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Barnes' Multiple Zeta Function and Apostol's Generalized Dedekind Sum

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0. Introduction

The present paper can be viewed as a continuation of [6], where we treated the case of Barnes' double zeta function. Here we handle mainly the case of Barnes' triple zeta function and prove the three term formula for odd Apostol's generalized Dedekind sum. It seems to be new.

Our method is the same as [6], namely we compute the contour integral representation of the Barnes' triple zeta function. Section 1 concerns the general Barnes multiple and twisted multiple Bernoulli polynomials. This, in section 2, enables us to get generally residues at poles of the integrand of the contour integral of the Barnes multiple zeta function. In section 3, we shall quote results of Apostol[1] for later use and derive some formulas relative to Lambert series and Apostol's generalized Dedekind sum.

In the last of section 3, we derive the formula to be called the "three term formula" for odd Apostol's generalized Dedekind sum.

1. Multiple Bernoulli numbers and polynomials twisted by $\tilde{\alpha}$

1.1. Barnes' polynomial. Barnes [2] (cf. [7]) introduced r-ple Bernoulli polynomials ${}_rS_n(u;\tilde{\omega})$ by

$$\frac{(-1)^r te^{-ut}}{\prod_{i=1}^r(1-e^{-\omega_i t})} = \sum_{k=1}^{r}\frac{(-1)^k {}_rS_1^{(k+1)}(u;\tilde{\omega})}{t^{k-1}} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}{}_rS_n'(u;\tilde{\omega})}{n!}t^n \tag{1.1.1}$$

for $|t| < \min\{|2\pi/\omega_1|, \cdots, |2\pi/\omega_r|\}$. Here $\omega_1, \cdots, \omega_r$ are complex numbers with positive real parts and $\tilde{\omega} = (\omega_1, \cdots, \omega_r)$. ${}_rS_1^{(k)}(u;\tilde{\omega})$ means the k-th derivative of ${}_rS_1(u;\tilde{\omega})$ with respect to u.

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The n-th Bernoulli polynomial $B_n(u)$ is defined by

$$\frac{te^{ut}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(u)}{n!}t^n\ .$$

$B_n=B_n(0)$ is the n-th Bernoulli number. Then

$$B_n(u)=(B+u)^n\ ,$$

where we understand $B^j=B_j$, the j-th Bernoulli number, in the binomial expansion of the right-hand side.

Observing that

$$\frac{(-1)^r te^{-ut}}{\prod_{i=1}^r(1-e^{-\omega_i t})}=\frac{\prod_{i=1}^r(-\omega_i t)}{\prod_{i=1}^r(e^{-\omega_i t}-1)}e^{-ut}\frac{1}{t^{r-1}}\cdot\frac{(-1)^r}{\prod_{i=1}^r\omega_i}$$
$$=\exp(-({}^1B\omega_1+\cdots+{}^rB\omega_r+u)t)\frac{1}{t^{r-1}}\cdot\frac{(-1)^r}{\prod_{i=1}^r\omega_i}\ ,$$

we have for $n\geqq 1$

(1.1.2) $$ {}_rS'_n(u;\tilde{\omega})=\frac{({}^1B\omega_1+\cdots+{}^rB\omega_r+u)^{n+r-1}n!}{\prod_{i=1}^r\omega_i\cdot(n+r-1)!}$$

where in the multinomial expansion of the right,

$$({}^iB)^j(=j\text{-th power of }{}^iB)=B_j\ ,$$

but

$$({}^iB)^j\cdot({}^{i'}B)^k\neq B_{j+k}\quad\text{for}\quad i\neq i'\ .$$

1.2. Twisted Bernoulli for $r=1$. Let $\alpha\neq 1$ be a complex number. We define the Bernoulli number $B[\alpha]_n$ twisted by α (simply twisted Bernoulli number) by

(1.2.1) $$(1-\alpha)\frac{1}{1-\alpha e^{-t}}=\sum_{n=0}^{\infty}\frac{B[\alpha]_n}{n!}t^n\ .$$

Then $B[\alpha]_0=1$. Differentiating (1.2.1) with respect to t, we have

(1.2.2) $$-\alpha(1-\alpha)\frac{e^{-t}}{(1-\alpha e^{-t})^2}=\sum_{n=0}^{\infty}\frac{B[\alpha]_{n+1}}{n!}t^n\ .$$

The left hand side is

$$-\alpha(1-\alpha)\frac{1}{1-\alpha e^{-t}}\cdot\frac{1}{1-\alpha e^{-t}}e^{-t}$$
$$=\frac{-\alpha}{1-\alpha}\sum_{n=0}^{\infty}\frac{{}^1B[\alpha]_n}{n!}t^n\cdot\sum_{n=0}^{\infty}\frac{{}^2B[\alpha]_n}{n!}t^n\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n$$

$$= \frac{\alpha}{\alpha - 1} \exp(({}^1B[\alpha] +{}^2 B[\alpha] - 1)t) .$$

Hence

PROPOSITION 1.2.1

$$B[\alpha]_{n+1} = \frac{\alpha}{\alpha - 1}({}^1B[\alpha] +{}^2 B[\alpha] - 1)^n \quad for \ \ n \geqq 0 .$$

In the above,

$${}^iB[\alpha]^j (j - \text{th power of } {}^iB[\alpha]) = B[\alpha]_j \quad \text{for } i = 1, 2 ,$$

but

$${}^1B[\alpha]^j \cdot {}^2 B[\alpha]^k \neq B[\alpha]_{j+k} .$$

From the proposition, we have for example,

$$B[\alpha]_1 = \frac{\alpha}{\alpha - 1} , \quad B[\alpha]_2 = \frac{\alpha(1+\alpha)}{(\alpha - 1)^2} , \quad B[\alpha]_3 = \frac{\alpha(\alpha^2 + 4\alpha + 1)}{(\alpha - 1)^3} ,$$

$$B[\alpha]_4 = \frac{\alpha(\alpha^3 + 11\alpha^2 + 11\alpha + 1)}{(\alpha - 1)^4} , \quad B[\alpha]_5 = \frac{\alpha(\alpha^4 + 26\alpha^3 + 66\alpha^2 + 26\alpha + 1)}{(\alpha - 1)^5} ,$$

$$B[\alpha]_6 = \frac{\alpha(\alpha^5 + 57\alpha^4 + 302\alpha^3 + 302\alpha^2 + 57\alpha + 1)}{(\alpha - 1)^6} ,$$

and

$$B[\alpha]_7 = \frac{\alpha(\alpha^6 + 120\alpha^5 + 1191\alpha^4 + 2416\alpha^3 + 1191\alpha^2 + 120\alpha + 1)}{(\alpha - 1)^7} .$$

Put

$$P_n(\alpha) = \frac{(\alpha - 1)^{n+1}}{\alpha} B[\alpha]_{n+1} \quad for \ \ n \geqq 0 . \tag{1.2.3}$$

$P_n(\alpha)$ is called the Euler polynomial. It is first introduced by Euler in his "Institutiones Calculi Differentialis, 1755" (cf. [3]). In [3],

$$\psi_n(p) = (-1)^n P_{n-1}(-p)$$

is called the Eulerian polynomial.

It is known that with respect to α, $P_n(\alpha)$ is a reciprocal and monic polynomial of degree n with positive integral coefficients.

The Bernoulli polynomial $B[\alpha]_n(u)$ twisted by α is defined by

$$(1-\alpha)\frac{e^{-ut}}{1 - \alpha e^{-t}} = \sum_{n=0}^{\infty} \frac{B[\alpha]_n(u)}{n!} t^n .$$

Then the left hand side is

$$\sum_{n=0}^{\infty} \frac{B[\alpha]_n}{n!} t^n \cdot \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} t^n .$$

Hence we have

$$B[\alpha]_n(u) = (B[\alpha] - u)^n . \tag{1.2.4}$$

In (1.2.4), we mean that $(B[\alpha])^k = B[\alpha]_k$.

1.3. Twisted Bernoulli for $r \geqq 1$**.** Let $\alpha_1, \cdots, \alpha_r$ be complex numbers such that

$$\begin{cases} |\alpha_i| \leqq 1 , \quad \alpha_i \neq 1 \quad \text{for } i = 1, 2, \cdots, p , \\ \alpha_i = 1 \quad \text{for } i = p+1, \cdots, p+q (= r) . \end{cases} \tag{1.3.1}$$

Further, let $\omega_1, \cdots, \omega_r$ be complex numbers with positive real parts. Then we define twisted multiple Bernoulli polynomials $B[\tilde{\alpha}]_N(u; \tilde{\omega})$ by

$$\frac{\prod_{i=1}^{p}(1-\alpha_i)e^{-ut}t^q}{\prod_{i=1}^{r}(1-\alpha_i e^{-\omega_i t})} = \sum_{N=0}^{\infty} \frac{B[\tilde{\alpha}]_N(u; \tilde{\omega})}{N!} t^N , \tag{1.3.2}$$

where $\tilde{\alpha} = (\alpha_1, \cdots, \alpha_r)$ and $\tilde{\omega} = (\omega, \cdots, \omega_r)$.

When $p = 0$, i.e., all α_i are 1, the relation with ${}_rS'_n(u; \tilde{\omega})$ is given by

$$B[(1, \cdots, 1)]_N(u; \tilde{\omega}) = \frac{(-1)^N {}_rS'_{N-r+1}(u; \tilde{\omega}) \cdot N!}{(N-r+1)!} .$$

The left hand side of (1.3.2) is equal to

$$\begin{aligned} &\prod_{i=1}^{p} \frac{1-\alpha_i}{1-\alpha_i e^{-\omega_i t}} \cdot \prod_{i=p+1}^{p+q} \frac{(-\omega_i t)}{e^{-\omega_i t}-1} \cdot e^{-ut} \frac{1}{\prod_{i=p+1}^{p+q} \omega_i} \\ &= \prod_{i=1}^{p} \left(\sum_{n=0}^{\infty} \frac{B[\alpha_i]_n \cdot (\omega_i)^n}{n!} t^n \right) \cdot \prod_{i=p+1}^{p+q} \left(\sum_{n=0}^{\infty} \frac{B_n \cdot (-\omega_i)^n}{n!} t^n \right) \cdot \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} t^n \\ &\quad \cdot \frac{1}{\prod_{i=p+1}^{p+q} \omega_i} \\ &= \exp(({}^1B[\alpha_1]\omega_1 + \cdots + {}^pB[\alpha]_p\omega_p - {}^{p+1}B\omega_{p+1} - \cdots - {}^{p+q}B\omega_{p+q} - u)t) \\ &\quad \cdot \frac{1}{\prod_{i=p+1}^{p+q} \omega_i} . \end{aligned}$$

By (1.3.2), we have

PROPOSITION 1.3.1

$$B[\tilde{\alpha}]_N(u;\tilde{\omega}) = \frac{1}{\prod_{i=p+1}^{p+q}\omega_i}$$
$$\cdot\,(B[\alpha_1]\omega_1 + \cdots + B[\alpha_p]\omega_p \,{-}^{p+1}B\omega_{p+1} - \cdots {-}^{p+q}B\omega_{p+q} - u)^N\,.$$

In the multinomial expansion of the right, the general term is of the form

$$(-1)^{k_1+\cdots+k_{r+1}}\frac{N!}{k_1!\cdots k_{r+1}!}B[\alpha_1]_{k_1}\cdots B[\alpha_p]_{k_p}B_{k_{p+1}}\cdots B_{k_r}\cdot u^{k_{r+1}}\,.$$

and

$$B[\alpha_i]_{k_i} = \alpha P_{k_i-1}(\alpha_i)/(\alpha_i-1)^{k_i}$$

with Euler polynomial $P_k(\alpha)$.

In particular, this means that $B[\tilde{\alpha}]_N(1;\tilde{\omega})$ is a polynomial of

$$\frac{1}{1-\alpha_1},\cdots,\frac{1}{1-\alpha_p}\,,$$

since we can write $P_k(\alpha_i)$ as a polynomial of $1-\alpha_i$ and we have

$$\frac{\alpha_i}{(1-\alpha_i)^j} = -\frac{1-\alpha_i-1}{(1-\alpha_i)^j} = \frac{1}{(1-\alpha_i)^j} - 1\,.$$

2. Barnes' multiple zeta functions

2.1. Barnes' r-ple zeta functions. Let $u,\omega_1,\cdots,\omega_r$ be complex numbers with positive real parts. Barnes' r-ple zeta function $\zeta_r(s;u,\tilde{\omega})$ ([2]. cf. [7]) is defined by

$$\zeta_r(s;u,\tilde{\omega}) = \sum_{m_1,\cdots,m_r=0}^{\infty}\frac{1}{(u+m_1\omega_1+\cdots+m_r\omega_r)^s}\qquad \mathrm{Re}(s) > r\,.$$

Here $w^s = \exp(s\log w)$ and $\log w = \log|w| + i\arg w$ with $-\pi < \arg w < \pi$ for any complex number w not on the non-positive real line. We have a contour integral representation of ζ_r:

$$\zeta_r(s;u,\tilde{\omega}) = \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-ut}t^{s-1}}{\prod_{i=1}^{r}(1-e^{-\omega_i t})}dt \tag{2.1.1}$$

where $I(\lambda,\infty)$ is the path consisting of the real line from $+\infty$ to λ, the circle around 0 of radius λ counter-clock wise from λ to λ and the real line from λ to $+\infty$. Via (2.1.1), $\zeta_r(s;u,\tilde{\omega})$ can be continued holomorphically to the whole complex plane, except for simple poles at $s = 1,\cdots,r$.

We have, for a positive integer m,

$$(2.1.2)\qquad \begin{aligned}\zeta_r(1-m;u,\tilde{\omega}) &= \frac{(-1)^r {}_rS'_m(u;\tilde{\omega})}{m}\\ &= \frac{(-1)^r({}^1B\omega_1+\cdots+{}^rB\omega_r+u)^{m+r-1}}{m(m+1)\cdots(m+r-1)\cdot\prod_{i=1}^r\omega_i}\end{aligned}$$

2.2. Representation of Barnes' multiple zeta function by residues. Let $h_1,\cdots,h_r$ be integers such that $(h_i,h_j)=1$ for $i\neq j$. We take $\omega_i=1/h_i$ and write $\tilde{h}=(1/h_1,\cdots,1/h_r)$.

We consider the integral of $f(t,\tilde{h})t^{s-1}$ with

$$f(t,\tilde{h})=\frac{e^{-t}}{\prod_{i=1}^r(1-e^{-t/h_i})},$$

along the path W, not going through any poles of the integrand, consisting of the rectangle G and the bottle B with the edge P, as indicated in Fig. 1:

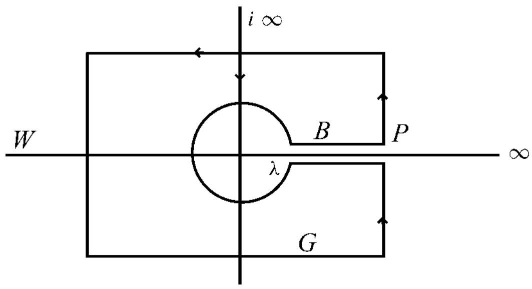

FIGURE 1.

Then we have

$$\int_G+\int_B f(t,\tilde{h})t^{s-1}dt \sum_{\substack{\text{all poles}\\ \text{inside}W}} \text{Res of } f(t,\tilde{h})t^{s-1}.$$

We can show that for $P\to\infty$,

$$\int_G f(t,\tilde{h})t^{s-1}dt\to 0$$

in the same way as in Siegel [9]. Hence

$$(2.2.1)\qquad \frac{-1}{2\pi i}\int_{I(\lambda,\infty)} f(t,\tilde{h})t^{s-1}dt=\sum_{\substack{\text{all poles}\\ \neq 0}} \text{Res of } f(t,\tilde{h})t^{s-1}$$

if the right hand side is convergent.

Now for q with $0 \leqq q \leqq r$, $t = -2\pi i n h_1 \cdots h_q$ with

$$n \not\equiv (\bmod\, h_j) \quad \text{for } q+1 \leqq j \leqq r \tag{2.2.2}$$

is a pole of order q of $f(t, \tilde{h})$.

PROPOSITION 2.2.1. *Put* $\alpha_k = e^{2\pi i n h_1 \cdots h_q / h_{q+k}}$, $k = 1, \cdots, p$, $p + q = r$. *The residue of* $f(t, \tilde{h}) t^{s-1}$ *at* $t = -2\pi i n h_1 \cdots h_q$ *with* (2.2.2) *is given by*

$$\sum_{k=1}^{q} c_{-q+k-1} \frac{(s-1)\cdots(s-q+k)}{(q-k)!} (-2\pi i n h_1 \cdots h_q)^{s-q-1+k} \tag{2.2.3}$$

with

$$\begin{aligned} c_{-q+k-1} = {} & \frac{\prod_{i=1}^{q} h_i}{\prod_{i=1}^{p}(1-\alpha_i)\cdot(k-1)!} \\ & \cdot \left(B[\alpha_1]\frac{1}{h_{q+1}} + \cdots + B[\alpha_p]\frac{1}{h_{p+q}} -^{p+1} B\frac{1}{h_1} - \cdots -^{p+q} B\frac{1}{h_q} - 1 \right)^{k-1} . \end{aligned}$$

PROOF. For simplicity, put $h(q) = h_1 \cdots h_q$. Let the Laurent expansion of $f(t, \tilde{h})$ at $t = -2\pi i n h(q)$ be

$$\begin{aligned} f(t, \tilde{h}) = {} & \frac{c_{-q}}{(t + 2\pi i n h(q))^q} + \cdots + \frac{c_{-1}}{t + 2\pi i n h(q)} \\ & + c_0 + c_1(t + 2\pi i n h(q)) + \cdots . \end{aligned} \tag{2.2.4}$$

Put $\tau = t + 2\pi i n h(q)$. Then (2.2.4) becomes

$$\begin{aligned} & \frac{e^{-\tau}}{(1-\alpha_1 e^{-\tau/h_{q+1}})\cdots(1-\alpha_p e^{-\tau/h_r})(1-e^{-\tau/h_1})\cdots(1-e^{-\tau/h_q})} \\ & = \frac{c_{-q}}{\tau^q} + \cdots + \frac{c_{-1}}{\tau} + c_0 + c_1\tau + \cdots . \end{aligned}$$

By (1.3.2), the left hand side of this times $(1-\alpha_1)\cdots(1-\alpha_p)\tau^q$ is

$$= \sum_{N=0}^{\infty} \frac{B[(\alpha_1, \cdots, \alpha_p, 1, \cdots, 1)]_N(1; \tilde{h}')}{N!} \tau^N$$

with $\tilde{h}' = (1/h_{q+1}, \cdots, 1/h_r, 1/h_1, \cdots, 1/h_q)$. Hence we have c_{-q+k-1} in the Proposition. Since t^{s-1} has the Taylor expanssion

$$t^{s-1} = \sum_{k=0}^{\infty} \frac{(s-1)\cdots(s-k)}{k!} (-2\pi i n h(q))^{s-k-1} (t + 2\pi i n h(q))^k ,$$

we have (2.2.3) by Proposition 1.3.1.

3. Generalized Dedekind sum

3.1. Some computations. Our intention is to compute (2.2.1) for the case $r = 3$. Proposition 2.2.1. shows that the residue of $f(t, \tilde{h})t^{s-1}$ is a polynomial of $\dfrac{1}{1-\alpha^n}$, $\dfrac{1}{1-\beta^n}$ and $\dfrac{1}{1-\gamma^n}$ of degree at most 2. Hence only the three types

$$\frac{1}{1-\alpha^n}, \quad \frac{1}{(1-\alpha^n)(1-\beta^n)}, \quad \frac{1}{(1-\alpha^n)^2}$$

appear. Here

$$\alpha = e^{2\pi i/h_j}, \quad \beta = e^{2\pi i/h_k} \quad (j, k = 1, 2, 3).$$

We gather here the fundamental computation necessary in the following.

Let p be an integer$\geqq 1$. In what follows, we denote by

$$\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \quad \text{or} \quad \sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}'$$

the sum over all nonzero n such that

$$n \not\equiv 0 \ (\mathrm{mod}\, k) \quad \text{or} \quad \text{“}n \not\equiv 0 \ (\mathrm{mod}\, k) \ \text{and} \ \not\equiv 0 \ (\mathrm{mod}\, h)\text{”}.$$

The $'$ of $\sum'$ means “omitting $n = 0$”.

Here and hereafter, we understand that for $p = 1$,

$$\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' = \lim_{M\to 0} \sum_{\substack{n=-M \\ (k)}}^{n=M}{}' \quad \text{and} \quad \sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}' = \lim_{M\to 0} \sum_{\substack{n=-M \\ (k),(h)}}^{n=M}{}'.$$

LEMMA 1. *Let α, β be the complex numbers such that $\alpha \neq 1, \beta \neq 1$ and $\alpha^k = 1, \beta^h = 1$. Then we have*

(i) $$\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)} = \frac{1}{2} \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p} \quad \textit{for } p \textit{ even},$$

(ii) $$\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{\alpha^n}{n^p(1-\alpha^n)} = \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)} \quad \textit{for } p \textit{ odd},$$

(iii) $$\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)(1-\beta^n)}$$

$$= \frac{1}{2} \sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p(1-\alpha^n)} + \frac{1}{2} \sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p(1-\beta^n)} \quad \textit{for } p \textit{ odd},$$

(iv) $$\sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{\alpha^n}{n^p(1-\alpha^n)^2} = 0 \quad \textit{for } p \textit{ odd}.$$

PROOF OF (i). Since

$$\frac{1}{1-\alpha^n} = 1 - \frac{1}{1-\alpha^{-n}},$$

we have

$$\sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{1}{n^p(1-\alpha^n)} = \sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{1}{n^p} - \sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{1}{n^p(1-\alpha^{-n})}$$

$$= \sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{1}{n^p} - \sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{1}{n^p(1-\alpha^n)}.$$

PROOF OF (ii). This follows from

$$\sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{\alpha^n}{n^p(1-\alpha^n)} = (-1)^p \sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{\alpha^{-n}}{n^p(1-\alpha^{-n})}$$

$$= -(-1)^p \sideset{}{'}\sum_{\substack{n=-\infty \\ (k)}}^{\infty} \frac{1}{n^p(1-\alpha^n)}.$$

PROOF OF (iii). We have

$$\frac{1}{(1-\alpha^n)(1-\beta^n)} = 1 - \frac{1}{1-\alpha^{-n}} - \frac{1}{1-\beta^{-n}} + \frac{1}{(1-\alpha^{-n})(1-\beta^{-n})}.$$

Hence

$$\sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p(1-\alpha^n)(1-\beta^n)} = \sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p} - \sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p(1-\alpha^{-n})}$$

$$- \sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p(1-\beta^{-n})} + \sideset{}{'}\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty} \frac{1}{n^p(1-\alpha^{-n})(1-\beta^{-n})}$$

$$= \sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)} + \sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}' \frac{1}{n^p(1-\beta^n)}$$

$$- \sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)(1-\beta^n)} ,$$

since

$$\sum_{\substack{n=-\infty \\ (k),(h)}}^{\infty}{}' \frac{1}{n^p} = 0 \quad \text{for odd } \ p .$$

Then (iii) follows.

PROOF OF (iv). From

$$\frac{\alpha^n}{(1-\alpha^n)^2} = -\frac{1}{1-\alpha^n} + \frac{1}{(1-\alpha^n)^2} ,$$

follows

(3.1.1) $$\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{\alpha^n}{n^p(1-\alpha^n)^2} = -\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)} + \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)^2} .$$

Since p is odd, we have

$$\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)^2} = \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)}$$

by putting $\alpha = \beta$ in (iii). Put this into (3.1.1) to get (iv).

Further we need the following ([1]) : for x_0 such that $x_0^k = 1,\ x_0 \neq 1$,

(3.1.2) $$\frac{1}{1-x_0} = -\sum_{\nu=1}^{k-1} \frac{\nu}{k} x_0^{\nu} \quad \text{holds} .$$

This is easily derived by operating $x\dfrac{d}{dx}$ to

$$1 + x + x^2 + \cdots + x^{k-1} = \frac{1-x^k}{1-x}$$

and putting $x = x_0$.

3.2. Apostol's generalized Dedekind sum. $B_p(x)$ being the p-th Bernoulli polynomial, we put

$$\bar{B}_p(x) = B_p(x-[x]) , \quad \text{for } \ p > 1 \ \text{ and } \ p = 1 , \quad x \notin \mathbf{Z} ,$$

$$\bar{B}_1(x) = 0 \quad \text{for } x \in \mathbf{Z},$$

where $[x]$ means the greatest integer not exceeding x. It is well known that

$$\bar{B}_p(x) = -\frac{p!}{(2\pi i)^p} \sum_{n=-\infty}^{\infty}{}' \frac{1}{n^p} e^{2\pi i n x}, \quad \text{for } 0 \leqq x < 1, \tag{3.2.1}$$

under the convention for the sum, aforementioned for the case $p = 1$.

Apostol [1] generalized Dedekind sum to

$$s_p(h,k) = \sum_{\nu=1}^{k-1} \frac{\nu}{k} \bar{B}_p(h\nu/k), \tag{3.2.2}$$

where h and k are positive integers.

$s_1(k,h) = s(h,k)$ is the original Dedekind sum. (3.2.2) is called the Apostol-Dedekind sum.

In [1], he proved "the reciprocity formula" for odd p and $(h,k) = 1$:

$$(p+1)\{hk^p s_p(h,k) + kh^p s_p(k,h)\} = (Bh - Bk)^{p+1} + pB_{p+1}. \tag{3.2.3}$$

Further Apostol derived the Lambert series expression for (3.2.2):

$$s_p(h,k) = \frac{p!}{(2\pi i)^p} \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{1}{n^p} \cdot \frac{1}{1 - e^{2\pi i n h/k}}, \quad \text{for odd } p \geqq 1. \tag{3.2.4}$$

For $p = 1$, (3.2.4) has been found by Dedekind from the transformation theory of Dedekind η-function.

It is known by Rademacher [5] the three term relation for the original Dedekind sum $s(h,k)$ (also see Berndt [4]):

$$s(bc',a) + s(ca',b) + s(ab',c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right), \tag{3.2.5}$$

where $(a,b) = (b,c) = (c,a) = 1$, $aa' \equiv 1 (\bmod b)$, $bb' \equiv 1 (\bmod c)$ and $cc' \equiv 1 (\bmod a)$.

Further the following three term congruence relation is known [5]:

$$\begin{aligned} &\left(s(bc,a) - \frac{bc}{12a}\right) + \left(s(ca,b) - \frac{ca}{12b}\right) + \left(s(ab,c) - \frac{ab}{12c}\right) \\ &\equiv -\frac{1}{4} - \frac{abc}{12} + \frac{1}{12abc} \pmod 2. \end{aligned} \tag{3.2.6}$$

3.3. Lambert series and Apostol's generalized Dedekind sum. Here h_1, h_2, h_3 are positive integers with $(h_i, h_j) = 1$ for $i \neq j$.

For brevity, we write (1) or (1, 2), for example, under $\sum$ instead of writing the sum condition $n \not\equiv 0 \ (\bmod h_1)$ or "$n \not\equiv 0 \ (\bmod h_1)$ and $(\bmod h_2)$" respectively.

LEMMA 2. *For odd $p \geqq 1$,*

$$\sum_{\substack{n=-\infty \\ (2),(3)}}^{\infty}{}' \frac{1}{n^p(1-e^{2\pi i n h_1/h_2})} = \frac{(2\pi i)^p}{p!}\left\{s_p(h_1,h_2) - \frac{1}{h_3^p}s_p(h_1,h_3,h_2)\right\},$$

PROOF. Put $\alpha = e^{2\pi i h_1/h_2}$. Then $\alpha^{h_2} = 1$ and $\alpha \neq 1$. By (3.1.2) and (3.2.1), we have

$$\begin{aligned}
\sum_{\substack{n=-\infty \\ (2),(3)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)} &= -\sum_{\substack{n=-\infty \\ (2),(3)}}^{\infty}{}' \frac{1}{n^p}\sum_{\mu=1}^{h_2-1}\frac{\mu}{h_2}\alpha^{\mu n} \\
&= -\sum_{\mu=1}^{h_2-1}\frac{\mu}{h_2}\left\{\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^p}\alpha^{\mu n} - \sum_{n=-\infty}^{\infty}{}' \frac{1}{(h_2 n)^p}\right. \\
&\qquad \left. - \sum_{n=-\infty}^{\infty}{}' \frac{1}{(h_3,n)^p}\alpha^{\mu h_3 n} + \sum_{n=-\infty}^{\infty}{}' \frac{1}{(h_2 h_3 n)^p}\right\} \\
&= \sum_{\mu=1}^{h_2-1}\frac{\mu}{h_2}\frac{(2\pi i)^p}{p!}\left\{\bar{B}_p(h_1\mu/h_2) - \frac{1}{h_3^p}\bar{B}_p(h_1h_3\mu/h_2)\right\}
\end{aligned}$$

since $\sum_{n=-\infty}^{\prime\infty} \dfrac{1}{n^p} = 0$ for odd p and we get the Lemma.

By Lemma 1, (iii) and Lemma 2, we get easily the following

LEMMA 3. *For odd $p \geqq 1$,*

$$\begin{aligned}
&\sum_{\substack{n=-\infty \\ (2),(3)}}^{\infty}{}' \frac{1}{n^p(1-e^{2\pi i n h_1/h_2})(1-e^{2\pi i n h_1/h_3})} \\
&\quad = \frac{(2\pi i)^p}{2p!}\left\{s_p(h_1,h_2) + s_p(h_1,h_3) - \frac{s_p(h_1h_2,h_3)}{h_2^p} - \frac{s_p(h_1h_3,h_2)}{h_3^p}\right\}
\end{aligned}$$

LEMMA 4. *For odd $p \geqq 1$,*

$$\sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \frac{1}{n^p(1-e^{2\pi i n h_1 h_2/h_3})} = \frac{(2\pi i)^p}{p!}s_p(h_1h_2,h_3)\,.$$

PROOF. Put $\alpha = e^{2\pi i h_1 h_2/h_3}$. By (3.1.2) and (3.2.1), we have

$$\sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)} = -\sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \frac{1}{n^p}\sum_{\mu=1}^{h_3-1}\frac{\mu}{h_3}\alpha^{n\mu}$$

$$= -\sum_{\mu=1}^{h_3-1} \frac{\mu}{h_3}\left\{ \sum_{n=-\infty}^{\infty}{}' \frac{1}{n^p}\alpha^{n\mu} - \sum_{n=-\infty}^{\infty}{}' \frac{1}{(h_3 n)^p} \right\}$$

$$= \frac{(2\pi i)^p}{p!} \sum_{\mu=1}^{h_3-1} \frac{\mu}{h_3} B_p(h_1 h_2 \mu / h_3) = \frac{(2\pi i)^p}{p!} s_p(h_1 h_2, h_3)\,.$$

3.4. Poles. We shall compute both hands of

$$(3.4.1)\qquad \begin{aligned} -\frac{e^{(1-p)\pi i}}{\Gamma(p)}\zeta_3(1-p;1;\tilde{h}) &= -\frac{1}{2\pi i}\int_{I(\lambda,\infty)} f(t,\tilde{h})t^{-p}dt\,,\\ &= \sum_{\substack{\text{all poles}\\ \neq 0}} \text{Res of } f(t,\tilde{h})t^{-p} \end{aligned}$$

where

$$f(t,\tilde{h}) = \frac{e^{-t}}{(1-e^{-t/h_1})(1-e^{-t/h_2})(1-e^{-t/h_3})}$$

By (2.1.2), the left hand side of (3.4.1) is equal to

$$(3.4.2)\qquad \frac{(-1)^{1-p}}{(p+2)!} h_1 h_2 h_3 \left({}^1B\frac{1}{h_1} + {}^2B\frac{1}{h_2} + {}^3B\frac{1}{h_3} + 1 \right)^{p+2}.$$

There are three types of poles of $f(t,\tilde{h})$:

$$\begin{array}{lll} \text{Type I.} & t = -2\pi i n h_1\,, & n \not\equiv 0 \ (\text{mod}\, h_2) \text{ and } (\text{mod}\, h_3)\,,\\ & t = -2\pi i n h_2\,, & n \not\equiv 0 \ (\text{mod}\, h_3) \text{ and } (\text{mod}\, h_1)\,,\\ & t = -2\pi i n h_3\,, & n \not\equiv 0 \ (\text{mod}\, h_1) \text{ and } (\text{mod}\, h_2)\,, \quad n \in \mathbf{Z}\,. \end{array}$$

All of these are poles of order 1

$$\begin{array}{lll} \text{Type II.} & t = -2\pi i n h_1 h_2 & n \not\equiv 0 \ (\text{mod}\, h_3)\,,\\ & t = -2\pi i n h_2 h_3 & n \not\equiv 0 \ (\text{mod}\, h_1)\,,\\ & t = -2\pi i n h_3 h_1 & n \not\equiv 0 \ (\text{mod}\, h_2)\,, \quad n \in \mathbf{Z}\,. \end{array}$$

All of these are poles of order 2.

$$\text{Type III.} \quad t = -2\pi i n h_1 h_2 h_3\,, \qquad n \neq 0\,, \quad n \in \mathbf{Z}\,.$$

All of these are poles of order 3.

(I) For poles of type I, we have, putting $\alpha = e^{2\pi i h_1/h_3}$, $\beta = e^{2\pi i h_2/h_3}$,

$$(3.4.3)\qquad \sum_{\substack{n=-\infty\\(2),(3)}}^{\infty}{}' \operatorname*{Res}_{t=-2\pi i n h_1} f(t,\tilde{h})t^{-p} = h_1^{1-p}(-2\pi i)^{-p} \sum_{\substack{n=-\infty\\(2),(3)}}^{\infty}{}' \frac{1}{n^p(1-\alpha^n)(1-\beta^n)}\,.$$

(II) Put $t = -2\pi i n h_1 h_2$, with $n \not\equiv 0 \pmod{h_3}$. This is a pole of $f(t, \tilde{h})$ of order 2. Hence f has the Laurent expansion, around there:

$$f(t, \tilde{h}) = \frac{c_{-2}}{(t + 2\pi i n h_1 h_2)^2} + \frac{c_{-1}}{t + 2\pi i n h_1 h_2} + c_0 + \cdots .$$

By Propositon 2.2.1

$$\begin{aligned}(3.4.4)\quad &\sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \operatorname*{Res}_{t=-2\pi i n h_1 h_2} f(t, \tilde{h})^{-p} \\ &= \sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' (-c_{-2}(-2\pi i n h_1 h_2)^{-1-p} p + c_{-1}(-2\pi i n h_1 h_2)^{-p})\end{aligned}$$

with

$$c_{-2} = \frac{h_1 h_2}{1 - \alpha^n}, \quad c_{-1} = -\frac{h_1 h_2}{h_3} \cdot \frac{\alpha^n}{(1 - \alpha^n)^2} + \frac{\frac{1}{2}h_1 + \frac{1}{2}h_2 - h_1 h_2}{1 - \alpha^n}$$
$$\alpha = e^{2\pi i h_1 h_2 / h_3} .$$

Hence

$$\begin{aligned}(3.4.4) = &-\frac{h_1 h_2}{h_3}(2\pi i h_1 h_2)^{-p} \sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \frac{\alpha^n}{n^p (1 - \alpha^n)^2} \\ &+ \left(\frac{1}{2}h_1 + \frac{1}{2}h_2 - h_1 h_2\right)(-2\pi i h_1 h_2)^{-p} \sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \frac{1}{n^p (1 - \alpha^n)} \\ &- p\frac{(-2\pi i h_1 h_2)^{-p}}{(-2\pi i)} \sum_{\substack{n=-\infty \\ (3)}}^{\infty}{}' \frac{1}{n^{p+1}(1 - \alpha^n)} .\end{aligned}$$

(III) $t = -2\pi i n h_1 h_2 h_3$ is a pole of order 3 of $f(t, \tilde{h})$. Hence

$$f(t, \tilde{h}) = \frac{c_{-3}}{(t + 2\pi i n h_1 h_2 h_3)^3} + \frac{c_{-2}}{(t + 2\pi i n h_1 h_2 h_3)^2} + \frac{c_{-1}}{t + 2\pi i n h_1 h_2 h_3} + c_0 \cdots .$$

By Proposition 2.2.1, we have

$$\begin{aligned}(3.4.5)\quad \sum_{n=-\infty}^{\infty}{}' \operatorname*{Res}_{t=-2\pi i n h_1 h_2 h_3} f(t, \tilde{h}) t^{-p} = \sum_{n=-\infty}^{\infty}{}' \Bigg(&c_{-3}\frac{1}{2}p(p+1)(-2\pi i n h_1 h_2 h_3)^{-p-2} \\ &- p c_{-2}(-2\pi i n h_1 h_2 h_3)^{-p-1} + c_{-1}(-2\pi i n h_1 h_2 h_3)^{-p} \Bigg)\end{aligned}$$

with

$$
\begin{aligned}
c_{-3} &= h_1h_2h_3\,,\\
c_{-2} &= -h_1h_2h_3\left({}^1B\frac{1}{h_1}+{}^2B\frac{1}{h_2}+{}^3B\frac{1}{h_3}+1\right)\\
&= h_1h_2h_3\left(\frac{1}{2h_1}+\frac{1}{2h_2}+\frac{1}{2h_3}-1\right),\\
c_{-1} &= \frac{1}{2}h_1h_2h_3\left({}^1B\frac{1}{h_1}+{}^2B\frac{1}{h_2}+{}^3B\frac{1}{h_3}+1\right)^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(3.4.5) = \frac{1}{2}&p(p+1)h_1h_2h_3(-2\pi i h_1h_2h_3)^{-p-2}\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^{p+2}}\\
&- ph_1h_2h_3\left(\frac{1}{2h_1}+\frac{1}{2h_2}+\frac{1}{2h_3}-1\right)(-2\pi i h_1h_2h_3)^{-p-1}\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^{p+1}}\\
&+ \frac{1}{2}h_1h_2h_3\left({}_1B\frac{1}{h_1}+{}^2B\frac{1}{h_2}+{}^3B\frac{1}{h_3}+1\right)^2(-2\pi i h_1h_2h_3)^{-p}\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^{p}}\,.
\end{aligned}
$$

3.5. Three term formula of Apostol's generalized Dedekind sum for odd p. In this subsection, p is assumed to be odd throughout and we shall compute the right hand side of (3.4.1). Because of Lemma 1(iv), the case of odd p can be handled easily.

For poles $t=-2\pi i n h_1$ of type (I), we have by Lemma 3,

$$
\begin{aligned}
(3.4.3) = h_1^{1-p}&(-2\pi i)^{-p}\\
&\cdot\frac{(2\pi i)^p}{2p!}\left\{s_p(h_1,h_2)+s_p(h_1,h_3)-\frac{s_p(h_1h_2,h_3)}{h_2^p}-\frac{s_p(h_1h_3,h_2)}{h_3^p}\right\}
\end{aligned}
$$

For other poles of type I, we compute in the same way, on cycling 1, 2, 3 to 2,3, 1 or 3, 2, 1. Hereafter we use the symbol $\sum^c$ to denote the cycling sum on 1, 2, 3. For example,

$$\textstyle\sum^c s_p(h_1,h_2)=s_p(h_1,h_2)+s_p(h_2,h_3)+s_p(h_3,h_1)\,.$$

Then the contribution of all poles of type I to the right hand side of (3.4.1) is

$$
\begin{aligned}
&-\frac{1}{2p!}\textstyle\sum^c h_1^{1-p}\{s_p(h_1,h_2)+s_p(h_1,h_3)\}\\
(3.5.1)\qquad&\quad+\frac{1}{2p!}\textstyle\sum^c h_1^{1-p}\left\{\dfrac{s_p(h_1h_2,h_3)}{h_2^p}+\dfrac{s_p(h_1h_3,h_2)}{h_3^p}\right\}.
\end{aligned}
$$

We continue our computation: by the reciprocity formula (3.2.3), the first cycling sum of (3.5.1) is

$$\begin{aligned}\sum\nolimits^c h_1^{1-p}\{s_p(h_1,h_2)+s_p(h_1,h_3)\} &= h_1^{1-p}s_p(h_1,h_2)+h_1^{1-p}s_p(h_1,h_3)\\ &\quad + h_2^{1-p}s_p(h_2,h_3)+h_2^{1-p}s_p(h_2,h_1)\\ &\quad + h_3^{1-p}s_p(h_3,h_1)+h_3^{1-p}s_p(h_3,h_2)\\ &= \frac{1}{(h_1h_2)^p}\{h_1h_2^p s_p(h_1,h_2)+h_2h_1^p s_p(h_2,h_1)\}\\ &\quad + \frac{1}{(h_2h_3)^p}\{h_2h_3^p s_p(h_2,h_3)+h_3h_2^p s_p(h_3,h_2)\}\\ &\quad + \frac{1}{(h_3h_1)^p}\{h_3h_1^p s_p(h_3,h_1)+h_1h_3^p s_p(h_1,h_3)\}\\ &= \frac{1}{p+1}\sum\nolimits^c \frac{({}^1Bh_1 - {}^2Bh_2)^{p+1}}{(h_1h_2)^p} + \frac{p}{p+1}B_{p+1}\sum\nolimits^c \frac{1}{(h_1h_2)^p}\end{aligned}$$

The second cycling sum of (3.5.1) is

$$= \sum\nolimits^c \frac{h_1+h_2}{(h_1h_2)^p}s_p(h_1h_2,h_3)\,.$$

Thus

$$\begin{aligned}\sum\nolimits^c \text{ of } (3.4.3) &= -\frac{1}{2\cdot(p+1)!}\sum\nolimits^c \frac{({}^1Bh_1 - {}^2Bh_2)^{p+1}}{(h_1h_2)^p}\\ &\quad - \frac{p}{2(p+1)!}B_{p+1}\sum\nolimits^c \frac{1}{(h_1h_2)^p}\\ &\quad + \frac{1}{2p!}\sum\nolimits^c \frac{h_1+h_2}{(h_1h_2)^p}s_p(h_1h_2,h_3)\,.\end{aligned} \tag{3.5.2}$$

Here we used

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^{p+1}} = \zeta(p+1) = \frac{(2\pi)^{p+1}(-1)^{(p+3)/2}}{2(p+1)!}\,.$$

For poles $t = -2\pi i n h_1 h_2$ of type (II), we have by Lemma 4 and Lemma 1 (i),

$$(3.4.4) = \frac{-\left(\frac{1}{2}h_1 + \frac{1}{2}h_2 - h_1h_2\right)}{p!(h_1h_2)^p}s_p(h_1h_2,h_3) + \frac{p(1-1/h_3^{p+1})}{2(p+1)!(h_1h_2)^p}B_{p+1}\,.$$

We compute residues at other poles of type (II) in the same way. Thus the contribution of all poles of type (II) to the right side of (3.4.1) is

$$\sum{}^{c} \text{ of } (3.4.4) = \frac{-1}{p!}\sum{}^{c} \frac{\left(\frac{1}{2}h_1 + \frac{1}{2}h_2 - h_1h_2\right)}{(h_1h_2)^p} s_p(h_1h_2, h_3) + \frac{pB_{p+1}}{2(p+1)!}\sum{}^{c} \frac{(1 - 1/h_3^{p+1})}{(h_1h_2)^p}\,. \tag{3.5.3}$$

For poles $t = -2\pi i n h_1 h_2 h_3$ of type (III), the first and the third sums of (3.4.5) vanish. Hence the contribution of all poles of type (III) equals

$$\frac{-p}{(h_1h_2h_3)^p(p+1)!}\left(1 - \frac{1}{2h_1} - \frac{1}{2h_2} - \frac{1}{2h_3}\right)B_{p+1}\,. \tag{3.5.4}$$

By (3.4.1), (3.4.2), (3.5.2), (3.5.3) and (3.5.4), we have, after a straight-forward calculation, the following

THEOREM (THE THREE TERM FORMULA). For odd $p \geqq 1$ and integers h_1, h_2, h_3 such that $(h_i, h_j) = 1$, $i \neq j$,

$$(p+1)\sum{}^{c} h_1h_2h_3^p s_p(h_1h_2, h_3) = \frac{(h_1h_2h_3)^{p+1}}{p+2}\left({}^1B\frac{1}{h_1} + {}^2B\frac{1}{h_2} + {}^3B\frac{1}{h_3} + 1\right)^{p+2}$$
$$+ \frac{1}{2}(h_3^p({}^1Bh_1 - {}^2Bh_2)^{p+1} + h_1^p({}^1Bh_2 - {}^2Bh_3)^{p+1} + h_2^p({}^1Bh_3 - {}^2Bh_1)^{p+1})$$
$$+ pB_{p+1}\,.$$

Note that this is different from Rademacher's (3.2.5) even in the case $p = 1$.

It may be worth noting that the right hand side of the formula in the Theorem can be written as a linear combination of values of Barnes' triple, double zeta functions and ordinary Riemann zeta function ($p > 1$): namely

$$\sum{}^{c} h_1h_2h_3^p s_p(h_1h_2, h_3) = -p(h_1h_2h_3)^p\zeta_3(1-p; 1; 1/h_1, 1/h_2, 1/h_3)$$
$$+ p\sum{}^{c}\frac{h_3^p}{h_1h_2}\tilde{\zeta}_2(1-p; 1/h_1, 1/h_2) - p\zeta(-p)\,.$$

In the above, $\tilde{\zeta}_2$ was introduced in [8]:

$$\tilde{\zeta}_2(s; h, k) = \sum_{m,n=0}^{\infty}{}' \frac{1}{(mh + nk)^s}$$

where $'$ means that $(m, n) = (0, 0)$ is omitted from the sum.

References

[1] T. M. APOSTOL, Generalizaed Dedekind sums and transformation formulae of certain Lambert series, Duke Math. J. **17** (1950), 147–157.

[2] E. W. BARNES, The theory of the double gamma functions, Proc. London Math. Soc. **31** (1899), 358–381.

[3] B. C. BERNDT, *Ramanujan's Notebooks II*, Springer (1989).

[4] B. C. BENNDT, Reciprocity Theorems for Dedekind Sums and Generalizations, Adv. Math. **23** (1977), 285–316.

[5] E. GROSSWALD and H. RADEMACHER, Dedekind sums, *The Carus Math. Monograph 16*, Math. Assoc. Amer. (1972).

[6] K. KATAYAMA, Barnes' double zeta function, the Dedekind sum and Ramanujan's formula, Tokyo J. Math. **27** (2004), 41–56.

[7] K. KATAYAMA and M. OHTSUKI, On the multiple gamma-functions, Tokyo J. Math. **21** (1998), 159–182.

[8] Y. NAGASAKA, K. OTA and C. SEKINE, Generalizations of Dedekind sums and their reciprocity laws, Acta Arith. **106** (2003), 355–378.

[9] C. L. SIEGEL, *Analytische Zahlentheorie I*, Göttingen (1963).

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