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On the Periodicity of Planes with Parallel Mean Curvature Vector in $\mathbf{C}H^2$

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Abstract. We give a simple criterion for when an isometric immersion from the real two-plane into the two-dimensional complex hyperbolic space with non-zero parallel mean curvature vector is invariant under the action of some lattice of $\mathbf{R}^2$.

1. Introduction

There are many minimal immersions of a flat plane into complex projective space $\mathbf{C}P^n$ [4]. Jensen and Liao [3] obtained a simple criterion for deciding when the immersion is doubly periodic. Such an immersion into complex Euclidean space $\mathbf{C}^n$ must be totally geodesic, and into complex hyperbolic space $\mathbf{C}H^n$ cannot exist. Ohnita [7] classified minimal surfaces with constant Gaussian curvature and constant Kähler angle in complex space forms.

We would like to consider immersions of a flat plane into complex space forms with non-zero parallel mean curvature vector. If the ambient spaces are $\mathbf{C}P^2$ or $\mathbf{C}^2$, then these problems are already settled: The main theorem of Dajczer and Tojeiro in [2] implies as a special case that such a map into $\mathbf{C}P^2$ must be doubly periodic. In the $\mathbf{C}^2$ case, Chen [1] showed that it is either doubly periodic or singly periodic.

In this paper we solve the problem in the $\mathbf{C}H^2$ case. Let $\mathbf{R}^2$ be the real two-plane with a flat metric and $x : \mathbf{R}^2 \to \mathbf{C}H^2$ an isometric immersion with non-zero parallel mean curvature vector from $\mathbf{R}^2$ into $\mathbf{C}H^2$ of constant holomorphic sectional curvature 4ρ (< 0). Kenmotsu and Zhou [5] proved that this map x is uniquely determined by two real numbers b and t, which are defined by the second fundamental form of the immersion and satisfy $0 < b$ and $0 \le t < 2\pi$, up to holomorphic isometries of $\mathbf{C}H^2$. The real number b has clear geometric meaning, as it is defined to be half of the length of the mean curvature vector. The other constant t is determined from x as follows: For the given $x : \mathbf{R}^2 \to \mathbf{C}H^2$, let e_4 be the unit normal vector field along $x(\mathbf{R}^2)$ which is orthogonal to $e_3 = -H/|H|$ and compatible to the orientation of $\mathbf{C}H^2$. Let $\{e_1, e_2\}$ be an oriented orthonormal frame field on $\mathbf{R}^2$, and denote by h^r_{ij}, $r = 3, 4$, $1 \le i, j \le 2$, the coefficients of the second fundamental form of x with

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respect to these e_1, e_2, e_3, e_4. Two quadratic differentials $\varphi_r = (h^r_{11} - h^r_{22} - \sqrt{-1}2h^r_{12})dz^2$ are globally defined on $\mathbf{R}^2$. It follows from (2.8) of [5] that $\varphi = \varphi_3/\varphi_4$ is constant and $\chi = \sqrt{1+\rho/2b^2}(\sqrt{-1}+\varphi)/(\sqrt{-1}-\varphi)$ has absolute value 1. We define t so that the argument of χ is equal to t.

To state the main result of this paper, we need a real valued function $f(t,\xi)$ on $(t,\xi) \in [0, 2\pi) \times [0, 1]$ defined by

$$f(t,\xi) = 27\xi^4 - 18\xi^2 + 8\cos t \cdot \xi - 1\,, \tag{1}$$

and we let $\omega(t)$ be the unique positive solution of the equation $f(t,\xi) = 0$ for each $t \in [0, 2\pi)$. We prove the following:

THEOREM 1. *Let $x : \mathbf{R}^2 \to \mathbf{C}H^2$ be an isometric immersion with non-zero parallel mean curvature vector determined by b and t. Then*

(1) *x is doubly periodic if and only if b and t satisfy $\omega(t) < \sqrt{1+\rho/2b^2}$;*

(2) *x is singly periodic if and only if b and t satisfy $\sqrt{1+\rho/2b^2} \le \omega(t)$ and $(t, \sqrt{1+\rho/2b^2}) \ne (0, \omega(0))$;*

(3) *x is an embedding if and only if b and t satisfy $(t, \sqrt{1+\rho/2b^2}) = (0, \omega(0))$.*

The main interest of Theorem 1 is that the periodicity of the map x is decided by its second order invariants. By [5], the immersions in Theorem 1 are totally real.

2. Planes with parallel mean curvature vector

In this section, following [5], we explain that the immersion x described in the introduction is determined by two real numbers b and t satisfying $0 < b$ and $0 \le t < 2\pi$. To do so, we need the bundle description of $\mathbf{C}H^2$. Let $\mathbf{C}^3$ be complex three space equipped with its Hermitian form $F(\mathbf{u},\mathbf{v}) = {}^t\mathbf{u}\,T\,\bar{\mathbf{v}}$, where

$$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{u} = (u^0, u^1, u^2)\,, \quad \mathbf{v} = (v^0, v^1, v^2) \in \mathbf{C}^3\,.$$

We set $U(1,2) = \{A \in GL(3,\mathbf{C}) \mid F(A\mathbf{u}, A\mathbf{v}) = F(\mathbf{u},\mathbf{v}),\ \mathbf{u},\mathbf{v} \in \mathbf{C}^3\}$. Let S be a real hypersurface in $\mathbf{C}^3$ defined by $F(\mathbf{u},\mathbf{u}) = -1$. Then $U(1,2)$ acts on S transitively and $S^1 = \{e^{\sqrt{-1}\theta} \mid \theta \in \mathbf{R}\}$ acts on S freely by $\mathbf{u} \to e^{\sqrt{-1}\theta}\mathbf{u}$. Let S' be the base manifold of a principal fiber bundle S with the structure group S^1 and $\pi : S \to S'$ the projection. Then we have $S' = \mathbf{C}H^2 = S/S^1 = SU(1,2)/S(U(1) \times U(2))$. The tangent space of $\mathbf{C}H^2$ at $\pi(\mathbf{u})$, ($\mathbf{u} \in S$), is given by $T_{\pi(\mathbf{u})}\mathbf{C}H^2 = \{\mathbf{v} \in \mathbf{C}^3 \mid F(\mathbf{u},\mathbf{v}) = 0\}$. The metric of $\mathbf{C}H^2$ is given by

$$-\frac{1}{\rho}\mathrm{Re}(F(\mathbf{u}',\mathbf{v}'))\,, \quad \mathbf{u}',\mathbf{v}' \in T_{\pi(\mathbf{u})}\mathbf{C}H^2\,.$$

We denote the standard complex coordinate of $\mathbf{R}^2$ by z. The following Theorem 2 was proved in [5].

THEOREM 2. *Let $x : \mathbf{R}^2 \to \mathbf{C}H^2$ be a totally real isometric immersion from $\mathbf{R}^2$ into $\mathbf{C}H^2$ with non-zero parallel mean curvature vector H. Then, there are a $U(1,2)$ valued function $X(z)$ on $\mathbf{R}^2$ and a real number t with $0 \le t < 2\pi$ such that $X(z) = \exp(zA + \bar{z}B)$, and x is the composition of the projection π and the first row X_0 of X, where we choose b and c so that $2b = |H|$, $c = \sqrt{b^2 + \rho/2}\, e^{\sqrt{-1}t}$, and we set*

$$A = \begin{pmatrix} 0 & \sqrt{-\rho/2} & 0 \\ 0 & b & \bar{c} \\ \sqrt{-\rho/2} & -b & b \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \sqrt{-\rho/2} \\ \sqrt{-\rho/2} & -b & b \\ 0 & -c & -b \end{pmatrix}. \tag{2}$$

REMARK. By (2.7) of [5], there exist some unitary coframes ω_i, $(i = 1, 2)$ and unitary connection forms ω_{ij}, $(i, j = 1, 2)$ on $\mathbf{C}H^2$ such that

$$\begin{aligned} \omega_1 &= \frac{1}{\sqrt{2}}\, dz\,, \quad \omega_2 = \frac{1}{\sqrt{2}}\, d\bar{z}\,, \\ \omega_{11} &= -b(dz - d\bar{z})\,, \quad \omega_{22} = -b(dz - d\bar{z})\,, \\ \omega_{12} &= b\, dz + c\, d\bar{z}\,, \quad \omega_{21} = -\bar{\omega}_{12}\,, \end{aligned}$$

along $x(\mathbf{R}^2)$. In fact, this system is obtained by putting $a(u) = -b$, $\theta(u) = \pi/2$, and $\phi = dz$ in (2.7) of [5]. It is easily integrated by the same method as in [5, pp. 313–315] (see also [4, pp. 679–681]).

3. Periodicity of the immersion

In the previous section, we recalled how the moduli of isometric immersions from $\mathbf{R}^2$ into $\mathbf{C}H^2$ with non-zero parallel mean curvature vector is parametrized by two real numbers b and t. We now give a simple criterion for when such a map is invariant under some lattice $\Lambda \subset \mathbf{R}^2$ and prove Theorem 1 of this paper by studying five cases for the eigenvalues and eigenvectors of the constant matrices A and B of Section 2. First we study basic properties of the eigenvalues of A. Let $\omega = \sqrt{1 + \rho/2b^2}$. Then the characteristic equation of A, $\det(\mu I - A) = 0$, is given by

$$\mu^3 - 2b\mu^2 + b^2(1 + e^{-\sqrt{-1}t}\omega)\mu + b^3 e^{-\sqrt{-1}t}\omega(\omega^2 - 1) = 0\,. \tag{3}$$

The discriminant of $\det(\mu I - A)$ is $b^6 e^{-2\sqrt{-1}t}\omega^2 f(t, \omega)$ by (1), hence if $\omega \neq 0$, then A has a multiple eigenvalue if and only if $f(t, \omega) = 0$. The equation $f(t, \xi) = 0$ can be implicitly solved by the following lemma.

LEMMA 1. *For each t with $0 \le t < 2\pi$, there is a unique solution $\omega(t)$ of the equation $f(t, \xi) = 0$ satisfying $0 < \omega(t) \le 1$. Moreover, for $t \ge 0$, $\omega(t) = 1$ if and only if $t = \pi$.*

PROOF. If $t = 0$, then $f(0, \xi)$ is monotone increasing on $[0, 1]$ and the equation $f(0, \xi) = 0$ has a unique solution $\omega(0) = 1/3$ in the interval $[0, 1]$.

For fixed t_0 with $0 < t_0 < 2\pi$, we know that $f(t_0, \xi) < f(0, \xi) \le f(0, 1/3) = 0$ on $\xi \in (0, 1/3]$, and $\partial^2 f/\partial\xi^2(t_0, \xi) = 36(9\xi^2 - 1) > 0$ on $\xi \in (1/3, 1]$. Hence $f(t_0, \xi)$ is concave for $1/3 < \xi \le 1$. By $f(t_0, 1) = 8(1 + \cos t_0) \ge 0$, the equation $f(t_0, \xi) = 0$ has a unique solution $\omega(t_0)$ in $(0, 1]$. The last statement is proved by direct computation. This completes the proof of Lemma 1.

By Lemma 1, the domain of $f(t, \xi)$ is divided by five parts: $[0, 2\pi) \times [0, 1] = D_+ \cup D_0 \cup D_- \cup \{(t, 0) \mid 0 \le t < 2\pi\} \cup \{(t, 1) \mid 0 \le t < 2\pi\}$, where $D_+ = \{(t, \xi) \in [0, 2\pi) \times [0, 1) \mid \xi > \omega(t)\}$, $D_0 = \{(t, \xi) \in [0, 2\pi) \times [0, 1) \mid \xi = \omega(t)\}$, and $D_- = \{(t, \xi) \in [0, 2\pi) \times [0, 1) \mid 0 < \xi < \omega(t)\}$. For any $\xi \in [0, 1)$, take a positive number b such that $\xi = \sqrt{1 + \rho/2b^2}$. For $\xi = 1$, there is no such a positive number because $\rho < 0$. Then by Theorem 2, $(t, \xi) \in [0, 2\pi) \times [0, 1)$ defines an isometric immersion from $\mathbf{R}^2$ into $\mathbf{C}H^2$ with non-zero parallel mean curvature vector determined by b and t.

The eigenvalues of B are determined from A as follows. Let μ_0, μ_1, and μ_2 be eigenvalues of A and $\mathbf{v}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ eigenvectors of A corresponding to these μ_0, μ_1, and μ_2, respectively. Since $X(z)$ in Theorem 2 takes value in $U(1, 2)$, we know $B = -T^t\bar{A}T$ and hence $-\bar{\mu}_0$, $-\bar{\mu}_1$, and $-\bar{\mu}_2$ are eigenvalues of B. It follows from the integrability condition of X, $d^2X = 0$, that we have $AB = BA$ and so $B\mathbf{v}_i$ is an eigenvector of μ_i.

Now, we study the case that A has three distinct eigenvalues, that is, $(t, \xi) \in D_+ \cup D_-$. Since the eigenspace of A is one-dimensional, we have $B\mathbf{v}_i = k_i\mathbf{v}_i$ for some complex number k_i $(i = 0, 1, 2)$. Changing indices if necessary, we know that there are three cases:

Case (1) $k_i = -\bar{\mu}_i$, $(i = 0, 1, 2)$,

Case (2) $k_0 = -\bar{\mu}_0$, $k_1 = -\bar{\mu}_2$, and $k_2 = -\bar{\mu}_1$,

Case (3) $k_0 = -\bar{\mu}_1$, $k_1 = -\bar{\mu}_2$, and $k_2 = -\bar{\mu}_0$.

In order to find a correspondence between points of $[0, 2\pi) \times [0, 1)$ and Cases (1), (2), and (3) above, we first prove the following three lemmas.

LEMMA 2. *If A has eigenvalue 0 or b, then $\omega = 0$.*

PROOF. If A has eigenvalue 0, then the characteristic equation (3) with $0 \le \omega < 1$ gives $\omega = 0$. If A has eigenvalue b, similar computations using (3) imply $\omega = 0$, proving Lemma 2.

It follows from Lemma 2 and (3) that $\mu_i \ne 0$ and $\mu_i \ne b$, $(i = 0, 1, 2)$ hold under the assumption that A has three distinct eigenvalues.

LEMMA 3. *We can assume that the first component of $\mathbf{v}_i$ is 1 for $i = 0, 1, 2$.*

PROOF. For fixed i, let $\mathbf{v}_i = {}^t(\alpha, \beta, \gamma)$. By (2), we have

$$\mu_i\alpha - \sqrt{-\frac{\rho}{2}}\,\beta = 0\,, \quad (\mu_i - b)\beta - \bar{c}\gamma = 0\,, \quad \sqrt{-\frac{\rho}{2}}\,\alpha - b\beta - (\mu_i - b)\gamma = 0\,. \tag{4}$$

If we assume $\alpha = 0$, then (4) and Lemma 2 imply $\mathbf{v}_i = 0$, a contradiction. Hence we can set $\alpha = 1$, showing Lemma 3.

LEMMA 4. *The equation $B\mathbf{v}_i = -\bar{\mu}_j\mathbf{v}_i$ holds for some i and j if and only if*

$$\bar{c}\,\bar{\mu}_j + (\mu_i - b)\mu_i = 0,\ \bar{c}\,\bar{\mu}_i + (\mu_j - b)\mu_j = 0\,.$$

PROOF. For $\mathbf{v}_i$, we use the same notation as in the proof of Lemma 3, so we have the equations in (4) with $\alpha = 1$. If $\mathbf{v}_i$ satisfies $B\mathbf{v}_i = -\bar{\mu}_j\mathbf{v}_i$, then the formula (2) implies

$$\bar{\mu}_j + \sqrt{-\frac{\rho}{2}}\,\gamma = 0\,, \quad c\beta - (\bar{\mu}_j - b)\gamma = 0\,, \quad \sqrt{-\frac{\rho}{2}} + (\bar{\mu}_j - b)\beta + b\,\gamma = 0\,. \tag{5}$$

From (4) we know $\beta = \mu_i/\sqrt{-\rho/2}$, $\gamma = (\mu_i - b)\mu_i/(\sqrt{-\rho/2}\,\bar{c})$ and $\rho/2 + b\mu_i + (\mu_i - b)^2\mu_i/\bar{c} = 0$. These equations and (5) yield

$$\bar{c}\,\bar{\mu}_j + (\mu_i - b)\mu_i = 0\,, \quad |c|^2 + (-\bar{\mu}_j + b)(\mu_i - b) = 0\,, \tag{6}$$

which proves Lemma 4.

The following Proposition 1 implies, in particular, that Case (3) does not happen if A has three distinct eigenvalues.

PROPOSITION 1. *If $(t, \xi) \in D_+$, then Case* (1) *holds, and if $(t, \xi) \in D_-$, then Case* (2) *holds.*

PROOF. By Lemma 4, $B\mathbf{v}_i = -\bar{\mu}_j\mathbf{v}_i$ implies $B\mathbf{v}_j = -\bar{\mu}_i\mathbf{v}_j$, hence Case (3) does not occur. Next, we show that on D_+ and D_- only one of Case (1) and Case (2) holds. We prove this by showing that if Case (1) changes to Case (2) at a point (t_0, ξ_0) of the domain D_+ or D_-, then A has a multiple eigenvalue at the point. Indeed, since Case (2) holds at the point, we have $\bar{c}\,\bar{\mu}_1 + (\mu_2 - b)\mu_2 = 0$ at (t_0, ξ_0), by Lemma 4. On the other hand, since Case (1) changes to Case (2) at (t_0, ξ_0), there is a subset of D around (t_0, ξ_0) on which Case (1) holds and hence we have $\bar{c}\,\bar{\mu}_2 + (\mu_2 - b)\mu_2 = 0$, by Lemma 4. Since μ_i is continuous in ξ and t, we have $\bar{c}\,\bar{\mu}_2 + (\mu_2 - b)\mu_2 = 0$ at (t_0, ξ_0). From these two equations, we have $\mu_1 = \mu_2$ at (t_0, ξ_0). Thus on D_+ and D_- only one of Case (1) and Case (2) can hold.

At a point $(0, 1/2) \in D_+$, A has eigenvalues $b/2$, $(3 \pm \sqrt{-3})b/4$ by (3). Since they satisfy $\bar{c}\,\bar{\mu}_i + (\mu_i - b)\mu_i = 0$, $(i = 0, 1, 2)$, Case (1) holds in D_+. At a point $(\pi, 1/3) \in D_-$, A has eigenvalues $4b/3$, $(1 \pm \sqrt{3})b/3$ by (3). Since they do not satisfy $\bar{c}\,\bar{\mu}_i + (\mu_i - b)\mu_i = 0$, $(i = 0, 1, 2)$, Case (2) holds in D_-. This proves Proposition 1.

Since we have obtained all necessary properties of the eigenvalues and eigenvectors of A and B to prove Theorem 1 of this paper, we now study periodicity of the immersion $x = \pi \circ X_0 : \mathbf{R}^2 \to \mathbf{C}H^2$.

PROPOSITION 2. *If $(t, \omega) \in D_+$, then x is doubly periodic.*

PROOF. In this case, Case (1) holds by Proposition 1. Since we have $X(z) = \exp(zA + \bar{z}B)$, we get $X(z)(\mathbf{v}_0\,\mathbf{v}_1\,\mathbf{v}_2) = (e^{\mu_0 z - \bar{\mu}_0\bar{z}}\mathbf{v}_0\; e^{\mu_1 z - \bar{\mu}_1\bar{z}}\mathbf{v}_1\; e^{\mu_2 z - \bar{\mu}_2\bar{z}}\mathbf{v}_2)$. Since the first component of $\mathbf{v}_i$ $(i = 0, 1, 2)$ is 1 by Lemma 3, we have

$$X_0(z) = (e^{\mu_0 z - \bar{\mu}_0 \bar{z}},\ e^{\mu_1 z - \bar{\mu}_1 \bar{z}},\ e^{\mu_2 z - \bar{\mu}_2 \bar{z}})(\mathbf{v}_0\ \mathbf{v}_1\ \mathbf{v}_2)^{-1}\,. \tag{7}$$

If $x(z) = x(w)$ holds for points $z, w \in \mathbf{R}^2$, then $X_0(z) = \gamma X_0(w)$ for some $\gamma \in S^1$, which is equivalent to $\exp(\mu_i z - \bar{\mu}_i \bar{z})$=$\gamma \exp(\mu_i w - \bar{\mu}_i \bar{w})$ by (7). By deleting γ from these equations, we have

$$\exp\{(\mu_0 - \mu_1)(z - w) - (\bar{\mu}_0 - \bar{\mu}_1)(\bar{z} - \bar{w})\} = 1\,,$$

$$\exp\{(\mu_0 - \mu_2)(z - w) - (\bar{\mu}_0 - \bar{\mu}_2)(\bar{z} - \bar{w})\} = 1\,.$$

It follows that $x(z) = x(w)$ holds if and only if $z - w \in \Lambda$, where Λ is a lattice of $\mathbf{R}^2$ of rank 2 defined by $\Lambda = \{z \in \mathbf{R}^2 \mid \mathrm{Im}(\mu_0 - \mu_1)z = n\pi,\ \mathrm{Im}(\mu_0 - \mu_2)z = m\pi,\ n, m \in \mathbf{Z}\}$. Thus the map $x : \mathbf{R}^2 \to \mathbf{C}H^2$ descends to a torus $\mathbf{R}^2/\Lambda$. This proves Proposition 2.

PROPOSITION 3. *If $(t, \omega) \in D_-$, then x is singly periodic.*

PROOF. In this case, Case (2) holds by Proposition 1. By a computation similar to that in Proposition 2, we have

$$X_0(z) = (e^{\mu_0 z - \bar{\mu}_0 \bar{z}},\ e^{\mu_1 z - \bar{\mu}_2 \bar{z}},\ e^{\mu_2 z - \bar{\mu}_1 \bar{z}})(\mathbf{v}_0\ \mathbf{v}_1\ \mathbf{v}_2)^{-1}\,.$$

If $x(z) = x(w)$ holds for points $z, w \in \mathbf{R}^2$, then we have

$$\exp\{(\mu_0 - \mu_1)(z - w) - (\bar{\mu}_0 - \bar{\mu}_2)(\bar{z} - \bar{w})\} = 1\,.$$

It follows that $x(z) = x(w)$ holds if and only if $z - w \in \Lambda$, where Λ is a lattice of $\mathbf{R}^2$ of rank 1 defined by $\Lambda = \{z \in \mathbf{R}^2 \mid (\mu_0 - \mu_1)z - (\bar{\mu}_0 - \bar{\mu}_2)\bar{z} = n \cdot 2\pi\sqrt{-1},\ n \in \mathbf{Z}\}$. Thus the map $x : \mathbf{R}^2 \to \mathbf{C}H^2$ descends to a cylinder $\mathbf{R}^2/\Lambda$. This proves Proposition 3.

Next we study the case where A has a multiple eigenvalue.

PROPOSITION 4. *If $(t, \omega) \in D_0 \setminus \{(0, \omega(0))\}$, then x is singly periodic.*

PROOF. In this case, A has a multiple eigenvalue μ_1 and no triple eigenvalue. We can take a vector $\mathbf{v}_2 \in \mathbf{C}^3$ such that $(A - \mu_1 I)\mathbf{v}_2 = \mathbf{v}_1$ and the first component of $\mathbf{v}_2$ is 1, where $\mathbf{v}_1$ is an eigenvector of μ_1. Then we know $\mathbf{C}^3 = \mathbf{C}\mathbf{v}_0 \oplus \mathbf{C}\mathbf{v}_1 \oplus \mathbf{C}\mathbf{v}_2$ and

$$(B + \bar{\mu}_0 I)\mathbf{v}_0 = 0\,, \quad (B + \bar{\mu}_1 I)\mathbf{v}_1 = 0\,, \quad (B + \bar{\mu}_1 I)\mathbf{v}_2 = \beta\mathbf{v}_1\,, \tag{8}$$

where β is a complex number. We now use that $|\beta| = 1$, as is proved in Lemma 5 below.

By computation similar to that in Proposition 2, we obtain $X_0(z) = (e^{\mu_0 z - \bar{\mu}_0 \bar{z}}, e^{\mu_1 z - \bar{\mu}_1 \bar{z}}, e^{\mu_1 z - \bar{\mu}_1 \bar{z}}(1 + z + \beta\bar{z}))(\mathbf{v}_0\ \mathbf{v}_1\ \mathbf{v}_2)^{-1}$. If we have $x(z) = x(w)$ for points $z, w \in \mathbf{R}^2$, then

$$1 + z + \beta\bar{z} = 1 + w + \beta\bar{w},\ \exp\{(\mu_0 - \mu_1)(z - w) - (\bar{\mu}_0 - \bar{\mu}_1)(\bar{z} - \bar{w})\} = 1\,.$$

Therefore $x(z) = x(w)$ holds if and only if $z - w \in \Lambda$, where Λ is a lattice of $\mathbf{R}^2$ of rank 1 defined by $\Lambda = \{z \in \mathbf{R}^2 \mid z + \beta\bar{z} = 0,\ \mathrm{Im}(\mu_0 - \mu_1)z = n\pi,\ n \in \mathbf{Z}\}$, because $\{z + \beta\bar{z} = 0\}$ is a line in $\mathbf{R}^2$ by $|\beta| = 1$. Thus the map $x : \mathbf{R}^2 \to \mathbf{C}H^2$ descends to a cylinder $\mathbf{R}^2/\Lambda$, proving Proposition 4.

LEMMA 5. *The absolute value of β is* 1.

PROOF. Since $\exp(zA+\bar{z}B)$ takes value in $U(1,2)$, we have $F(\exp(zA+\bar{z}B)\mathbf{v}_1, \exp(zA+\bar{z}B)\mathbf{v}_2)=F(\mathbf{v}_1,\mathbf{v}_2)$. On the other hand, by (8) we also have $F(\exp(zA+\bar{z}B)\mathbf{v}_1, \exp(zA+\bar{z}B)\mathbf{v}_2) = F(e^{\mu_1 z-\bar{\mu}_1\bar{z}}\mathbf{v}_1, e^{\mu_1 z-\bar{\mu}_1\bar{z}}(\mathbf{v}_2+(z+\beta\bar{z})\mathbf{v}_1))$. These two equations yield $(\bar{z}+\bar{\beta}z)F(\mathbf{v}_1,\mathbf{v}_1)=0$ for all $z\in\mathbf{R}^2$, which implies $F(\mathbf{v}_1,\mathbf{v}_1)=0$. Similarly, we obtain $\beta F(\mathbf{v}_1,\mathbf{v}_2)+F(\mathbf{v}_2,\mathbf{v}_1)=0$, by considering $F(\exp(zA+\bar{z}B)\mathbf{v}_2, \exp(zA+\bar{z}B)\mathbf{v}_2)$. Also we know $F(\mathbf{v}_0,\mathbf{v}_1)=0$ by considering $F(\exp(zA+\bar{z}B)\mathbf{v}_0, \exp(zA+\bar{z}B)\mathbf{v}_1)$. If we assume $F(\mathbf{v}_1,\mathbf{v}_2)=0$, then for all $\mathbf{v}\in\mathbf{C}^3$ we have $F(\mathbf{v},\mathbf{v}_1)=0$, which implies $\mathbf{v}_1=0$ and gives a contradiction. Hence we know $F(\mathbf{v}_1,\mathbf{v}_2)\neq 0$ and $\beta=-F(\mathbf{v}_2,\mathbf{v}_1)/F(\mathbf{v}_1,\mathbf{v}_2)$, showing $|\beta|=1$. This proves Lemma 5.

Finally, we study two exceptional cases.

PROPOSITION 5. *If $\omega=0$, then x is singly periodic.*

PROOF. In this case, A has eigenvalues 0 and b and $\mathbf{v}_0={}^t(1,1,0)$, $\mathbf{v}_1={}^t(0,0,1)$, $\mathbf{v}_2={}^t(1,0,-1)$ are eigenvectors of A. Then we obtain $X_0(z)=(e^{bz},\ e^{bz}-e^{b(z-\bar{z})},\ e^{-b\bar{z}})(\mathbf{v}_0\ \mathbf{v}_1\ \mathbf{v}_2)^{-1}$. If we have $x(z)=x(w)$ for points $z,w\in\mathbf{R}^2$, then $\exp(b(z-w))=1$. It follows that $x(z)=x(w)$ holds if and only if $z-w\in\Lambda$, where Λ is a lattice of $\mathbf{R}^2$ of rank 1 defined by $\Lambda=\{z\in\mathbf{R}^2 \mid bz=n\cdot 2\pi\sqrt{-1},\ n\in\mathbf{Z}\}$. Thus the map $x:\mathbf{R}^2\to\mathbf{C}H^2$ descends to a cylinder $\mathbf{R}^2/\Lambda$. This proves Proposition 5.

PROPOSITION 6. *If $(t,\omega)=(0,\omega(0))$, then x is an embedding.*

PROOF. In this case, we know $\omega(0)=1/3$ and A has a triple eigenvalue $2b/3$. Let $\mathbf{v}_0={}^t(1,1/\sqrt{2},-1/\sqrt{2})$, $\mathbf{v}_1={}^t(1,5\sqrt{2}/4,\sqrt{2}/4)$, $\mathbf{v}_2={}^t(1,5\sqrt{2}/4,5\sqrt{2}/2)$. Then we obtain

$$X_0(z)=e^{2b(z-\bar{z})/3}\left(1,1+b(z+\bar{z}),1+b(z+4\bar{z})+\frac{b^2(z+\bar{z})^2}{2}\right)(\mathbf{v}_0\ \mathbf{v}_1\ \mathbf{v}_2)^{-1}.$$

It is easily verified by this formula that $x:\mathbf{R}^2\to\mathbf{C}H^2$ is an embedding, proving Proposition 6.

By these Propositions 2 through 6, for any point $(t,\omega)\in[0,2\pi)\times[0,1)$, we can decide the periodicity of the map determined by b and t, where $\omega=\sqrt{1+\rho/2b^2}$. Therefore we have proven Theorem 1.

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ADDENDUM (December 2, 2003). In this paper, we refer to [5], in which they solve the overdetermined system in Ogata [6]. Recently, we found a gap in [6], but we can fill it to prove Theorem 1 of this paper.

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