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Irrationality of Certain Lambert Series

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Abstract. Let q with $|q| > 1$ be an integer in an algebraic number field $\mathbf{K}$ given below, and $\{a_n\}$ a periodic sequence in $\mathbf{K}$ of period two, not identically zero. Let $f(q) = \sum_{n=1}^{\infty} a_n/(1-q^n)$. We prove that (i) If $\mathbf{K}$ is either the rational number field $\mathbb{Q}$ or an imaginary quadratic field, then $f(q) \notin \mathbf{K}$. (ii) For an algebraic integer q such that $|q| > 1$ and $|q^\sigma| < 1$ for any $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $q^\sigma \neq q$, if $k = \mathbb{Q}(q)$, then $f(q) \notin \mathbb{Q}(q)$. For example, the three numbers

$$1\,, \quad \sum_{n=1}^{\infty} \frac{1}{q^n - 1}\,, \quad \sum_{n=1}^{\infty} \frac{1}{q^n + 1}$$

are linearly independent over $\mathbb{Q}$ for every $q \in \mathbb{Z}$ with $|q| \geq 2$. Further, irrationality results of the special values of the functions

$$\sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}}\,, \qquad \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}R_{a(n+1)+b}} \quad (z \in \mathbb{C})$$

can be deduced, where $a > 0$, $b \geq 0$ are integers and R_n is a certain binary recurrence.

1. Introduction and the results

For any fixed $q \in \mathbb{C}$ with $|q| > 1$ and $z \in \mathbb{C}$, the q-logarithmic function $L_q(z)$ and the q-exponential $E_q(z)$ are defined by

$$L_q(z) := \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{z}{q^n - z} \quad (|z| < |q|)\,,$$

$$E_q(z) := 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q-1)\cdots(q^n - 1)} = \prod_{n=1}^{\infty}\left(1 + \frac{z}{q^n}\right),$$

respectively. Bézivin [2] showed that the numbers 1, $E_q^{(k)}(\alpha_i)$ ($i = 1, \cdots, m$, $k = 0, 1, \cdots, l$) are linearly independent over $\mathbb{Q}$, where $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $\alpha_i \in \mathbb{Q}^\times$ satisfy

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$\alpha_i \neq -q^\mu$ and $\alpha_i \neq \alpha_j q^\nu$ for all $\mu, \nu \in \mathbb{Z}$ with $\mu \geq 1$ and $i \neq j$. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{q^n + \alpha} \notin \mathbb{Q},$$

where $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $\alpha \in \mathbb{Q}^\times$ with $\alpha \neq -q^i$ $(i \geq 1)$. Under the same conditions on q and α, Borwein [3], [4] obtained irrationality measures for the numbers $\sum_{n=1}^{\infty} 1/(q^n + \alpha)$ and $\sum_{n=1}^{\infty} (-1)^n/(q^n + \alpha)$. These results include the irrationality of $L_2(1) = \sum_{n=1}^{\infty} 1/(2^n - 1)$ proved by Erdös [10]. Furthermore, Bundschuh and Väänaänen [6], and Matala-Aho and Väänänen [11] obtained quantitative irrationality results for the values of the q-logarithm both in the Archimedean and p-adic cases. In [7], Duverney generalized certain results obtained by Borwein [3], [4], and Bundschuh and Väänänen [6]. Recently, Van Assche [15] gave irrationality measures for the numbers $L_q(1)$ and $L_q(-1)$ by using little q-Legendre polynomials. In this paper, we prove irrationality results for certain Lambert series, which in particular implies the linear independence of the numbers $1,\ L_q(1),\ L_q(-1)$ with $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ by developing Borwein's idea in [4].

Let R_n be a binary recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \ (n \geq 0), \quad A_1, A_2 \in \mathbb{Q}^\times,\ R_0,\ R_1 \in \mathbb{Q}.$$

André-Jeannin [1] proved for some R_n the irrationality of the value of the function $f(x) = \sum_{n=1}^{\infty} x^n/R_n$ at a nonzero rational integer x in the disk of convergence of f, which gave the first proof of the irrationality of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/L_n$, where F_n and L_n are Fibonacci numbers and Lucas numbers, respectively. Prévost [13] extended this result to any rational x in the domain of meromorphy of f. Recently, Matala-aho and Prévost [12] obtained for some type of R_n irrationality measures for the number $\sum_{n=1}^{\infty} \gamma^n/R_{an}$, where γ belongs to an imaginary quadratic field, and $a > 0$ is an integer. We will prove for some R_n the irrationality of the numbers $\sum_{n=1}^{\infty} \gamma^n/R_{an+b}$ and $\sum_{n=1}^{\infty} \gamma^n/R_{an+b}R_{a(n+1)+b}$, where $a > 0,\ b \geq 0$ are integers and γ is a certain number in a real quadratic field (see Corollaries 2 and 3, below).

For an algebraic number α, we denote by $\overline{|\alpha|}$ the maximum of absolute values of its conjugates and by $\mathrm{den}\alpha$ the least positive integer such that $\alpha \cdot \mathrm{den}\alpha$ is an algebraic integer. We put $\mathbb{N} = \{0, 1, 2, \cdots\}$.

THEOREM 1. *Let* $\mathbf{K}$ *be either* $\mathbb{Q}$ *or an imaginary quadratic field. Assume that* q *is an integer in* $\mathbf{K}$ *with* $|q| > 1$ *and* $\{a_n\}$ *a periodic sequence in* $\mathbf{K}$ *of period two, not identically zero. Then*

$$\theta = \sum_{n=1}^{\infty} \frac{a_n}{1 - q^n} \notin \mathbf{K}.$$

COROLLARY 1. *Let $q \in \mathbb{Z}$ with $|q| \geq 2$ and $\{a_n\}, \{b_n\}$ be periodic sequences in $\mathbb{Q}$ of period two, not identically zero. Then the numbers*

$$1\,, \quad \sum_{n=1}^{\infty} \frac{a_n}{q^n - 1}\,, \quad \sum_{n=1}^{\infty} \frac{b_n}{q^n - 1}$$

are linearly independent over $\mathbb{Q}$ if and only if $\{a_n\}$ and $\{b_n\}$ are linearly independent over $\mathbb{Q}$.

PROOF. This follows immediately from Theorem 1.

EXAMPLE 1. Let $q \in \mathbb{Z}$ with $|q| \geq 2$. Then

$$1\,, \quad L_q(1) = \sum_{n=1}^{\infty} \frac{1}{q^n - 1}\,, \quad L_q(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{-1}{q^n + 1}$$

are linearly independent over $\mathbb{Q}$.

THEOREM 2. *Let q be an algebraic integer such that $|q| > 1$ and $|q^\sigma| < 1$ for any $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $q^\sigma \neq q$ and $\{a_n\}$ be a periodic sequence in $\mathbb{Q}(q)$ of period two, not identically zero. Then*

$$\theta = \sum_{n=1}^{\infty} \frac{a_n}{1 - q^n} \notin \mathbb{Q}(q)\,.$$

EXAMPLE 2. Let $\varepsilon\,(> 1)$ be the fundamental unit in a real quadratic field $\mathbb{Q}(\sqrt{m})$. Then

$$1\,, \quad \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n - 1}\,, \quad \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n + 1}$$

are linearly independent over $\mathbb{Q}(\sqrt{m})$. For example, the following numbers are linearly independent over $\mathbb{Q}(\sqrt{2})$.

$$1\,, \quad \sum_{n=1}^{\infty} \frac{1}{(1+\sqrt{2})^n - 1}\,, \quad \sum_{n=1}^{\infty} \frac{1}{(1+\sqrt{2})^n + 1}\,.$$

THEOREM 3. *Let q be a quadratic integer satisfying $|q| > 1$ and $|q^\sigma| < 1$ for any $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $q^\sigma \neq q$, γ a unit in $\mathbb{Q}(q)$ with $|\gamma| \leq 1$, and $\alpha \in \mathbb{Q}(q)^\times$ with $(\mathrm{den}(q^l\alpha))^4 < |q|$ for some $l \in \mathbb{N}$. Then*

$$\xi = \sum_{n=1}^{\infty} \frac{\gamma^n}{1 - \alpha q^n} \notin \mathbb{Q}(q)\,,$$

provided that $\alpha q^n \neq 1$ for all $n \geq 1$.

In the following Corollaries 2 and 3, we consider the binary recurrences $\{R_n\}_{n\geq 0}$ defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n\,, \quad A_1, A_2 \in \mathbb{Z}\setminus\{0\}\,, R_0,\ R_1 \in \mathbb{Z}\,.$$

We suppose that $R_n \neq 0$ for all $n \geq 1$, the corresponding polynomial $\Phi(X) = X^2 - A_1 X - A_2$ is irreducible in $\mathbb{Q}[X]$, and $\triangle = A_1^2 + 4A_2 > 0$. We can write R_n as

$$R_n = g_1\rho_1^n + g_2\rho_2^n\ (n \geq 0), \quad g_1, g_2 \in \mathbb{Q}(\rho_1)^\times\,, \tag{1}$$

where ρ_1 and ρ_2 are the roots of $\Phi(X)$. We may assume $|\rho_1| > |\rho_2|$, since $A_1 \neq 0$ and $\triangle > 0$.

For $a, b \in \mathbb{N}$ with $a \neq 0$, we define

$$R(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} \quad (|z| < |\rho_1|^a)\,.$$

This function can be extended to a meromorphic function on the whole complex plane $\mathbb{C}$ with poles $\{(\rho_1^{n+1}/\rho_2^n)^a \mid n \geq 0\}$, since

$$\sum_{n=1}^{\infty} \frac{z^n}{1-\alpha q^n} = \sum_{m=1}^{\infty} \frac{\alpha^{-m} z}{z - q^m} \quad (|z| < |q|)$$

for any complex numbers q and α with $|q| > 1$ and $|\alpha| \geq 1$, and so

$$\sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} = \sum_{n=1}^{i} \frac{z^n}{R_{an+b}} - \frac{z^{i+1}}{g_1\rho_1^{ai+b}} \sum_{n=0}^{\infty} \frac{(-(g_2/g_1)(\rho_2/\rho_1)^{ai+b})^n}{z - \rho_1^a(\rho_1/\rho_2)^{an}}\,, \tag{2}$$

where i is chosen as $|(g_2/g_1)(\rho_2/\rho_1)^{ai+b}| < 1$. We denote the function again by $R(z)$.

COROLLARY 2. *Let R_n be a binary recurrence given by* (1) *and $a, b \in \mathbb{N}$ with $a \neq 0$. Assume that g_1/g_2 and ρ_1/ρ_2 are units in $\mathbb{Q}(\rho_1)$ and $\gamma \in \mathbb{Q}(\rho_1)^\times$ is not a pole of $R(z)$ with $(\mathrm{den}(\rho_1^a/\gamma))^4 < |\rho_1/\rho_2|^a$. Then we have $R(\gamma) \notin \mathbb{Q}(\rho_1)$.*

PROOF. Apply Theorem 3 to the last sum in (2).

EXAMPLE 3. Let F_n and L_n be Fibonacci numbers and Lucas numbers defined by $F_{n+2} = F_{n+1} + F_n\ (n \geq 0),\ F_0 = 0,\ F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n\ (n \geq 0),\ L_0 = 2,\ L_1 = 1$, respectively. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{F_{an+b}}\,, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{an+b}}\,, \quad \sum_{n=1}^{\infty} \frac{1}{L_{an+b}}\,, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{an+b}} \notin \mathbb{Q}(\sqrt{5})\,.$$

André-Jeannin[1] proved that each of these numbers is irrational. We remark that the numbers $\sum_{n=1}^{\infty} 1/F_{2n+1}$ and $\sum_{n=1}^{\infty} 1/L_{2n}$ are transcendental (cf. [8], [9]).

EXAMPLE 4. Let R_n be defined by $R_{n+2} = 4R_{n+1} - 2R_n$ $(n \geq 0)$, $R_0 = 1$, $R_1 = 2$, so that $R_n = (2+\sqrt{2})^n/2 + (2-\sqrt{2})^n/2$ $(n \geq 0)$. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{R_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{R_{an+b}} \notin \mathbb{Q}(\sqrt{2}).$$

Next we consider, for any given $a, b \in \mathbb{N}$ with $a \neq 0$, the function

$$S(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}R_{a(n+1)+b}} \quad (|z| < |\rho_1|^{2a}),$$

which can be extended to a meromorphic function on the whole plane with the poles $\{(\rho_1^{n+2}/\rho_2^n)^a \mid n \geq 0\}$. We denote it again by $S(z)$.

COROLLARY 3. *Let R_n be a binary recurrence given by* (1) *and $a, b \in \mathbb{N}$ with $a \neq 0$. Assume that g_1/g_2 and ρ_1/ρ_2 are units in $\mathbb{Q}(\rho_1)$. If $\gamma \in \mathbb{Q}(\rho_1)^\times$ is not a pole of $S(z)$ with $(\mathrm{den}(\rho_1^{2a}/\gamma))^4 < |\rho_1/\rho_2|^a$, then $S(\gamma) \notin \mathbb{Q}(\rho_1)$, provided $\gamma \neq (-A_2)^a$. If $\gamma = (-A_2)^a$, we have*

$$S(\gamma) = \sum_{n=1}^{\infty} \frac{\gamma^n}{R_{an+b}R_{a(n+1)+b}} = \frac{\rho_2^a}{g_1\rho_1^b(\rho_1^a - \rho_2^a)R_{a+b}} \in \mathbb{Q}(\rho_1).$$

PROOF. Since $\rho_1\rho_2 = -A_2$, we have

$$\rho_1{}^a R_{a(n+1)+b} - (-A_2)^a R_{an+b} = g_1\rho_1{}^b(\rho_1^{2a} - (-A_2)^a)(\rho_1^a)^n.$$

We multiply both sides by $z^n/(\rho_1^{a(n+1)}R_{an+b}R_{a(n+1)+b})$ and sum up from n equals 1 to infinity. Then, we get

$$g_1\rho_1{}^b(\rho_1^a - \rho_2^a)\sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}R_{a(n+1)+b}} = \frac{\rho_2^a}{R_{a+b}} + \left(1 - \frac{(-A_2)^a}{z}\right)\sum_{n=1}^{\infty} \frac{(z/\rho_1^a)^n}{R_{an+b}}$$

$(|z| < |\rho_1|^{2a})$. Hence, the left-hand side can be extended to a meromorphic function, and Corollary 3 follows from Corollary 2.

EXAMPLE 5. Let F_n be Fibonacci numbers. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{F_{(2a-1)n+b}F_{(2a-1)(n+1)+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{2an+b}F_{2a(n+1)+b}} \notin \mathbb{Q}(\sqrt{5}).$$

The same holds for Lucas numbers. We put

$$T_l := \sum_{n=1}^{\infty} \frac{1}{F_nF_{n+l}}, \quad T_l^* := \sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+l}} \quad (l \geq 1).$$

Then Brousseau [5] and Rabinowitz [14] proved that

$$T_{2l} = \frac{1}{F_{2l}} \sum_{n=1}^{l} \frac{1}{F_{2n-1}F_{2n}}\,, \quad T_{2l+1} = \frac{1}{F_{2l+1}}\left(T_1 - \sum_{n=1}^{l} \frac{1}{F_{2n}F_{2n+1}}\right),$$

$$T_l^* = \frac{1}{F_l}\left(\frac{1-\sqrt{5}}{2}l + \sum_{n=1}^{l} \frac{F_{n-1}}{F_n}\right),$$

so that $T_{2l} \in \mathbb{Q}$ and $T_l^* \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$ for all $l \geq 1$. We see that $T_{2l+1} \notin \mathbb{Q}(\sqrt{5})$ for all $l \geq 0$, since the first sum in this example with $a = 1,\ b = 0$ implies

$$T_1 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} \notin \mathbb{Q}(\sqrt{5})\,.$$

EXAMPLE 6. Let R_n be the binary recurrence given in Example 4. Then

$$\sum_{n=1}^{\infty} \frac{2^n}{R_{2n-1}R_{2n+1}}\,, \quad \sum_{n=1}^{\infty} \frac{(-2)^n}{R_{2n-1}R_{2n+1}} \notin \mathbb{Q}(\sqrt{2})\,.$$

2. Lemmas

For the proof of theorems, we prepare some lemmas. Let $\{a_m\}_{m\geq 1}$ be a periodic sequence of complex numbers of period two, not identically zero. We put

$$\theta = \sum_{m=1}^{\infty} \frac{a_m}{1-q^m}\,,$$

where $q \in \mathbb{C}$ with $|q| > 1$. We start with the integral

$$F_n(q) = \frac{1}{2\pi i} \int_{|t|=1} \frac{(-1/t)\prod_{k=1}^{2n}(1-q^k/t)}{\prod_{k=1}^{n}(1-q^{2k}t)} \sum_{m=1}^{\infty} \frac{a_m}{1-q^m/t}dt, \tag{3}$$

which is a variant of that used by Borwein [4]. We note that the integrand is meromorphic in t provided $|q| > 1$. We use the notations

$$[n]_q! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n}\,, \quad [0]_q := 1\,,$$

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{[n]_q!}{[i]_q![n-i]_q!} \in \mathbb{Z}[q]\,.$$

In what follows, we denote $c_1, c_2, \cdots$ positive constants independent of n.

LEMMA 1.

$$F_n(q) = \sum_{i=1}^{n} \frac{\prod_{k=1}^{2n}(1-q^{k+2i})}{\prod_{\substack{k=1\\k\neq i}}^{n}(1-q^{2k-2i})}\left(\theta - \sum_{m=1}^{2i}\frac{a_m}{1-q^m}\right)$$

$$- \frac{1}{(2n-1)!}\left(\prod_{k=1}^{2n}\left(t-q^k\right)\prod_{k=1}^{n}\left(1-q^{2k}t\right)^{-1}\sum_{m=1}^{\infty}\frac{a_m}{t-q^m}\right)^{(2n-1)}\Bigg|_{t=0} \qquad (4)$$

PROOF. This can be proved by using the residue theorem similarly as the proof of Lemma 1 in [4].

We put $D_n(q) := \prod_{k=n+1}^{2n}(1-q^{2k})$. Then we have

$$|D_n(q)| \le c_1|q|^{3n^2+n}\,. \qquad (5)$$

LEMMA 2.

$$D_n(q)F_n(q) = A_n(q)\theta + B_n(q)\,, \qquad (6)$$

where $A_n(q),\ B_n(q) \in \mathbb{Z}[a_1, a_2, q]$.

PROOF. Since

$$\frac{1}{\prod_{\substack{k=1\\k\neq i}}^{n}(1-q^{2k-2i})} = \frac{q^{i(i-1)}}{\prod_{k=1}^{i-1}(q^{2k}-1)\prod_{k=1}^{n-i}(1-q^{2k})}\,,$$

we have by (4)

$$F_n(q) = \frac{1}{\prod_{k=1}^{n-1}\left(1-q^{2k}\right)}\sum_{i=1}^{n}(-1)^{i-1}q^{i(i-1)}\begin{bmatrix} n-1\\ i-1\end{bmatrix}_{q^2}\prod_{k=1}^{2n}(1-q^{k+2i})\left(\theta - \sum_{m=1}^{2i}\frac{a_m}{1-q^m}\right)$$

$$-\sum_{\substack{\lambda,\mu,\nu\ge 0\\ \lambda+\mu+\nu=2n-1}}\frac{1}{\lambda!\mu!\nu!}\left(\prod_{k=1}^{2n}(t-q^k)\right)^{(\lambda)}\Bigg|_{t=0}\left(\prod_{k=1}^{n}(1-q^{2k}t)^{-1}\right)^{(\mu)}\Bigg|_{t=0}\left(\sum_{m=1}^{\infty}\frac{a_m}{t-q^m}\right)^{(\nu)}\Bigg|_{t=0}$$

with

$$\left(\prod_{k=1}^{2n}(t-q^k)\right)^{(\lambda)}\Bigg|_{t=0} = \lambda!(-1)^{2n-\lambda} \sum_{\substack{\lambda_1+\cdots+\lambda_{2n}=2n-\lambda \\ \lambda_i=0,1}} q^{\lambda_1+2\lambda_2+\cdots+2n\lambda_{2n}}\,,$$

$$\left(\prod_{k=1}^{n}(1-q^{2k}t)^{-1}\right)^{(\mu)}\Bigg|_{t=0} = \mu! \sum_{\substack{\mu_1+\cdots+\mu_n=\mu \\ \mu_i\geq 0}} q^{2(\mu_1+2\mu_2+\cdots+n\mu_n)}\,,$$

$$\left(\sum_{m=1}^{\infty}\frac{a_m}{t-q^m}\right)^{(\nu)}\Bigg|_{t=0} = -\nu!\sum_{m=1}^{\infty}\frac{a_m}{(q^{\nu+1})^m} = \nu!(a_1q^{\nu+1}+a_2)\frac{1}{1-q^{2(\nu+1)}}\,.$$

Hence we get

$$F_n(q) = \frac{1}{\prod_{k=1}^{n-1}\left(1-q^{2k}\right)} \sum_{i=1}^{n}(-1)^{i-1}q^{i(i-1)}\begin{bmatrix} n-1 \\ i-1\end{bmatrix}_{q^2}\prod_{k=1}^{2n}(1-q^{k+2i})\left(\theta-\sum_{m=1}^{2i}\frac{a_m}{1-q^m}\right)$$
$$+\sum_{\substack{\lambda+\mu+\nu=2n-1 \\ \lambda,\mu,\nu\geq 0}} Q_{\lambda\mu\nu}(q)\frac{1}{1-q^{2(\nu+1)}} \tag{7}$$

with $Q_{\lambda\mu\nu}(q)$ a polynomial in $\mathbb{Z}[a_1,a_2,q]$ for all $\lambda,\mu,\nu\geq 0$. Here we note that

$$\prod_{k=1}^{2n}(1-q^{k+2i})\sum_{m=1}^{2i}\frac{a_m}{1-q^m}\in\mathbb{Z}[a_1,a_2,q]\,,\qquad i=1,2,\cdots,n\,,$$

and each of $\prod_{k=1}^{n-1}(1-q^{2k})$ and $1-q^{2l}$ $(l=1,\cdots,2n)$ divides $D_n(q)$ in $\mathbb{Z}[q]$. Therefore the lemma follows from (7).

LEMMA 3. *For large n, we have*

$$0<|F_n(q)|\leq c_3|q|^{-3n^2-2n}\,. \tag{8}$$

PROOF. Similarly to the proof of Lemma 4 in [4], the residue theorem applied exterior to the circle $|t|=1$ shows that

$$F_n(q)=\sum_{m=2n+1}^{\infty} I_m\,,\quad I_m=a_m\frac{\prod_{k=1}^{2n}(1-q^{k-m})}{\prod_{k=1}^{n}(1-q^{2k+m})}$$

for large n. Since $|I_m|\leq c_2|q|^{-n^2-n(m+1)}$, we get the upper bound for $|F_n(q)|$. Furthermore, if $a_1\neq 0$, it follows that,

$$F_n(q)=a_1\frac{\prod_{k=1}^{2n}(1-q^{k-2n-1})}{\prod_{k=1}^{n}(1-q^{2k+2n+1})}\left(1+\sum_{l=1}^{\infty}b_{nl}\right)$$

with

$$b_{nl} = \frac{a_{l+1}}{a_1} \prod_{k=1}^{n} \left(\frac{1 - q^{2k+2n+1}}{1 - q^{2k+2n+l+1}} \right) \prod_{k=1}^{2n} \left(\frac{1 - q^{k-2n-l-1}}{1 - q^{k-2n-1}} \right),$$

where $|b_{nl}| \le c_4 |q^{-n}|^l$. Hence we have $F_n(q) \ne 0$, since $\sum_{l=1}^{\infty} |b_{nl}| < 1$ for large n. The proof is similar in the case of $a_1 = 0, a_2 \ne 0$.

3. Proofs of Theorems 1 and 2

PROOF OF THEOREM 1. Let $\mathbf{K}$, q, and $\{a_m\}$ be as in Theorem 1. We may suppose that a_1 and a_2 are integers in $\mathbf{K}$. Assume that $\theta \in \mathbf{K}$ and let $d = \mathrm{den}\theta$. Then by (5), (6), and (8), we have

$$0 < d|A_n(q)\theta + B_n(q)| \le dc_5 |q|^{-n}$$

for large n; which is a contradiction.

PROOF OF THEOREM 2. Let q and $\{a_m\}$ be as in Theorem 2. We may suppose that a_1 and a_2 are integers in $\mathbb{Q}(q)$. Then we have again by (5), (6), and (8)

$$0 < |A_n(q)\theta + B_n(q)| \le c_6 |q|^{-n}$$

for large n. We assume that $\theta \in \mathbb{Q}(q)$ and evaluate the upper bound of $|A_n(q)^\sigma \theta^\sigma + B_n(q)^\sigma|$ for all $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $q^\sigma \ne q$. By (6) and (7), we have

$$A_n(q) = (1 - q^{2n}) \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \sum_{i=1}^{n} (-1)^{i-1} q^{i(i-1)} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_{q^2} \prod_{k=1}^{2n} (1 - q^{k+2i}) .$$

Since $|q^\sigma| < 1$, we get $|A_n(q)^\sigma| \le c_7 n$ for all $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $q^\sigma \ne q$. In the same way, we see that $|B_n(q)^\sigma| \le c_8 n^2$. Hence, $|A_n(q)^\sigma \theta^\sigma + B_n(q)^\sigma| \le c_9 n^2$ for all $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $q^\sigma \ne q$. Therefore, we have

$$1 \le |\mathbf{N}_{\mathbb{Q}(q)/\mathbb{Q}} \mathrm{den}\theta (A_n(q)\theta + B_n(q))| \le c_{10} n^{c_{11}} |q|^{-n}$$

for large n; which is a contradiction, and the proof is completed.

4. Proof of Theorem 3

Let q, α, and γ be as in Theorem 3. Since

$$\sum_{m=1}^{\infty} \frac{\gamma^m}{1 - \alpha q^l q^m} = \gamma^{-l} \left(\sum_{m=1}^{\infty} \frac{\gamma^m}{1 - \alpha q^m} - \sum_{m=1}^{l} \frac{\gamma^m}{1 - \alpha q^m} \right) \quad (l \ge 1) ,$$

we can assume that α satisfies $|\alpha| > 1$ and $|\alpha^\sigma| < 1$ for any $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\alpha^\sigma \neq \alpha$, by replacing α by $q^l\alpha$ with suitable l. We modify Borwein's integral in [4] as follows:

$$G_n(q,\alpha,\gamma) = \frac{1}{2\pi i}\int_{|t|=1}\prod_{k=1}^{n-1}\left(\frac{1-\alpha q^k/t}{1-q^k t}\right)\frac{-1/t}{1-q^n t}\sum_{m=1}^{\infty}\frac{\gamma^m}{1-\alpha q^m/t}dt\,.$$

By the residue theorem, we have

$$G_n(q,\alpha,\gamma) = \sum_{i=1}^{n}\prod_{k=1}^{n-1}\left(\frac{1-\alpha q^{k+i}}{1-q^k}\right)(-1)^{i-1}q^{i(i-1)/2}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q \gamma^{-i}\left(\xi - \sum_{m=1}^{i}\frac{\gamma^m}{1-\alpha q^m}\right)$$

$$+ \sum_{\substack{\lambda+\mu+\nu=n-2 \\ \lambda,\mu,\nu\geq 0}}(-1)^{n-\lambda}Q_{1\lambda}(q)Q_{2\mu}(q)\frac{\alpha^\mu}{1-\gamma^{-1}q^{\nu+1}}\,,$$

where

$$Q_{1\lambda}(q) = \sum_{\substack{\lambda_1+\cdots+\lambda_{n-1}=n-1-\lambda \\ \lambda_i=0,1}} q^{\lambda_1+2\lambda_2+\cdots+(n-1)\lambda_{n-1}}\,,$$

$$Q_{2\mu}(q) = \sum_{\substack{\mu_1+\cdots+\mu_n=\mu \\ \mu_i\geq 0}} q^{\mu_1+2\mu_2+\cdots+n\mu_n}\,, \qquad \lambda,\mu = 0,1,\cdots,n-2\,.$$

We put

$$D_n(q,\alpha,\gamma) := \prod_{k=1}^{n-1}(1-q^k)\prod_{k=1}^{n}(1-\alpha q^k)\prod_{k=1}^{n-1}(1-\gamma^{-1}q^k) \in \mathbb{Z}[q,\ \alpha,\ \gamma]\,.$$

Then we have $|D_n(q,\alpha,\gamma)| \leq c_{12}|\alpha\gamma^{-1}|^n|q|^{\frac{3}{2}n^2-\frac{1}{2}n}$. In the same way as the proof of Lemma 3, we have $0 < |G_n(q,\alpha,\gamma)| \leq c_{13}|\alpha^{-1}\gamma|^n|q|^{-\frac{3}{2}n^2-\frac{1}{2}n}$ for large n. Hence

$$|D_n(q,\alpha,\gamma)G_n(q,\alpha,\gamma)| = |A_n(q,\alpha,\gamma)\xi + B_n(q,\alpha,\gamma)| \leq c_{14}|q|^{-n}$$

for large n, where $A_n, B_n \in \mathbb{Z}[q,\alpha,\gamma]$, of degree at most $2n$ in α.

Now we assume $\xi \in \mathbb{Q}(q)$. Noting that $|q^\sigma| < 1$, $|\alpha^\sigma| < 1$, and $|\gamma^\sigma| \geq 1$ we have $|A_n(q,\alpha,\gamma)^\sigma|, |B_n(q,\alpha,\gamma)^\sigma| \leq c_{15}n^2$ for $\sigma \in \mathrm{Aut}(\mathbb{Q}(q)/\mathbb{Q})$ with $q^\sigma \neq q$. Therefore, we have

$$1 \leq |\mathbf{N}_{\mathbb{Q}(q)/\mathbb{Q}}\mathrm{den}\xi(\mathrm{den}\alpha)^{2n}(A_n(q,\alpha,\gamma)\xi + B_n(q,\alpha,\gamma))| \leq c_{16}n^2|(\mathrm{den}\alpha)^4 q^{-1}|^n$$

for large n; which is a contradiction, and the proof is completed.

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