bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)]/Q^n) = \mathrm{e}_Q^1(A)(1-n)$.

Hence $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)$ for all $n \geq 1$ if and only if $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)$ for some $n \geq 2$, or equivalently $Q^2 = \bigcap_{\alpha \in \Lambda_{2,2}}(\underline{a};\alpha)$.

PROOF. Let $e_i = \mathrm{e}_Q^i(A)$ $(i = 0, 1, 2)$. Then by Proposition 4.3 and (4.4) we get

$$\begin{aligned}\ell_A\Bigg(A\Big/\bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)\Bigg) &= \sum_{i=1}^{n}[(n+1-i)ie_0 + \mathrm{I}(A)] - \sum_{i=1}^{n-1}[(n-i)ie_0 + \mathrm{I}(A)] \\ &= \frac{n(n+1)}{2}e_0 + \mathrm{I}(A) \\ &= e_0\binom{n+1}{2} + \mathrm{I}(A)\,.\end{aligned}$$

Hence by (4.5)

$$\begin{aligned}\ell_A\Bigg(\Big[\bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)\Big]\Big/Q^n\Bigg) &= \Bigg[e_0\binom{n+1}{2} - e_1 n + e_2\Bigg] - \Bigg[e_0\binom{n+1}{2} + \mathrm{I}(A)\Bigg] \\ &= e_1(1-n)\,,\end{aligned}$$

because $\mathrm{I}(A) = e_2 - e_1$. □

We now come to the main result of this section. The authors do not know whether similar characterizations still hold true for higher-dimensional generalized Cohen-Macaulay rings.

THEOREM 4.7. *Suppose A is a generalized Cohen-Macaulay local ring with* $\dim A = 2$. *Let $\underline{a} = a_1, a_2$ be a standard system of parameters in A and put $Q = (a_1, a_2)$. Then the following conditions are equivalent.*

(1) $A/\mathrm{H}_{\mathfrak{m}}^0(A)$ *is a Cohen-Macaulay ring.*

(2) $\sup_{n>0} \ell_A([\bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)]/Q^n) < \infty$.

(3) $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)$ *for all $n \geq 1$.*

(4) $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}}(\underline{a};\alpha)$ *for some $n \geq 2$.*

(5) $Q^2 = \bigcap_{\alpha \in \Lambda_{2,2}}(\underline{a};\alpha)$.

(6) $(a_1) \cap (a_2) \subseteq (a_1, a_2)^2$.

PROOF. The ring $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is a Cohen-Macaulay ring if and only if $h^1(A) = 0$. Since $\mathrm{e}^1_Q(A) = -h^1(A)$, by Corollary 4.6 (2) the latter condition is equivalent to saying that

$$\sup_{n>0} \ell_A\Big(\Big[\bigcap_{\alpha\in\Lambda_{2,n}} (\underline{a};\alpha)\Big]\Big/Q^n\Big) = \sup_{n>0} \mathrm{e}^1_Q(A)(1-n) < \infty\,.$$

We then have by Corollary 4.6 (2) the equivalences (1)⇔(2)⇔(3)⇔(4)⇔(5). Since

$$\bigcap_{\alpha\in\Lambda_{2,2}} (\underline{a};\alpha) = (a_1^2, a_2)\cap(a_1, a_2^2) = (a_1^2, a_2^2) + [(a_1)\cap(a_2)]\,,$$

we get the equivalence (5)⇔(6). □

5. An example

Let us explore one example to illustrate our theorems. The example shows also that the *parametric* decomposition of powers of an ideal $(\underline{a})$ depends on the choice of systems of generators for the ideal $(\underline{a})$.

Let R be a three-dimensional regular local ring with the maximal ideal $\mathfrak{n}$ and let $\mathfrak{n} = (X, Y, Z)$ with $X, Y, Z \in R$. We put

$$A = R/(X,Y)\cap(Z)\,.$$

Let x, y and z denote the reduction of X, Y and Z mod $(X,Y)\cap(Z)$, respectively. Let $Q = (x+z, y)$. We then have the following.

EXAMPLE 5.1. (1) $Q^n = \bigcap_{\alpha\in\Lambda_{2,n}}(x+z, y;\alpha)$ and $\ell_A(A/Q^n) = (n^2+3n)/2$ for all $n \geq 1$.

(2) Let $b_1 = x+z$ and $b_2 = x+y+z$. Then $Q = (b_1, b_2)$ and for every $n \geq 1$

$$\ell_A\Big(A\Big/\bigcap_{\alpha\in\Lambda_{2,n}} (\underline{b};\alpha)\Big) = \begin{cases} \dfrac{n^2+2n}{2} & \text{if } n \text{ is even}\,,\\[2ex] \dfrac{(n+1)^2}{2} & \text{if } n \text{ is odd}\,.\end{cases}$$

Hence the function $\ell_A(A/\bigcap_{\alpha\in\Lambda_{2,n}}(\underline{b};\alpha))$ is not the polynomial in n, $Q^n \neq \bigcap_{\alpha\in\Lambda_{2,n}}(\underline{b};\alpha)$ for *any* $n \geq 2$, and

$$\sup_{n>0} \ell_A\Big(\Big[\bigcap_{\alpha\in\Lambda_{2,n}} (\underline{b};\alpha)\Big)\Big]\Big/Q^n\Big) = \infty\,.$$

PROOF. (1) Letting $I = (z)$, the first equality follows from Proposition 2,2, because $I \cong R/(X,Y)$ and $A/(z) = R/(Z)$. We put $a_1 = x+z$ and $a_2 = y$ and look at the exact sequence

$$0 \to R/(Y,Z) \xrightarrow{\ \phi\ } A \to R/(Z) \to 0 \tag{5.2}$$

of R-modules, where the homomorphism ϕ is defined by $\phi(1) = z$. Let $\ell, m \geq 1$ be integers. Then the sequence a_1^ℓ, a_2^m is $R/(Z)$-regular and so by (5.2), we get the exact sequence

$$0 \to R/(X^\ell, Y, Z) \to A/(a_1^\ell, a_2^m) \to R/(X^\ell, Y^m, Z) \to 0\,.$$

Hence $\ell_A(A/(a_1^\ell, a_2^m)) = \ell(m+1)$, so that by Proposition 4.3

$$\ell_A(A/Q^n) = \ell_A\left(A\Big/ \bigcap_{\alpha \in \Lambda_{2,n}} (\underline{a}; \alpha)\right) = \frac{n^2+3n}{2}$$

for all $n \geq 1$.

(2) Let $b_1 = x + z$ and $b_2 = x + y + z$. Then $Q = (b_1, b_2)$. Let $\ell, m \geq 1$ be integers. Then by (5.2) we have the exact sequence

$$0 \to R/((Y, Z) + (X^\ell, X^m)) \to A/(b_1^\ell, b_2^m) \to R/((Z) + (X^\ell, (X+Y)^m)) \to 0\,,$$

whence $\ell_A(A/(b_1^\ell, b_2^m)) = \ell m + \min\{\ell, m\}$. Consequently by Proposition 4.3 we get

$$\begin{aligned} \ell_A\left(A\Big/ \bigcap_{\alpha \in \Lambda_{2,n}} (\underline{b}; \alpha)\right) &= \frac{n^2+n}{2} + \sum_{i=1}^{n} \min\{n+1-i, i\} - \sum_{i=1}^{n-1} \min\{n-i, i\} \\ &= \frac{n^2+n}{2} + \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

Thus the function $\ell_A(A/[\bigcap_{\alpha \in \Lambda_{2,n}} (\underline{b}; \alpha)])$ of n is not the polynomial in n and $Q^n \neq \bigcap_{\alpha \in \Lambda_{2,n}} (\underline{b}; \alpha)$ for any $n \geq 2$. Letting $n = 2\ell$ with $\ell \geq 1$, we have

$$\ell_A\left(\left[\bigcap_{\alpha \in \Lambda_{2,n}} (\underline{b}; \alpha)\right]\Big/ Q^n\right) = \frac{n^2+3n}{2} - \frac{n^2+2n}{2} = \ell\,.$$

Hence $\sup_{n>0} \ell_A([\bigcap_{\alpha \in \Lambda_{2,n}} (\underline{b}; \alpha)]/Q^n) = \infty$. □

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