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On Abelian p-Extensions of Formal Power Series Fields

Koji SEKIGUCHI

Kochi University of Technology

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Introduction

Let p be a prime number. Then a field k is said to be p-quasifinite, if k is a perfect field of characteristic p and $\mathrm{Gal}(k_{sep}^{[p]}/k) \cong \mathbf{Z}_p$. Here $k_{sep}^{[p]}$ is the maximal separable p-extension of k and $\mathbf{Z}_p$ is the ring of p-adic integers.

Suppose that k is a p-quasifinite field, $n \geqq 1$ and $K = k((t_n)) \cdots ((t_1))$ is a formal power series field in n variables with coefficient field k. Then the nth Milnor K-group $K_n^M K$ of K gives rise to a topological group by introducing the weak topology (see §4). Moreover we put $\Gamma K = \mathrm{Gal}(K_{ab}^{[p]}/K)$, where $K_{ab}^{[p]}$ is the maximal abelian p-extension of K. Then the following results are obtained.

Main Theorem. *Let k be a p-quasifinite field, $n \geqq 1$ and $K = k((t_n)) \cdots ((t_1))$. Then*

(i) *for any element $F \in \Gamma k$ having the property $\Gamma k = F^{\mathbf{Z}_p}$, there exists a homomorphism*

$$\rho_K : K_n^M K \longrightarrow \Gamma K$$

of topological groups which satisfies the following two conditions:

(1) *Take any finite separable p-extension K'/K of fields. Then*

$$\overline{N_{K'/K} K_n^M K'} = \rho_K^{-1}(\mathrm{Gal}(K_{ab}^{[p]}/K' \cap K_{ab}))\,.$$

Moreover, ρ_K induces an isomorphism:

$$K_n^M K / \overline{N_{K'/K} K_n^M K'} \cong \mathrm{Gal}(K' \cap K_{ab}/K)$$

of abelian groups. Here "overline" means the closure of $K_n^M K$ with respect to the weak topology.

(2) *Take any $\alpha \in K_n^M K$. Then*

$$\rho_K(\alpha)\Big|_{k_{ab}^{[p]}} = F^{\ell(\alpha)}\,.$$

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For the mapping $\ell : K_n^M K \to \mathbf{Z}$, *see Lemma* 15, (ii).

(ii) *The mapping from the set of finite abelian* p*-extensions* L *over* K *to the set of open subgroups of* $K_n^M K$ *defined by*

$$L \longmapsto \overline{N_{L/K} K_n^M L}$$

is an inclusion-reversing bijection, and

$$K_n^M K / \overline{N_{L/K} K_n^M L} \cong \mathrm{Gal}(L/K)\,.$$

(iii) *For any finite abelian* p*-extension* L *over* K, *we obtain*

$$L/K \text{ is unramified} \iff U_K^{(0)} \subset \overline{N_{L/K} K_n^M L}\,.$$

Here $U_K^{(0)} = \mathrm{Ker}\,\ell$.

COROLLARY. *The first inequality*:

$$(K_n^M K : N_{K'/K} K_n^M K') \geqq [K' \cap K_{ab} : K]$$

holds for any finite separable p*-extension* K'/K *of fields.*

REMARK. (i) If the second inequality:

$$(K_n^M K : N_{K'/K} K_n^M K') \leqq [K' \cap K_{ab} : K]$$

holds, then the Main Theorem gives rise to the fundamental theorem of class field theory for p-extensions.

(ii) The second inequality is already proved in the case when k is finite or $n = 1$. See [4] and [8].

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1. Here we shall define two rings $((A))^n$, $[[A]]_n$ for a ring A and $n \geqq 1$, and study the fundamental properties of these rings.

Let A be a ring and Γ a totally ordered abelian group. For $x \in A^\Gamma$, we put

$$s(x) = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}\,.$$

Here A^Γ denotes the set of mappings from Γ to A. Then the set

$$A((\Gamma)) = \{x \in A^\Gamma \mid s(x) \text{ is a well-ordered subset of } \Gamma\}$$

is a sub A-module of A^Γ. For $x, y \in A((\Gamma))$, we define $xy \in A((\Gamma))$ by

$$xy : \begin{array}{ccc} \Gamma & \longrightarrow & A \\ \Psi & & \Psi \\ \gamma & \longmapsto & \displaystyle\sum_{\alpha \in \Gamma} x(\alpha) y(\gamma - \alpha). \end{array}$$

Then $A((\Gamma))$ turns out to be a ring with this product (see [1, Chapter 6, §3, Exercise 2]). Moreover we put

$$A[[\Gamma]] = \{x \in A((\Gamma)) \mid x(\gamma) \neq 0 \Rightarrow \gamma \geqq 0\},$$

$$\mathfrak{m} = \{x \in A[[\Gamma]] \mid x(0) = 0\},$$

then $A[[\Gamma]]$ is a subring of $A((\Gamma))$, $\mathfrak{m}$ is an ideal of $A[[\Gamma]]$ and $A[[\Gamma]] = A \oplus \mathfrak{m}$.

For $\alpha \in \Gamma$, we define $t_\alpha \in A((\Gamma))$ by $t_\alpha : \gamma \longmapsto t_\alpha(\gamma) = \delta_{\alpha,\gamma}$. Then $(t_\alpha x)(\gamma) = x(\gamma - \alpha)$ for any $x \in A((\Gamma))$, and the mapping:

$$\begin{array}{ccc} \Gamma & \longrightarrow & A((\Gamma))^\times \\ \Psi & & \Psi \\ \alpha & \longmapsto & t_\alpha \end{array}$$

is an injective homomorphism of groups. In what follows, we denote by t_Γ the image of this mapping. The ring $A((\Gamma))$ is complete with respect to the linear topology with fundamental system of neighborhoods $\Sigma = \{t_\alpha \mathfrak{m} \mid \alpha \in \Gamma, \alpha \geqq 0\}$ of 0.

For a ring A and a totally ordered abelian group Γ, we introduce the mapping

$$\mathrm{ord}_{A[[\Gamma]]} : \begin{array}{ccc} A((\Gamma)) & \longrightarrow & \Gamma \cup \{+\infty\} \\ \Psi & & \Psi \\ x & \longmapsto & \min s(x). \end{array}$$

Here we put $\min \varnothing = +\infty$.

LEMMA 1. *Suppose that A is an integral ring and Γ is a totally ordered abelian group. Then for any $x, y \in A((\Gamma))$, we have*

$$\mathrm{ord}_{A[[\Gamma]]}(x) = +\infty \iff x = 0,$$

$$\mathrm{ord}_{A[[\Gamma]]}(xy) = \mathrm{ord}_{A[[\Gamma]]}(x) + \mathrm{ord}_{A[[\Gamma]]}(y),$$

$$\mathrm{ord}_{A[[\Gamma]]}(x + y) \geqq \min\{\mathrm{ord}_{A[[\Gamma]]}(x),\ \mathrm{ord}_{A[[\Gamma]]}(y)\}.$$

Moreover

$$A[[\Gamma]] = \{x \in A((\Gamma)) \mid \mathrm{ord}_{A[[\Gamma]]}(x) \geqq 0\},$$

$$\mathfrak{m} = \{x \in A((\Gamma)) \mid \mathrm{ord}_{A[[\Gamma]]}(x) > 0\}.$$

COROLLARY. $A((\Gamma))^\times = t_\Gamma \times A[[\Gamma]]^\times$.

Next we introduce the notion of strong homomorphisms of A-modules as follows.

Let A be a ring and Γ_1, Γ_2 totally ordered abelian groups. Then a mapping $\psi : A((\Gamma_1)) \to A((\Gamma_2))$ is said to be a *strong homomorphism* of A-modules, if the following three conditions are satisfied: For any well-ordered subset I of Γ_1,

- $\{\alpha \in I \mid \gamma \in s(\psi(t_\alpha))\}$ is a finite subset of Γ_1 for any $\gamma \in \Gamma_2$,
- $\bigcup_{\alpha \in I} s(\psi(t_\alpha))$ is a well-ordered subset of Γ_2,
- $\psi\left(\sum_{\alpha \in I} a_\alpha t_\alpha\right) = \sum_{\alpha \in I} a_\alpha \psi(t_\alpha)$ for any $(a_\alpha)_{\alpha \in I} \in A^I$.

Let $\text{st.Hom}_A(A(\!(\Gamma_1)\!), A(\!(\Gamma_2)\!))$ denote the set of strong homomorphisms of A-modules from $A(\!(\Gamma_1)\!)$ to $A(\!(\Gamma_2)\!)$. Then the set $\text{st.Hom}_A(A(\!(\Gamma_1)\!), A(\!(\Gamma_2)\!))$ is a sub $A(\!(\Gamma_2)\!)$-module of $\text{Hom}_A(A(\!(\Gamma_1)\!), A(\!(\Gamma_2)\!))$. Moreover we get

$$\psi \in \text{st.Hom}_A(A(\!(\Gamma_1)\!), A(\!(\Gamma_2)\!)),\ \bigoplus_{\alpha \in \Gamma_1} At_\alpha \subset \operatorname{Ker} \psi \implies \psi = 0\,.$$

EXAMPLE 1. Suppose that A is a ring and Γ is a totally ordered abelian group. Then for any $\gamma \in \Gamma$, $x \in A(\!(\Gamma)\!)$, the mapping:

$$\begin{array}{ccc} A(\!(\Gamma)\!) & \longrightarrow & A \\ \cup & & \cup \\ y & \longmapsto & (xy)(\gamma) \end{array}$$

is a strong homomorphism of A-modules.

LEMMA 2. *Let A be a ring and Γ_1, Γ_2 totally ordered abelian groups. Then, for any strong homomorphism $\psi : A(\!(\Gamma_1)\!) \to A(\!(\Gamma_2)\!)$ of A-modules, ψ is a ring homomorphism if and only if $\psi(t_{\alpha+\beta}) = \psi(t_\alpha)\psi(t_\beta)$ for any $\alpha, \beta \in \Gamma_1$ and $\psi(1) = 1$.*

The proof is similar to the case of group rings.

A mapping $\psi : A(\!(\Gamma_1)\!) \to A(\!(\Gamma_2)\!)$ is said to be a strong homomorphism of A-rings, if ψ is a strong homomorphism of A-modules and is a ring homomorphism.

Moreover, for a ring A and totally ordered abelian groups Γ_1, Γ_2, the definition of a mapping $\psi : A[[\Gamma_1]] \to A[[\Gamma_2]]$ to be a strong homomorphism of A-modules or A-rings is similar to the case that $\psi : A(\!(\Gamma_1)\!) \to A(\!(\Gamma_2)\!)$.

For $n \geqq 1$, we define two functors $(\!(\ )\!)^n$, $[[\ ]]_n$: (Rings) $\to$ (Rings) by putting $(\!(A)\!)^n = A(\!(\mathbf{Z}^n)\!)$, $[[A]]_n = A[[\mathbf{Z}^n]]$ for a ring A. Here $\mathbf{Z}^n$ is a totally ordered abelian group with the lexicographical order. Especially if we write $(\!(\ )\!) = (\!(\ )\!)^1$, $[[\ ]] = [[\ ]]_1$, then

$$(\!(\ )\!)^n = (\!(\ )\!) \circ (\!(\ )\!) \circ \cdots \circ (\!(\ )\!) \quad (n\text{th composite})$$

for any $n \geqq 1$.

LEMMA 3. *For a ring A and $n \geqq 1$, we put $t_1 = t_{(1,0,\cdots,0)}, \cdots, t_n = t_{(0,\cdots,0,1)} \in (\!(A)\!)^n$. Then*

(i) *t_i is transcendental over $A(\!(t_n)\!) \cdots (\!(t_{i+1})\!)$ for any $i \in \{1, \cdots, n\}$, and*

$$(\!(A)\!)^n = A(\!(t_n)\!) \cdots (\!(t_1)\!)\,,$$

$$[[A]]_n = A \oplus \bigoplus_{i=1}^{n} t_i A((t_n)) \cdots ((t_{i+1}))[[t_i]]\,,$$

$$\mathfrak{m} = \bigoplus_{i=1}^{n} t_i A((t_n)) \cdots ((t_{i+1}))[[t_i]]\,.$$

Moreover we obtain $t_\Gamma = t_1^{\mathbf{Z}} \times \cdots \times t_n^{\mathbf{Z}}$.

(ii) *If we put* $D = \{x \in ((A))^n \mid x(\gamma) \neq 0 \Rightarrow \gamma < 0\}$, *then*

$$((A))^n = D \oplus [[A]]_n = D \oplus A \oplus \mathfrak{m}\,,$$

$$D = \bigoplus_{i=1}^{n} t_i^{-1} A((t_n)) \cdots ((t_{i+1}))[t_i^{-1}]\,.$$

COROLLARY 1. *Suppose that A is integral. Then*

$$((A))^{n\times} = t_1^{\mathbf{Z}} \times \cdots \times t_n^{\mathbf{Z}} \times [[A]]_n^{\times}\,,$$

$$[[A]]_n^{\times} = A^{\times} \times (1 + \mathfrak{m})\,,$$

$$1 + \mathfrak{m} = \prod_{i=1}^{n} (1 + t_i A((t_n)) \cdots ((t_{i+1}))[[t_i]])\,.$$

COROLLARY 2. *Let A be a field. Then*

(i) $((A))^n$ *is also a field.*

(ii) $[[A]]_n$ *is a strictly complete valuation ring with quotient field* $((A))^n$, *residue field A and value group* $\mathbf{Z}^n$.

(iii) $\mathrm{ord}_{[[A]]_n}$ *is an additive valuation of* $((A))^n$ *corresponding to* $[[A]]_n$.

LEMMA 4. *Let A be an integral ring and* $n \geqq 1$.

(i) *If we write* $[[A]] = A[[t]]$, *then the following three conditions for* $x \in [[A]]_n$ *are equivalent:*

(a) *There exists a strong homomorphism* $\psi : [[A]] \to [[A]]_n$ *of A-rings such that* $\psi(t) = x$.

(b) $\{i \in \mathbf{N} \mid \gamma \in s(x^i)\}$ *is a finite set for any* $\gamma \in \mathbf{Z}^n$, *and* $\bigcup_{i \in \mathbf{N}} s(x^i)$ *is a well-ordered subset of* $\mathbf{Z}^n$.

(c) $x \in \mathfrak{m}$.

(ii) *If* $x \in \mathfrak{m}$ *and* $x \neq 0$, *then the mapping* ψ *in* (a) *is injective.*

PROOF. (i) In general, the following claim is proved:

CLAIM 1. *Put* $\Gamma = \mathbf{Z}^n$. *Let N be a well-ordered subset of* $\Gamma^+ = \{\gamma \in \Gamma \mid \gamma > 0\}$. *Then* $\{i \in \mathbf{N} \mid \gamma \in iN\}$ *is a finite set for any* $\gamma \in \Gamma$, *and* $\bigcup_{i \in \mathbf{N}} iN$ *is a well-ordered subset of* Γ.

If we put $N = s(x)$ in Claim 1, then we can prove (c) $\Rightarrow$ (b). The proof of (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) and (ii) are easy. □

At the end of this section, we consider the principle of substitution and the change of variables in $((A))^n$, by the use of strong homomorphisms of A-rings.

LEMMA 5. *Let A be an integral ring and $n \geqq 1$. For $u_1, \cdots, u_n \in ((A))^n - \{0\}$, we put*

$$M = \begin{bmatrix} \mathrm{ord}_{[[A]]_n}(u_1) \\ \vdots \\ \mathrm{ord}_{[[A]]_n}(u_n) \end{bmatrix} \in M(n, \mathbf{Z}),$$

$$\ell(u_1, \cdots, u_n) = \det \begin{bmatrix} \mathrm{ord}_{[[A]]_n}(u_1) \\ \vdots \\ \mathrm{ord}_{[[A]]_n}(u_n) \end{bmatrix} \in \mathbf{Z},$$

and take the elements $t_1, \cdots, t_n \in ((A))^n$ defined in Lemma 3. Then

(i) *the following three conditions for $u_1, \cdots, u_n \in ((A))^n$ are equivalent:*

(a) *There exists a strong homomorphism $\psi : ((A))^n \to ((A))^n$ of A-rings such that $\psi(t_i) = u_i$ for any $i \in \{1, \cdots, n\}$.*

(b) *u_i is transcendental over $A((u_n)) \cdots ((u_{i+1}))$ for any $i \in \{1, \cdots, n\}$, and $A((u_n)) \cdots ((u_1))$ is a subring of $((A))^n$.*

(c) *$u_1, \cdots, u_n \in ((A))^{n\times}$ and M is an upper triangular matrix such that all the diagonal elements are positive.*

(ii) *Suppose that $u_1, \cdots, u_n$ satisfy the condition (a). Then*

$$\psi \text{ is surjective } \iff \ell(u_1, \cdots, u_n) = 1 .$$

Therefore the following three conditions for $u_1, \cdots, u_n \in ((A))^n$ are also equivalent:

(a_0) *There exists a strong isomorphism $\psi : ((A))^n \to ((A))^n$ of A-rings such that $\psi(t_i) = u_i$ for any $i \in \{1, \cdots, n\}$.*

(b_0) *u_i is transcendental over $A((u_n)) \cdots ((u_{i+1}))$ for any $i \in \{1, \cdots, n\}$, and $A((u_n)) \cdots ((u_1)) = ((A))^n$.*

(c_0) *$u_1, \cdots, u_n \in ((A))^{n\times}$ and M is an upper triangular matrix such that all the diagonal elements are 1.*

PROOF. (i) Using Claim 1 described in the proof of Lemma 4, we can prove (c) $\Rightarrow$ (a). The proof of (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) and (ii) are easy. □

Suppose that $u_1, \cdots, u_n \in ((A))^n$ satisfy the condition (a) in Lemma 5, (i). Then we write

$$((A))^n_u = A((u_n)) \cdots ((u_1)) = \mathrm{Im}\, \psi .$$

Note that $((A))^n = ((A))^n_t$.

2. Here we shall define an $((A))^n$-module $\Omega^n A$ for a ring A and $n \geqq 1$, and study the fundamental properties of this module.

For a ring A and an A-ring B, let $\mathrm{Der}_A B$ denote the set of A-derivations of B and $\Omega_{B/A}$ the B-module of regular differential forms of B over A. Moreover for $n \geqq 1$, we put $\Omega^n_{B/A} = \Omega_{B/A} \wedge \cdots \wedge \Omega_{B/A}$ (nth exterior power as B-modules). For a ring A and $n \geqq 1$, we define a functor Ω^n_A : (A-Rings) $\to$ (A-Mod.) by putting $\Omega^n_A B = \Omega^n_{B/A}$ for an A-ring B. We also write $\Omega_A = \Omega^1_A$. In the following, we consider the case that $B = ((A))^n$.

LEMMA 6. *Suppose that A is a ring and $n \geqq 1$.*

(i) *For any $i \in \{1, \cdots, n\}$, there exists $\partial_i \in \mathrm{Der}_A((A))^n$ such that*

$$(\partial_i x)(\gamma) = (\gamma_i + 1)x(\gamma + e_i) \quad (x \in ((A))^n,\ \gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbf{Z}^n)\,.$$

Here e_i is the ith unit vector: $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbf{Z}^n$.

(ii) *$\partial_1, \cdots, \partial_n \in \mathrm{Der}_A((A))^n$ are linearly independent over $((A))^n$. Moreover $\mathrm{ord}_{[[A]]_n}(\partial_i x) \geqq \mathrm{ord}_{[[A]]_n}(x) - e_i$ for any $x \in ((A))^n$. Thus ∂_i is continuous. If we put*

$$\mathrm{st.Der}_A((A))^n = \mathrm{st.Hom}_A(((A))^n, ((A))^n) \cap \mathrm{Der}_A((A))^n\,,$$

then

$$\mathrm{st.Der}_A((A))^n = \bigoplus_{i=1}^n ((A))^n \partial_i\,.$$

PROOF. (i) Noting that $\partial_i = \dfrac{\partial}{\partial t_i}$, we have $\partial_i \in \mathrm{Der}_A((A))^n$ ($1 \leqq i \leqq n$).

(ii) The linear independence of $\partial_1, \cdots, \partial_n$ is easily proved from $\partial_i t_j = \delta_{ij}$. Since $\partial_i(\sum_{\alpha\in I} a_\alpha t_\alpha) = \sum_{\alpha \in I} a_\alpha \partial_i(t_\alpha)$ holds for any well-ordered subset I of $\mathbf{Z}^n$ and $(a_\alpha)_{\alpha\in I} \in A^I$, we obtain $\partial_i \in \mathrm{st.Der}_A((A))^n$. □

Since the $((A))^n$-module $\Omega_{((A))^n/A}$ has the universal mapping property for A-derivations, there exists a homomorphism $\varphi_i : \Omega_{((A))^n/A} \to ((A))^n$ of $((A))^n$-modules such that $\varphi_i \circ d_{((A))^n/A} = \partial_i$ for any $i \in \{1, \cdots, n\}$. Here we define a homomorphism

$$\varphi : \begin{array}{ccc} \Omega_{((A))^n/A} & \longrightarrow & ((A))^n \times \cdots \times ((A))^n \\ \cup & & \cup \\ \omega & \longmapsto & (\varphi_1(\omega), \cdots, \varphi_n(\omega)) \end{array}$$

of $((A))^n$-modules. In what follows we write $d = d_{((A))^n/A}$.

LEMMA 7. *Let A be a ring and $n \geqq 1$. Take the elements $t_1, \cdots, t_n$ defined in Lemma 3. Then $\omega - \sum_{i=1}^n \varphi_i(\omega)dt_i \in \mathrm{Ker}\,\varphi$ for any $\omega \in \Omega_{((A))^n/A}$. Thus*

$$\Omega_{((A))^n/A} = \mathrm{Ker}\,\varphi \oplus ((A))^n dt_1 \oplus \cdots \oplus ((A))^n dt_n\,.$$

EXAMPLE 2. For any $f \in ((A))^n$, we have

$$\varphi(df) = \left(\frac{\partial f}{\partial t_1}, \cdots, \frac{\partial f}{\partial t_n}\right).$$

Therefore

$$df - \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} dt_i \in \operatorname{Ker} \varphi \,.$$

For a ring A and $n \geqq 1$, we consider the $(\!(A)\!)^n$-module

$$\Omega^n A = \Omega_A^n (\!(A)\!)^n = \Omega_{(\!(A)\!)^n/A}^{\;n} \,.$$

For $(\omega_1, \cdots, \omega_n) \in \Omega_{(\!(A)\!)^n/A} \times \cdots \times \Omega_{(\!(A)\!)^n/A}$, we put

$$[\varphi_j(\omega_i)] = \begin{bmatrix} \varphi_1(\omega_1) & \cdots & \varphi_n(\omega_1) \\ \vdots & & \vdots \\ \varphi_1(\omega_n) & \cdots & \varphi_n(\omega_n) \end{bmatrix} = \begin{bmatrix} \varphi(\omega_1) \\ \vdots \\ \varphi(\omega_n) \end{bmatrix} \in M(n, (\!(A)\!)^n)$$

and define the mapping

$$\Phi : \begin{array}{ccc} \Omega_{(\!(A)\!)^n/A} \times \cdots \times \Omega_{(\!(A)\!)^n/A} & \longrightarrow & (\!(A)\!)^n \\ \Cup & & \Cup \\ (\omega_1, \cdots, \omega_n) & \longmapsto & \det[\varphi_j(\omega_i)] \,. \end{array}$$

Since Φ is $(\!(A)\!)^n$-multilinear and alternating, there exists a homomorphism

$$\varphi_A^n : \Omega^n A \longrightarrow (\!(A)\!)^n$$

of $(\!(A)\!)^n$-modules such that $\Phi = \varphi_A^n \circ c$. Here $c : \Omega_{(\!(A)\!)^n/A} \times \cdots \times \Omega_{(\!(A)\!)^n/A} \to \Omega^n A$ is the canonical mapping. Thus $\varphi_A^n(\omega_1 \wedge \cdots \wedge \omega_n) = \Phi(\omega_1, \cdots, \omega_n)$.

LEMMA 8. *Let A be a ring and $n \geqq 1$. Take the elements $t_1, \cdots, t_n$ defined in Lemma 3. Then $\omega - \varphi_A^n(\omega) dt_1 \wedge \cdots \wedge dt_n \in \operatorname{Ker} \varphi_A^n$ for any $\omega \in \Omega^n A$. Thus*

$$\Omega^n A = \Omega_A^n (\!(A)\!)^n = \Omega_{(\!(A)\!)^n/A}^{\;n} = \operatorname{Ker} \varphi_A^n \oplus (\!(A)\!)^n dt_1 \wedge \cdots \wedge dt_n \,.$$

EXAMPLE 3. (i) For any $f_1, \cdots, f_n \in (\!(A)\!)^n$, we have

$$\varphi_A^n(df_1 \wedge \cdots \wedge df_n) = \det J(f/t) \,.$$

Therefore

$$df_1 \wedge \cdots \wedge df_n - \det J(f/t) dt_1 \wedge \cdots \wedge dt_n \in \operatorname{Ker} \varphi_A^n \,.$$

Here

$$J(f/t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial t_1} & \cdots & \dfrac{\partial f_1}{\partial t_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial t_1} & \cdots & \dfrac{\partial f_n}{\partial t_n} \end{bmatrix} \in M(n, (\!(A)\!)^n)$$

is the Jacobian matrix.

(ii) For $(m_1, \cdots, m_n) \in \mathbf{Z}^n$, we put $\omega = t_1^{m_1} \cdots t_n^{m_n} dt_1 \wedge \cdots \wedge dt_n \in \Omega^n A$. If there exists $j \in \{1, \cdots, n\}$ such that $m_j + 1 \in A^\times$, then there exist monomials $f_1, \cdots, f_n \in ((A))^n$ such that

$$\omega - df_1 \wedge \cdots \wedge df_n \in \operatorname{Ker} \varphi_A^n .$$

For a ring A and $n \geqq 1$, we introduce the mappings

$$\operatorname{ord}_{\Omega^n A} = \operatorname{ord}_{[[A]]_n} \circ \varphi_A^n : \Omega^n A \longrightarrow \mathbf{Z}^n \cup \{+\infty\}$$

and

$$\operatorname{res}_A^n : \begin{array}{ccc} \Omega^n A & \longrightarrow & A \\ \Psi & & \Psi \\ \omega & \longmapsto & \varphi_A^n(\omega)(-1, \cdots, -1). \end{array}$$

Then $\operatorname{ord}_{\Omega^n A}(\omega)$ is called the order of $\omega \in \Omega^n A$ and $\operatorname{res}_A^n(\omega)$ is called the residue of $\omega \in \Omega^n A$.

LEMMA 9. *Let A be a ring and $n \geqq 1$. Then*

(i) *the mapping $\operatorname{res}_A^n : \Omega^n A \to A$ is a homomorphism of A-modules.*

(ii) *For any $\omega \in \Omega^n A$, we have*

$$\operatorname{ord}_{\Omega^n A}(\omega) = +\infty \iff \omega \in \operatorname{Ker} \varphi_A^n ,$$
$$\operatorname{ord}_{\Omega^n A}(\omega) \geqq (-1, \cdots, -1, 0) \implies \operatorname{res}_A^n(\omega) = 0 .$$

(iii) *For any $f_1, \cdots, f_n \in ((A))^n$, we have*

$$\operatorname{res}_A^n(df_1 \wedge \cdots \wedge df_n) = 0 .$$

PROOF. The statements (i) and (ii) are obvious.

(iii) Note first that (iii) is proved in the case that $f_1, \cdots, f_n$ are all monomials. Since the mapping:

$$\begin{array}{ccc} ((A))^n & \longrightarrow & A \\ \Psi & & \Psi \\ f_1 & \longmapsto & \operatorname{res}_A^n(df_1 \wedge \cdots \wedge df_n) \end{array}$$

is a strong homomorphism of A-modules for any fixed $f_2, \cdots, f_n \in ((A))^n$, (iii) is valid for $f_1 \in ((A))^n$. Repeating this process, we obtain $\operatorname{res}_A^n(df_1 \wedge \cdots \wedge df_n) = 0$ for any $f_1, \cdots, f_n \in ((A))^n$. $\square$

LEMMA 10. *Let A be an integral ring and $n \geqq 1$. For $u_1, \cdots, u_n \in ((A))^{n\times}$ and $t_1, \cdots, t_n$ defined in Lemma* 3, *we put*

$$J^L(u/t) = \begin{bmatrix} \dfrac{t_1}{u_1}\dfrac{\partial u_1}{\partial t_1} & \cdots & \dfrac{t_n}{u_1}\dfrac{\partial u_1}{\partial t_n} \\ \vdots & & \vdots \\ \dfrac{t_1}{u_n}\dfrac{\partial u_n}{\partial t_1} & \cdots & \dfrac{t_n}{u_n}\dfrac{\partial u_n}{\partial t_n} \end{bmatrix} \in M(n, ((A))^n)\,.$$

Then

(i) *$J^L(u/t) \in M(n, [[A]]_n)$ and*

$$J^L(u/t)(0, \cdots, 0) = ME_{n,A} \in M(n, A)\,.$$

Here M is the matrix defined in Lemma 5 *and $E_{n,A}$ is the unit matrix in $M(n, A)$.*

(ii) $\det J^L(u/t) \in [[A]]_n$ *and*

$$\det J^L(u/t)(0, \cdots, 0) = \ell(u_1, \cdots, u_n) \cdot 1_A\,.$$

Here $\ell(u_1, \cdots, u_n)$ is the integer defined in Lemma 5 *and 1_A is the unity of A.*

(iii) *For any $x \in ((A))^n$, we obtain*

$$\mathrm{res}_A^n\left(x\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}\right) = (x \det J^L(u/t))(0, \cdots, 0)\,.$$

PROOF. (i) Since $\mathrm{ord}_{[[A]]_n}(\partial_j x) \geqq \mathrm{ord}_{[[A]]_n}(x) - \mathrm{ord}_{[[A]]_n}(t_j)$ for any $x \in ((A))^n$, we have $J^L(u/t) \in M(n, [[A]]_n)$ by putting $x = u_i$ $(1 \leqq i \leqq n)$. If we put $\gamma = \mathrm{ord}_{[[A]]_n}(x) - e_j$, then $(\frac{t_j}{x}\frac{\partial x}{\partial t_j})(0, \cdots, 0) = \frac{(\partial_j x)(\gamma)}{x(\gamma+e_j)} = (\gamma_j + 1) \cdot 1_A = (\mathrm{ord}_{[[A]]_n}(x))_j \cdot 1_A$. Therefore $J^L(u/t)(0, \cdots, 0) = ME_{n,A}$.

(ii) This statement is proved easily from (i).

(iii) By Example 3, (i), we get $x\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n} - x \det J^L(u/t)\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \in \mathrm{Ker}\,\varphi_A^n$. Thus $\mathrm{res}_A^n(x\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}) = \mathrm{res}_A^n(x \det J^L(u/t)\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}) = (x \det J^L(u/t))(0, \cdots, 0)$. □

Next we shall study the properties with respect to the change of variables.

LEMMA 11. *Let A be an integral ring and $n \geqq 1$. Suppose that $u_1, \cdots, u_n \in ((A))^n$ satisfy the condition* (a) *in Lemma* 5, (i).

(i) *For any $i \in \{1, \cdots, n\}$, there exists a unique $\dfrac{\partial}{\partial u_i} \in \mathrm{st.Der}_A((A))_u^n$ such that*

$$\frac{\partial}{\partial u_i}(u_j^m) = mu_j^{m-1}\delta_{ij} \quad (m \in \mathbf{Z},\ j = 1, 2, \cdots, n)\,.$$

(ii) $\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}$ *are linearly independent over* $(\!(A)\!)_u^n$ *and*

$$\mathrm{st.Der}_A (\!(A)\!)_u^n = \bigoplus_{i=1}^{n} (\!(A)\!)_u^n \frac{\partial}{\partial u_i} .$$

The proof is similar to that of Lemma 6.

Let A be an integral ring and $n \geqq 1$. Suppose that $u_1, \cdots, u_n \in (\!(A)\!)^n$ satisfy the condition (a) in Lemma 5, (i). Then there exists a homomorphism $\varphi_i^u : \Omega_{(\!(A)\!)_u^n/A} \to (\!(A)\!)_u^n$ of $(\!(A)\!)_u^n$-modules such that $\varphi_i^u \circ d = \frac{\partial}{\partial u_i}$ for any $i \in \{1, \cdots, n\}$. Here we define a homomorphism

$$\varphi^u = (\varphi_1^u, \cdots, \varphi_n^u) : \Omega_{(\!(A)\!)_u^n/A} \longrightarrow (\!(A)\!)_u^n \times \cdots \times (\!(A)\!)_u^n$$

of $(\!(A)\!)_u^n$-modules. Note that $\varphi_i = \varphi_i^t$ $(1 \leqq i \leqq n)$ and $\varphi = \varphi^t$.

Next we put

$$\Omega_u^n A = \Omega_A^n (\!(A)\!)_u^n = \Omega_{(\!(A)\!)_u^n/A}^{\,n} .$$

Then we can define an alternating $(\!(A)\!)_u^n$-multilinear mapping

$$\Phi^u : \Omega_{(\!(A)\!)_u^n/A} \times \cdots \times \Omega_{(\!(A)\!)_u^n/A} \longrightarrow (\!(A)\!)_u^n$$

and a homomorphism

$$\varphi_u^n : \Omega_u^n A \longrightarrow (\!(A)\!)_u^n$$

of $(\!(A)\!)_u^n$-modules by putting

$$\Phi^u(\omega_1, \cdots, \omega_n) = \det[\varphi_j^u(\omega_i)] ,$$

$$\varphi_u^n(\omega_1 \wedge \cdots \wedge \omega_n) = \Phi^u(\omega_1, \cdots, \omega_n)$$

for $\omega_1, \cdots, \omega_n \in \Omega_{(\!(A)\!)_u^n/A}$. Note that $\Omega^n A = \Omega_t^n A$, $\Phi = \Phi^t$ and $\varphi_A^n = \varphi_t^n$.

Moreover we introduce the mappings

$$\mathrm{ord}_{\Omega_u^n A} = \mathrm{ord}_{[[A]]_n^u} \circ \varphi_u^n : \Omega_u^n A \longrightarrow \mathbf{Z}^n \cup \{+\infty\}$$

and

$$\mathrm{res}_u^n : \begin{array}{ccc} \Omega_u^n A & \longrightarrow & A \\ \cup & & \cup \\ \omega & \longmapsto & \varphi_u^n(\omega)(-1, \cdots, -1) . \end{array}$$

Note that $\mathrm{res}_A^n = \mathrm{res}_t^n$.

LEMMA 12. *Suppose that A is an integral ring, $n \geqq 1$ and $u_1, \cdots, u_n \in ((A))^n$ satisfy the condition* (a) *in Lemma* 5, (i). *Let $i : ((A))^n_u \hookrightarrow ((A))^n$ denote the natural inclusion mapping. Then*

(i) *for any $\omega_0, \omega_1, \cdots, \omega_n \in \Omega_{((A))^n_u/A}$ and $\omega \in \Omega^n_u A$, we obtain*

$$(\varphi^t \circ \Omega_A i)(\omega_0) = \varphi^u(\omega_0) J(u/t)\,,$$

$$(\Phi^t \circ (\Omega_A i \times \cdots \times \Omega_A i))(\omega_1, \cdots, \omega_n) = \Phi^u(\omega_1, \cdots, \omega_n) \det J(u/t)\,,$$

$$(\varphi^n_t \circ \Omega^n_A i)(\omega) = \varphi^n_u(\omega) \det J(u/t)\,.$$

(ii) *For any $\omega \in \Omega^n_u A$, we have*

$$(\mathrm{ord}_{\Omega^n_t A} \circ \Omega^n_A i)(\omega) = \mathrm{ord}_{\Omega^n_u A}(\omega) M + \mathrm{ord}_{[[A]]_n}(\det J(u/t))\,.$$

Here M is the matrix defined in Lemma 5.

(iii) *For any $\omega \in \Omega^n_u A$, we have*

$$(\mathrm{res}^n_t \circ \Omega^n_A i)(\omega) = \ell(u_1, \cdots, u_n) \mathrm{res}^n_u(\omega)\,.$$

Here $\ell(u_1, \cdots, u_n)$ is the integer defined in Lemma 5.

PROOF. The statements (i) and (ii) are easy to verify.

(iii) For $(m_1, \cdots, m_n) \in \mathbf{Z}^n$, we put $\omega = u_1^{m_1} \cdots u_n^{m_n} du_1 \wedge \cdots \wedge du_n \in \Omega^n_u A$. By Example 3, (i) and Lemma 10, (iii), we have

$(m_1, \cdots, m_n) = (-1, \cdots, -1) \Longrightarrow (\mathrm{res}^n_t \circ \Omega^n_A i)(\omega) = \ell(u_1, \cdots, u_n) \cdot 1_A$.

Moreover, by Example 3, (ii) and the fact that res^n is a natural transformation, we obtain

$(m_1, \cdots, m_n) \neq (-1, \cdots, -1) \Longrightarrow (\mathrm{res}^n_t \circ \Omega^n_A i)(\omega) = 0$.

Next we put $\omega = f du_1 \wedge \cdots \wedge du_n \in \Omega^n_u A$ for $f \in ((A))^n_u$. Then both the mappings: $f \mapsto (\mathrm{res}^n_t \circ \Omega^n_A i)(\omega)$ and $f \mapsto \ell(u_1, \cdots, u_n)\mathrm{res}^n_u(\omega)$ are strong homomorphisms of A-modules. Therefore $(\mathrm{res}^n_t \circ \Omega^n_A i)(\omega) = \ell(u_1, \cdots, u_n)\mathrm{res}^n_u(\omega)$. Since $\Omega^n_u A = \mathrm{Ker}\ \varphi^n_u \oplus ((A))^n_u du_1 \wedge \cdots \wedge du_n$, we get $(\mathrm{res}^n_t \circ \Omega^n_A i)(\omega) = \ell(u_1, \cdots, u_n)\mathrm{res}^n_u(\omega)$ for any $\omega \in \Omega^n_u A$. □

COROLLARY. *Suppose that $u_1, \cdots, u_n$ satisfy the condition* (a_0) *in Lemma* 5, (ii). *Then*

(i) *for any $\omega_0, \omega_1, \cdots, \omega_n \in \Omega_{((A))^n/A}$ and $\omega \in \Omega^n A$, we obtain*

$$\varphi^t(\omega_0) = \varphi^u(\omega_0) J(u/t)\,,$$

$$\Phi^t(\omega_1, \cdots, \omega_n) = \Phi^u(\omega_1, \cdots, \omega_n) \det J(u/t)\,,$$

$$\varphi^n_t(\omega) = \varphi^n_u(\omega) \det J(u/t)\,.$$

(ii) *For any $\omega \in \Omega^n A$, we have*

$$\mathrm{ord}_{\Omega^n_t A}(\omega) = (\mathrm{ord}_{\Omega^n_u A}(\omega) + (1, \cdots, 1))M - (1, \cdots, 1)\,.$$

(iii) *For any $\omega \in \Omega^n A$, we have*

$$\mathrm{res}^n_t(\omega) = \mathrm{res}^n_u(\omega)\,.$$

3. Here we shall define the Milnor K-group M_nB for a ring B and $n \geqq 0$, and study the fundamental properties of this group.

For a ring B and $n \geqq 0$, we define the nth Milnor K-group M_nB of B as follows:

- If $n = 0$, then we put

$$M_0B = \mathbf{Z}.$$

- If $n \geqq 1$, then we put

$$M_nB = B^\times \otimes \cdots \otimes B^\times / I_B ,$$

where I_B is the subgroup of the nth tensor product $B^\times \otimes \cdots \otimes B^\times$ generated by the sets

$$\{a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_j \otimes \cdots \otimes a_n \mid a_i \in B^\times, a_i + a_j = 1 \text{ for some } i \neq j\}$$

and

$$\{a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_j \otimes \cdots \otimes a_n \mid a_i \in B^\times, a_i + a_j = 0 \text{ for some } i \neq j\}.$$

Then we obtain a functor M_n : (Rings) $\longrightarrow$ (C.Groups) for any $n \geqq 0$. For an inclusion mapping $i : A \hookrightarrow B$ of rings, we also write $M_{nA|B} = M_n i$.

REMARK. (i) The group operation in M_nB will be written multiplicatively for $n \geqq 1$, although $M_0B = \mathbf{Z}$. Especially if $n = 1$, then $I_B = 1$. Therefore $M_1B = B^\times$.

(ii) $M_nB \cong K_n^M B$ for a field B. See [2, Chapter IX, (1)].

LEMMA 13. *Suppose that B is a ring and $n \geqq 1$. For $a_1, \cdots, a_n \in B^\times$, we put*

$$\{a_1, \cdots, a_n\} = a_1 \otimes \cdots \otimes a_n \bmod I_B \in M_nB .$$

Then

(M-1) $\{a_1, \cdots, a_n\}$ $(a_1, \cdots, a_n \in B^\times)$ *generate* M_nB,

(M-2) $\{a_1, \cdots, a_{i-1}, bc, a_{i+1}, \cdots, a_n\}$
$= \{a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n\}\{a_1, \cdots, a_{i-1}, c, a_{i+1}, \cdots, a_n\}$,

(M-3) *if* $a_i + a_j = 0$ *or* 1 *for some* $i \neq j$, *then* $\{a_1, \cdots, a_n\} = 1$.

COROLLARY. $\{a_{\sigma(1)}, \cdots, a_{\sigma(n)}\} = \{a_1, \cdots, a_n\}^{\mathrm{sgn}(\sigma)}$ *for any* $\sigma \in S_n$.

Let B be an integral ring, t an indeterminate over B and $n \geqq 1$. Then there exists a unique homomorphism $\psi_t : M_{n-1}B((t)) \to M_nB((t))$ of groups such that $\psi_t(\{u_2, \cdots, u_n\}) = \{t, u_2, \cdots, u_n\}$ for any $u_2, \cdots, u_n \in B((t))^\times$. For $r \geqq 1$, let $U^{(r)}_{M_nB((t))}$ denote the subgroup of $M_nB((t))$ generated by $\{u_1, \cdots, u_n\}$, where $u_1 \in 1 + t^rB[[t]]$, $u_2, \cdots, u_n \in B((t))^\times$. Then we have $\psi_t(U^{(r)}_{M_{n-1}B((t))}) \subset U^{(r)}_{M_nB((t))}$.

LEMMA 14. *Suppose that B is an integral ring, t is an indeterminate over B and $n \geqq 1$. Let $\varphi_0 : B[[t]] \to B$ denote the ring homomorphism defined by $\varphi_0(f) = f(0)$ for any $f \in B[[t]]$. Then*

(i) *there exists a unique homomorphism* $\partial_1 : M_n B((t)) \to M_n B$ *of groups, which satisfies*

$$\partial_1(\{u_1, \cdots, u_n\}) = \{\varphi_0(u_1 t^{-\mathrm{ord}_{B[[t]]}(u_1)}), \cdots, \varphi_0(u_n t^{-\mathrm{ord}_{B[[t]]}(u_n)})\}$$

for any $u_1, \cdots, u_n \in B((t))^\times$. *Moreover we obtain* $\partial_1 \circ M_{nB|B((t))} = id_{M_n B}$. *Thus* $M_n B((t)) = \mathrm{Im}\, M_{nB|B((t))} \times \mathrm{Ker}\, \partial_1$.

(ii) *There exists a unique homomorphism* $\partial_2 : M_n B((t)) \to M_{n-1} B$ *of groups, which satisfies*

$$\partial_2(\{t, u_2, \cdots, u_n\}) = \{\varphi_0(u_2), \cdots, \varphi_0(u_n)\}$$

for any $u_2, \cdots, u_n \in B[[t]]^\times$ *and* $\mathrm{Im}\, M_{nB[[t]]|B((t))} \subset \mathrm{Ker}\, \partial_2$. *Moreover we obtain* $\partial_2 \circ \psi_t \circ M_{n-1B|B((t))} = id_{M_{n-1}B}$. *Thus* $M_n B((t)) = \mathrm{Im}(\psi_t \circ M_{n-1B|B((t))}) \times \mathrm{Ker}\, \partial_2$.

(iii) $\mathrm{Ker}\, \partial_1 = \mathrm{Im}(\psi_t \circ M_{n-1B|B((t))}) \times U^{(1)}_{M_n B((t))}$ *and* $\mathrm{Ker}\, \partial_2 = \mathrm{Im}\, M_{nB|B((t))} \times U^{(1)}_{M_n B((t))}$.

The proof is similar to the case that B is a field. See also [2, Chapter IX, (2.1), (2.2), (2.3)].

REMARK. In what follows, we shall regard $M_n B$ and $M_{n-1} B$ as subgroups of $M_n B((t))$ by the injections $M_{nB|B((t))}$ and $\psi_t \circ M_{n-1B|B((t))}$, respectively:

$$M_{nB|B((t))} : M_n B \hookrightarrow M_n B((t)) ,$$

$$\psi_t \circ M_{n-1B|B((t))} : M_{n-1} B \hookrightarrow M_n B((t)) .$$

Moreover we write

$$U^{(0)}_{M_n B((t))} = \mathrm{Ker}\, \partial_2 = M_n B \times U^{(1)}_{M_n B((t))} .$$

Then we have

$$M_n B((t)) = M_{n-1} B \times M_n B \times U^{(1)}_{M_n B((t))} = M_{n-1} B \times U^{(0)}_{M_n B((t))} .$$

LEMMA 15. *Suppose that* A *is an integral ring and* $n \geqq 1$. *Then*

(i) $$M_n((A))^n = \{t_1, \cdots, t_n\}^{\mathbf{Z}} \times \prod_{i=1}^{n} U^{(0)}_{M_{n-i+1}((A))^{n-i+1}}$$

$$= \prod_{i=1}^{n} M_{n-i+1}((A))^{n-i} \times \{t_1, \cdots, t_n\}^{\mathbf{Z}} \times \prod_{i=1}^{n} U^{(1)}_{M_{n-i+1}((A))^{n-i+1}} ,$$

where $((A))^{n-i+1} = A((t_n)) \cdots ((t_i))$ $(1 \leqq i \leqq n)$.

(ii) *There exists a unique homomorphism* $\ell : M_n((A))^n \to \mathbf{Z}$ *of groups, which satisfies*

$$\ell(\{u_1, \cdots, u_n\}) = \ell(u_1, \cdots, u_n)$$

for any $u_1, \cdots, u_n \in ((A))^{n\times}$. *Here* $\ell(u_1, \cdots, u_n)$ *is the integer defined in Lemma* 5. *Moreover we obtain* $M_n((A))^n = \{t_1, \cdots, t_n\}^{\mathbf{Z}} \times \mathrm{Ker}\,\ell$ *and*

$$\mathrm{Ker}\,\ell = \prod_{i=1}^{n} U^{(0)}_{M_{n-i+1}((A))^{n-i+1}} .$$

Therefore $\ell = \partial_2^1 \circ \cdots \circ \partial_2^n : M_n((A))^n \to \mathbf{Z}$ *for* $\partial_2^i : M_i((A))^i \to M_{i-1}((A))^{i-1}$ $(1 \leqq i \leqq n)$.

PROOF. We can prove (i) easily from Lemma 14 and $U^{(0)}_{M_{n-i+1}((A))^{n-i+1}} = M_{n-i+1}((A))^{n-i} \times U^{(1)}_{M_{n-i+1}((A))^{n-i+1}}$.

(ii) Since the mapping $\ell : ((A))^{n\times} \times \cdots \times ((A))^{n\times} \to \mathbf{Z}$ is multilinear, we have a homomorphism $\ell : ((A))^{n\times} \otimes \cdots \otimes ((A))^{n\times} \to \mathbf{Z}$ of groups. Then, by $I_{((A))^n} \subset \mathrm{Ker}\,\ell$, we obtain a homomorphism $\ell : M_n((A))^n \to \mathbf{Z}$ of groups. If we introduce the mapping $\psi : \mathbf{Z} \to M_n((A))^n$ defined by $\psi(m) = \{t_1, \cdots, t_n\}^m$ for any $m \in \mathbf{Z}$, then $\ell \circ \psi = id_{\mathbf{Z}}$. Thus $M_n((A))^n = \{t_1, \cdots, t_n\}^{\mathbf{Z}} \times \mathrm{Ker}\,\ell$. Moreover, by the definition of $U^{(0)}_{M_{n-i+1}((A))^{n-i+1}}$ and (i), we have $\mathrm{Ker}\,\ell = \prod_{i=1}^{n} U^{(0)}_{M_{n-i+1}((A))^{n-i+1}}$. This implies $\ell = \partial_2^1 \circ \cdots \circ \partial_2^n$. □

For an integral ring A and $n \geqq 1$, there exists a linear topology on $M_n((A))^n$ with fundamental system of neighborhoods $\Sigma = \{U^{(r)}_{M_n((A))^n} \mid r \geqq 1\}$ of 0. This topology is said to be the valuation topology on $M_n((A))^n$. Then $M_n((A))^n$ is a topological group with respect to the valuation topology.

4. Here we shall define a group pairing $\mathrm{Res}_\infty^{((A))^n} : M_n((A))^n \times W_\infty((A))^n \to W_\infty A$ for an integral ring A and $n \geqq 1$, and study the fundamental properties of this pairing.

In the following, we consider the ring of Witt vectors with respect to the prime number p. For a ring A, let $W_\infty A$ denote the ring of Witt vectors of infinite length over A. Then the next results are easily obtained from Lemma 3, (ii).

LEMMA 16. *Suppose that* A *is a ring and* $n \geqq 1$. *Then*

$$W_\infty((A))^n = W_\infty D \oplus W_\infty[[A]]_n = W_\infty D \oplus W_\infty A \oplus W_\infty \mathfrak{m} ,$$

$$W_\infty D = \bigoplus_{i=1}^{n} W_\infty(t_i^{-1} A((t_n)) \cdots ((t_{i+1}))[t_i^{-1}]) .$$

Here, let

$$0^* : W_\infty((A))^n \longrightarrow W_\infty A$$

denote the projection with respect to the decomposition: $W_\infty((A))^n = W_\infty D \oplus W_\infty A \oplus W_\infty \mathfrak{m}$.

For an integral ring A and $n \geqq 1$, we shall define a group pairing $\mathrm{Res}_\infty^{((A))^n} : M_n((A))^n \times W_\infty((A))^n \to W_\infty A$ as follows.

First we consider the case that $p \in A^\times$. For $u_1, \cdots, u_n \in ((A))^{n\times}$ and $b \in W_\infty((A))^n$, we put

$$c_i = \mathrm{res}_A^n\left(w_i(b)\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}\right) \in A \quad (i \geqq 0)$$

and define the mapping

$$\begin{array}{rccc} & ((A))^{n\times} \times \cdots \times ((A))^{n\times} \times W_\infty((A))^n & \longrightarrow & W_\infty A \\ \Psi_1 : & \Psi & & \Psi \\ & (u_1, \cdots, u_n, b) & \longmapsto & \theta_A^{-1}(c_0, c_1, \cdots) . \end{array}$$

Here w_i $(i \geqq 0)$ are the Witt polynomials and

$$\begin{array}{rccc} & W_\infty A & \longrightarrow & A^{\mathbf{N}} \\ \theta_A : & \Psi & & \Psi \\ & a & \longmapsto & (w_0(a), w_1(a), \cdots) . \end{array}$$

Since Ψ_1 is multilinear with respect to $u_1, \cdots, u_n \in ((A))^{n\times}$ for any fixed $b \in W_\infty((A))^n$, we obtain a mapping

$$\Psi_2 : ((A))^{n\times} \otimes \cdots \otimes ((A))^{n\times} \times W_\infty((A))^n \longrightarrow W_\infty A$$

by putting $\Psi_2(u_1 \otimes \cdots \otimes u_n, b) = \Psi_1(u_1, \cdots, u_n, b)$. Since

$$\Psi_2(I_{((A))^n} \times W_\infty((A))^n) = 0 ,$$

we can define a mapping

$$\mathrm{Res}_\infty^{((A))^n} : M_n((A))^n \times W_\infty((A))^n \longrightarrow W_\infty A$$

by $\mathrm{Res}_\infty^{((A))^n}(\{u_1, \cdots, u_n\}, b) = \Psi_1(u_1, \cdots, u_n, b)$. Therefore we have

$$w_i(\mathrm{Res}_\infty^{((A))^n}(\{u_1, \cdots, u_n\}, b)) = \mathrm{res}_A^n(w_i(b)\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n}) \in A \quad (i \geqq 0) .$$

LEMMA 17. *Let A be an integral ring and $n \geqq 1$. Assume that $p \in A^\times$.*

(i) *For any $\alpha, \alpha' \in M_n((A))^n$, $b, b' \in W_\infty((A))^n$, $c \in W_\infty A$, and for any ring homomorphism $\varphi : A \to B$, we obtain*

(R-1) $\mathrm{Res}_\infty^{((A))^n}(\alpha\alpha', b) = \mathrm{Res}_\infty^{((A))^n}(\alpha, b) + \mathrm{Res}_\infty^{((A))^n}(\alpha', b)$

(R-2) $\mathrm{Res}_\infty^{((A))^n}(\alpha, b + b') = \mathrm{Res}_\infty^{((A))^n}(\alpha, b) + \mathrm{Res}_\infty^{((A))^n}(\alpha, b')$

(R-3) $\mathrm{Res}_\infty^{((A))^n}(\alpha, cb) = c\mathrm{Res}_\infty^{((A))^n}(\alpha, b)$

(R-4) $\mathrm{Res}_\infty^{((A))^n}(\alpha, Vb) = V\mathrm{Res}_\infty^{((A))^n}(\alpha, b)$

(R-5) $\mathrm{Res}_\infty^{((A))^n} : M_n((A))^n \times W_\infty((A))^n \longrightarrow W_\infty A$
is continuous with respect to the valuation topology on $M_n((A))^n$

(R-6) $W_\infty\varphi(\mathrm{Res}_\infty^{((A))^n}(\alpha, b)) = \mathrm{Res}_\infty^{((B))^n}(M_n((\varphi))^n(\alpha), W_\infty((\varphi))^n(b))$

(R-7) $b \in W_\infty[[A]]_n \implies \mathrm{Res}_\infty^{((A))^n}(\alpha, b) = \ell(\alpha)\, 0^*(b)$.

(ii) *For any* $u_1, \cdots, u_n \in ((A))^{n\times}$ *and* $b \in W_\infty((A))^n$, *we have*

(R-8) $u_i \in A^\times$ *for some* $i \in \{1, \cdots, n\} \implies \mathrm{Res}_\infty^{((A))^n}(\{u_1, \cdots, u_n\}, b) = 0$

(R-9) $u_1, \cdots, u_n \in t_1^{\mathbf{Z}} \times \cdots \times t_n^{\mathbf{Z}}$
$\implies \mathrm{Res}_\infty^{((A))^n}(\{u_1, \cdots, u_n\}, b) = \ell(\{u_1, \cdots, u_n\})\, 0^*(b)$.

The proof is similar to the case that $n = 1$.

LEMMA 18. *Suppose that A is an integral ring,* $n \geqq 1$ *and* $u_1, \cdots, u_n \in ((A))^n$ *satisfy the condition* (a) *in Lemma* 5, (i). *If* $p \in A^\times$, *then the mapping*

$$\mathrm{Res}_\infty^{((A))^n_u} : M_n((A))^n_u \times W_\infty((A))^n_u \longrightarrow W_\infty A$$

is defined. Let $i : ((A))^n_u \hookrightarrow ((A))^n$ *denote the natural inclusion mapping. Then for any* $\alpha \in M_n((A))^n_u$, $b \in W_\infty((A))^n_u$, *we have*

(R-10) $\mathrm{Res}_\infty^{((A))^n_t}((M_n i)(\alpha), (W_\infty i)(b)) = \ell(u_1, \cdots, u_n)\mathrm{Res}_\infty^{((A))^n_u}(\alpha, b)$.

Here $\ell(u_1, \cdots, u_n)$ *is the integer defined in Lemma* 5.

The proof is induced from Lemma 12, (iii) and the definition of the mapping $\mathrm{Res}_\infty^{((A))^n_u}$.

COROLLARY. *Suppose that* $u_1, \cdots, u_n$ *satisfy the condition* (a_0) *in Lemma* 5, (ii). *Then for any* $\alpha \in M_n((A))^n$, $b \in W_\infty((A))^n$, *we obtain*

$$\mathrm{Res}_\infty^{((A))^n_t}(\alpha, b) = \mathrm{Res}_\infty^{((A))^n_u}(\alpha, b)\,.$$

Next we try to omit the condition $p \in A^\times$.

LEMMA 19. *Let* $i : A \hookrightarrow B$ *be an extension of integral rings and* $n \geqq 1$. *If* $p \in B^\times$, *then for any* $\alpha \in M_n((A))^n$, $b \in W_\infty((A))^n$, *we have*

$$\mathrm{Res}_\infty^{((B))^n}(M_n((i))^n(\alpha), W_\infty((i))^n(b)) \in W_\infty A\,.$$

PROOF. It suffices to prove this lemma in the case that $\alpha = \{u_1, \cdots, u_n\}$ for $u_1, \cdots, u_n \in ((A))^{n\times}$. Put $\alpha = \{t_1, \cdots, t_{i-1}, u_i, \cdots, u_n\}$ $(1 \leqq i \leqq n+1)$, and prove the assertion by induction on i. For $i = n+1$, it is easy from (R-9). Assume that the assertion holds for $i+1$. If $u_i, \cdots, u_n \in A((t_n))\cdots((t_{i+1}))$, then $\det J^L(u/t) = 0$, and hence $\mathrm{Res}_\infty^{((B))^n}(M_n((i))^n(\alpha), W_\infty((i))^n(b)) = 0$, by Lemma 16, (iii). Therefore we may assume $u_i \notin A((t_n))\cdots((t_{i+1}))$ by Corollary to Lemma 13. Moreover, by Corollary 1 to Lemma 3, we can also assume (1) $u_i \in t_j^{\mathbf{Z}}$ $(1 \leqq j \leqq i)$ or (2) $u_i \in [[A]]_n^\times$. In the case (1), we can put $u_i = t_j^m$ $(m \in \mathbf{Z})$. If $1 \leqq j \leqq i-1$, then $\alpha = \{t_1, \cdots, t_{i-1}, -1, u_{i+1}, \cdots, u_n\}^m$ by Lemma 13. This implies $\mathrm{Res}_\infty^{((B))^n}(M_n((i))^n(\alpha), W_\infty((i))^n(b)) = 0$ by (R-8). If $j = i$, then

$$\alpha = \{t_1, \cdots, t_i, u_{i+1}, \cdots, u_n\}^m\,.$$

Therefore $\mathrm{Res}_\infty^{((B))^n}(M_n((i))^n(\alpha), W_\infty((i))^n(b)) \in W_\infty A$ by the assumption of induction. In the case (2), if we put $t_i' = u_i t_i$, then

$$\mathrm{Res}_\infty^{((B))^n}(M_n((i))^n(\alpha), W_\infty((i))^n(b)) = \mathrm{Res}_\infty^{((B))^n}(\{t_1, \cdots, t_{i-1}, t_i', u_{i+1}, \cdots, u_n\}, b)$$

$$- \mathrm{Res}_{\infty}^{(\!(B)\!)^n}(\{t_1, \cdots, t_{i-1}, t_i, u_{i+1}, \cdots, u_n\}, b)\,.$$

Since $t_1, \cdots, t_{i-1}, t_i', t_{i+1}, \cdots, t_n$ satisfy the condition (a_0) in Lemma 5, (ii), we have $\mathrm{Res}_{\infty}^{(\!(B)\!)^n}(M_n(\!(i)\!)^n(\alpha), W_\infty(\!(i)\!)^n(b)) \in W_\infty A$ by Corollary to Lemma 18 and the assumption of induction. □

LEMMA 20. *Let A be an integral ring and $n \geqq 1$.*

(i) *Even if $p \notin A^\times$, the mapping $\mathrm{Res}_{\infty}^{(\!(A)\!)^n} : M_n(\!(A)\!)^n \times W_\infty(\!(A)\!)^n \to W_\infty A$ is defined and satisfies all the conditions* (R-1), $\cdots$, (R-9) *and* (R-10).

(ii) *If A is of characteristic p, then for any $\alpha \in M_n(\!(A)\!)^n$, $b \in W_\infty(\!(A)\!)^n$, we obtain*

(R-11) $\mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha, Pb) = P\mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha, b)$.

PROOF. (i) By Lemma 19, the assertion (i) is valid for an integral ring A of characteristic 0, especially for a polynomial ring A in countable variables with coefficient ring $\mathbf{Z}$. Therefore, by the similar method to the case that $n = 1$, we can define $\mathrm{Res}_{\infty}^{(\!(A)\!)^n}$ by the use of (R-6).

(ii) This statement follows from (R-2), (R-4) and $p = PV$. □

LEMMA 21. *Let A be an integral ring and $n \geqq 1$. Take any $\alpha \in M_n(\!(A)\!)^n$, $b \in W_\infty(\!(A)\!)^n$, and write $\alpha = \prod_{i=-1}^{n} \alpha(i)$, $b = \sum_{i=-1}^{n} b(i)$ following Lemma* 15, (i) *and Lemma* 16. *Here*

$$\alpha(-1) \in \prod_{i=1}^{n} M_{n-i+1}(\!(A)\!)^{n-i},\ \alpha(0) \in \{t_1, \cdots, t_n\}^{\mathbf{Z}},\ \alpha(i) \in U^{(1)}_{M_{n-i+1}(\!(A)\!)^{n-i+1}}\,,$$

$$b(-1) \in W_\infty \mathfrak{m},\ b(0) \in W_\infty A,\ b(i) \in W_\infty(t_i^{-1}(\!(A)\!)^{n-i}[t_i^{-1}])\,,$$

and $(\!(A)\!)^{n-i} = A(\!(t_n)\!) \cdots (\!(t_{i+1})\!)$ $(1 \leqq i \leqq n)$. If we put $\ell = \ell(\alpha)$, then $\alpha(0) = \{t_1, \cdots, t_n\}^{\ell}$ and

$$\mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha, b) = \ell b(0) + \sum_{i=1}^{n} \mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha(i), b(i))\,.$$

PROOF. By (R-7), we have $\mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha, b(-1)) = 0$ for any $\alpha \in M_n(\!(A)\!)^n$. If $u_1 = t_1$, $\cdots$, $u_{i-1} = t_{i-1}$, u_i, $\cdots$, $u_n \in (\!(A)\!)^{n-i}$ $(1 \leqq i \leqq n)$, then $\det J^L(u/t) = 0$. Therefore $\mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha(-1), b) = 0$ for any $b \in W_\infty(\!(A)\!)^n$, by Lemma 10, (iii). By (R-7), (R-9), we obtain

$$\mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha(0), b(0)) = \ell b(0)\,,$$

$$j \neq 0 \implies \mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha(0), b(j)) = 0\,,$$

$$i \neq 0 \implies \mathrm{Res}_{\infty}^{(\!(A)\!)^n}(\alpha(i), b(0)) = 0\,.$$

If $u_1 = t_1, \cdots, u_{i-1} = t_{i-1}, u_i \in 1 + t_i ((A))^{n-i}[[t_i]], u_{i+1}, \cdots, u_n \in ((A))^{n-i+1}$ $(1 \leqq i \leqq n)$, then $\det J^L(u/t) \in t_i ((A))^{n-i}[[t_i]]$, and if $b = b(j)$, then $w_k(b) \in t_j^{-1}((A))^{n-j}[t_j^{-1}]$ for any $k \geqq 0$. Therefore

$$i \neq j \implies \mathrm{Res}_\infty^{((A))^n}(\alpha(i), b(j)) = 0$$

for any $i, j \in \{1, \cdots, n\}$. □

In what follows, we consider the case that A is a ring of characteristic p. Then we introduce a homomorphism

$$\wp : W_\infty A \longrightarrow W_\infty A$$

of modules defined by $\wp(a) = Pa - a$ for any $a \in W_\infty A$. If we put

$$WA = W_\infty A/\wp(W_\infty A) \otimes \mathbf{Z}[\frac{1}{p}]/\mathbf{Z},$$

then we obtain a group pairing

$$\mathrm{Res}^{((A))^n} : M_n((A))^n \times W((A))^n \longrightarrow WA$$

by the method similar to that in the case when $n = 1$. See [6, Lemma 3.10] or [8, §3]. Here we induce the valuation topology on $M_n((A))^n$.

Let $\mathrm{Ann}^{((A))^n}$ denote the annihilator of the pairing $\mathrm{Res}^{((A))^n}$. Then there exists a linear topology on $M_n((A))^n$ with fundamental system of neighborhoods

$$\Sigma = \{\mathrm{Ann}^{((A))^n}(Q) \mid Q \text{ is a finite subset of } W((A))^n\}$$

of 0. This topology is said to be the weak topology on $M_n((A))^n$. Then $M_n((A))^n$ is a topological group with respect to the weak topology. Note that $M_n((A))^n$ is not separable with this topology.

LEMMA 22. *Suppose that A is an integral ring of characteristic p and $n \geqq 1$. Then we obtain $\wp(\mathfrak{m}) = \mathfrak{m}$. Therefore*

$$\wp(W_\infty \mathfrak{m}) = W_\infty \mathfrak{m}, \quad W\mathfrak{m} = 0$$

and

$$W((A))^n \cong WD \oplus WA \cong \bigoplus_{i=1}^{n} W(t_i^{-1} A((t_n)) \cdots ((t_{i+1}))[t_i^{-1}]) \oplus WA .$$

PROOF. By Lemma 4, we reduce to the case when $n = 1$. □

LEMMA 23. *For a ring A of characteristic p and indeterminate t over A, we put*

$$A((t^*)) = \bigoplus_{m \in \mathbf{N}_p} At^{-m} \oplus \prod_{m \in \mathbf{N}_p} At^m ,$$

where $\mathbf{N}_p = \mathbf{N} - p\mathbf{N}$. *Moreover, we take any submodule* A_0 *of* A *such that* $A = PA \oplus A_0$.

(i) *If we put* $A((t))_0 = \bigoplus_{m\in\mathbf{N}_p} At^{-m} \oplus \prod_{m\in\mathbf{N}_p} At^m \oplus \bigoplus_{n=1}^{\infty} A_0 t^{-np} \oplus A_0 \oplus \prod_{n=1}^{\infty} A_0 t^{np}$, *then we can write* $A((t))_0 = A((t^*)) \oplus A_0((t^p))$ *and* $A((t)) = P(A((t))) \oplus A((t))_0$.

(ii) $t^{-1}A[t^{-1}] = \wp(t^{-1}A[t^{-1}]) \oplus \bigoplus_{m\in\mathbf{N}_p} At^{-m} \oplus \bigoplus_{n=1}^{\infty} A_0 t^{-np}$.

PROOF. We can prove easily (i).

(ii) Put $x = at^{-mp^e}$ for $a \in A$, $m \in \mathbf{N}_p$, $e \geqq 0$, and prove $x \in \wp(t^{-1}A[t^{-1}]) \oplus \bigoplus_{m\in\mathbf{N}_p} At^{-m} \oplus \bigoplus_{n=1}^{\infty} A_0 t^{-np}$, by induction on e. For $e = 0$, it is obvious. Let $e \geqq 1$. If we put $a = b^p + c$ ($b \in A$, $c \in A_0$), then $x = (bt^{-mp^{e-1}})^p + ct^{-mp^e} = \wp(bt^{-mp^{e-1}}) + bt^{-mp^{e-1}} + ct^{-mp^e}$. Thus $x \in \wp(t^{-1}A[t^{-1}]) \oplus \bigoplus_{m\in\mathbf{N}_p} At^{-m} \oplus \bigoplus_{n=1}^{\infty} A_0 t^{-np}$. □

COROLLARY. *For a ring* A *of characteristic* p *and* $n \geqq 1$, *we obtain*

$$D = \wp(D) \oplus \bigoplus_{i=1}^{n} \bigoplus_{m\in\mathbf{N}_p} A((t_n))\cdots((t_{i+1}))t_i^{-m}$$

$$\oplus \bigoplus_{i=1}^{n} \bigoplus_{m\in\mathbf{N}_p} \bigoplus_{e=1}^{\infty} \bigoplus_{f=i+1}^{n} A((t_n))\cdots((t_{f+1}))((t_f^*))((t_{f-1}^p))\cdots((t_{i+1}^p))t_i^{-mp^e}$$

$$\oplus \bigoplus_{i=1}^{n} \bigoplus_{m\in\mathbf{N}_p} \bigoplus_{e=1}^{\infty} A_0((t_n^p))\cdots((t_{i+1}^p))t_i^{-mp^e}\,.$$

LEMMA 24. *Let* A *be an integral ring of characteristic* p *and* $n \geqq 1$. *Then*

(i) *the mapping* $\mathrm{Res}^{((A))^n} : M_n((A))^n \times W((A))^n \to WA$ *is continuous with respect to the weak topology on* $M_n((A))^n$. *Therefore* $\mathrm{Res}^{((A))^n}$ *is a group pairing.*

(ii) *The weak topology is weaker than the valuation topology on* $M_n((A))^n$.

(iii) *If* A *is a field,* $PA = A$ *and* $A \neq \wp(A)$, *then*

$$\mathrm{Ann}^{((A))^n}(M_n((A))^n) = 0\,.$$

PROOF. The statements (i) and (ii) are easy to verify.

(iii) Take any $\beta \in \mathrm{Ann}^{((A))^n}(M_n((A))^n)$. Then, by Lemma 21 and Lemma 22, we have $\beta \in WD$. Here we assume $\beta \neq 0$. Since WD is a torsion p-group, we may assume $p\beta = 0$. Then we can write $\beta = \phi_1(b)$, $b \in W_\infty D$, $b_0 \notin \wp(D)$. Noting that $A_0 = 0$, the monomial appeared in b_0 with order $\mathrm{ord}_{[[A]]_n}(b_0)$ is contained in $A((t_n))\cdots((t_{i+1}))t_i^{-m}$ or $A((t_n))\cdots((t_{f+1}))((t_f^*))((t_{f-1}^p))\cdots((t_{i+1}^p))t_i^{-mp^e}$, by Corollary to Lemma 23. Here we put $\gamma = -\mathrm{ord}_{[[A]]_n}(b_0)$, $x = t_\gamma$, $u_f = 1 + a_0 x$ for any $a_0 \in A$, $u_j = t_j$ ($j \neq f$), and define $\alpha = \{u_1, \cdots, u_n\}$. Then we obtain $\phi_1(\mathrm{Res}_\infty^{((A))^n}(\alpha, b)) = \mathrm{Res}^{((A))^n}(\alpha, \phi_1(b)) = \mathrm{Res}^{((A))^n}(\alpha, \beta) = 0$, that is, $\mathrm{Res}_\infty^{((A))^n}(\alpha, b) \in \mathrm{Ker}\,\phi_1 = \wp(W_\infty A) + pW_\infty A$. On the other hand, since we can write $\det J^L(u/t) = \frac{\gamma_s a_0 x}{1 + a_0 x}$ and $b_0 = c_0 x^{-1} + \cdots$ ($c_0 \neq 0$), we also have $\mathrm{Res}_\infty^{((A))^n}(\alpha, b)_0 =$

$\gamma_s a_0 c_0$, by Lemma 10, (iii). Therefore we obtain $a_0 c_0 \in \wp(A)$. Thus $A = \wp(A)$. This is a contradiction, and hence $\beta = 0$. $\square$

5. Here we shall define a mapping $\rho_K : M_n K \to \Gamma K$ for a formal power series field K in n variables with p-quasifinite coefficient field, and prove the Main Theorem.

First, note that the group pairings

$$\langle\ ,\ \rangle_\infty^{\Gamma K} : \Gamma K \times W_\infty K \longrightarrow W_\infty \mathbf{F}_p, \quad \langle\ ,\ \rangle^{\Gamma K} : \Gamma K \times W K \longrightarrow \mathbf{Q}/\mathbf{Z}$$

are defined in [8, §2] for any field K of characteristic p.

THEOREM 1. *For a perfect field k and $n \geqq 1$, we put $K = ((k))^n$. Then*

$$K'/K \text{ is an unramified extension } \iff K'/K \text{ is an extension of coefficient fields}$$

for any finite extension K'/K of fields.

The proof is induced from [11, Theorem 2].

COROLLARY. *Suppose that k is a perfect field of characteristic p $(p \neq 0)$. If we put*

$$K_{ur,ab}^{[p]} = k_{ab}^{[p]} K = K(\wp^{-1} W_\infty k)\,,$$

then $K_{ur,ab}^{[p]}$ is the maximal unramified abelian p-extension of K.

For a field k of characteristic p and $n \geqq 1$, we put $K = ((k))^n$. Then $M_n K = M_n((k))^n$ is a topological group by introducing the weak topology. Moreover, from the results in §4, we obtain the mappings

$$\mathrm{Res}_\infty^K : M_n K \times W_\infty K \longrightarrow W_\infty k, \quad \mathrm{Res}^K : M_n K \times W K \longrightarrow W k\,.$$

Suppose that k is a field of characteristic p having the property $k/\wp(k) \cong \mathbf{F}_p$. Then there exists $F \in \Gamma k$ such that $\Gamma k = F^{\mathbf{Z}_p} \cong \mathbf{Z}_p$, and the mapping

$$S_F : \begin{array}{ccc} W_\infty k & \longrightarrow & W_\infty \mathbf{F}_p \\ \Psi & & \Psi \\ b & \longmapsto & \langle F, b\rangle_\infty^{\Gamma k} \end{array}$$

is a surjective continuous homomorphism of $\mathbf{Z}_p$-modules and Ker $S_F = \wp(W_\infty k)$. Therefore we have $W_\infty k \cong \wp(W_\infty k) \oplus W_\infty \mathbf{F}_p$.

Next we define a mapping

$$\langle\ ,\ \rangle_\infty^{M_n K} : M_n K \times W_\infty K \longrightarrow W_\infty \mathbf{F}_p$$

by putting

$$\langle \alpha, b\rangle_\infty^{M_n K} = S_F(\mathrm{Res}_\infty^K(\alpha, b))$$

for $\alpha \in M_nK, b \in W_\infty K$. Then $\langle\ ,\ \rangle_\infty^{M_nK}$ is a group pairing with respect to the weak topology on M_nK. Similarly, from $Wk \cong \mathbf{Z}[\frac{1}{p}]/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$, we also obtain a group pairing

$$\langle\ ,\ \rangle^{M_nK} : M_nK \times WK \longrightarrow \mathbf{Q}/\mathbf{Z}\,.$$

Using these pairings, we can define a mapping

$$\rho_K : M_nK \longrightarrow \Gamma K$$

by putting $\langle \rho_K(\alpha), \beta\rangle^{\Gamma K} = \langle \alpha, \beta\rangle^{M_nK}$ for $\alpha \in M_nK$, $\beta \in WK$. Moreover, for any abelian p-extension L over K, we put $\rho_{L/K}(\alpha) = \rho_K(\alpha)\,|_L$ for $\alpha \in M_nK$. Then we obtain a mapping

$$\rho_{L/K} : M_nK \longrightarrow \mathrm{Gal}(L/K)\,.$$

Note that both the mappings ρ_K and $\rho_{L/K}$ are dependent on F.

LEMMA 25. *For a perfect field k of characteristic p which satisfies $k/\wp(k) \cong \mathbf{F}_p$ and $n \geqq 1$, we put $K = ((k))^n$.*

(i) *Let L, L' be abelian p-extensions over K, $H' = \mathrm{Gal}(K_{ab}^{[p]}/L')$ and $Q' =$ Ker $W_{K|L'}$. If $L' \subset L$, then*

$$\rho_{L/K}^{-1}(\mathrm{Gal}(L/L')) = \rho_K^{-1}(H') = \mathrm{Ann}^{M_nK}(Q')\,.$$

Here Ann^{M_nK} *denotes the annihilator of the pairing* $\langle\ ,\ \rangle^{M_nK}$.

(ii) *The mapping $\rho_K : M_nK \to \Gamma K$ is a continuous homomorphism of groups.*

(ii′) *The weak topology on M_nK is the induced topology of Krull topology on ΓK with respect to the mapping ρ_K.*

(iii) *For any subgroup A of M_nK, we have* $\mathrm{Ann}^{\Gamma K}(\rho_K(A)) = \mathrm{Ann}^{M_nK}(A)$. *Therefore we obtain* $\overline{\rho_K(A)} = \mathrm{Ann}^{\Gamma K}(\mathrm{Ann}^{M_nK}(A))$ *and*

$$\overline{A} = \rho_K^{-1}(\overline{\rho_K(A)}) = \mathrm{Ann}^{M_nK}(\mathrm{Ann}^{M_nK}(A))\,.$$

Here $\mathrm{Ann}^{\Gamma K}$ *denotes the annihilator of the pairing* $\langle\ ,\ \rangle^{\Gamma K}$ *and "overline" means the closure of topological spaces.*

(iv) *The mapping $\rho_K : M_nK \to \Gamma K$ is dominant.*

PROOF. The statement (i) is verified from the definitions of ρ_K, $\rho_{L/K}$ and $H' = \mathrm{Ann}^{\Gamma K}(Q')$.

(ii) It is easy to prove that ρ_K is a homomorphism of groups. The continuity of ρ_K and (ii′) are induced from

$$\{\rho_K^{-1}(H) \mid H \text{ is an open subgroup of } \Gamma K\} =$$

$$\{\mathrm{Ann}^{M_nK}(Q) \mid Q \text{ is a finite subgroup of } WK\}\,.$$

(iii) We can prove this statement easily from the definition of ρ_K and (ii′).

(iv) If we put $A = M_nK$ in (iii), then, by Lemma 24, (iii), we have

$$\overline{\rho_K(M_n K)} = \mathrm{Ann}^{\Gamma K}(\mathrm{Ann}^{M_n K}(M_n K)) = \mathrm{Ann}^{\Gamma K}(0) = \Gamma K .$$

□

COROLLARY. *For any closed subgroup A of $M_n K$, there exists an abelian p-extension L over K such that $A = \mathrm{Ker}\ \rho_{L/K}$. If A is open, then L is finite over K and is determined uniquely from A.*

THEOREM 2. *For a perfect field k of characteristic p which satisfies $k/\wp(k) \cong \mathbf{F}_p$ and $n \geqq 1$, we put $K = ((k))^n$. Take any element $F \in \Gamma k$ having the property $\Gamma k = F^{\mathbf{Z}_p}$, and define the mappings ρ_K and $\rho_{L/K}$. Then*

(i) *for any $\alpha \in M_n K$, we have*

$$\rho_K(\alpha)\Big|_{k_{ab}^{[p]}} = F^{\ell(\alpha)} .$$

(ii) *The mapping from the set of finite abelian p-extensions L over K to the set of open subgroups of $M_n K$ defined by*

$$L \longmapsto \mathrm{Ker}\ \rho_{L/K} = \rho_K^{-1}(\mathrm{Gal}(K_{ab}^{[p]}/L))$$

is an inclusion-reversing bijection, and

$$M_n K/\mathrm{Ker}\ \rho_{L/K} \cong \mathrm{Gal}(L/K) .$$

(iii) *For any finite abelian p-extension L over K, we obtain*

$$L/K \text{ is unramified} \iff U_K^{(0)} \subset \mathrm{Ker}\ \rho_{L/K} .$$

Here $U_K^{(0)} = \mathrm{Ker}\ \ell = \prod_{i=1}^n U^{(0)}_{M_{n-i+1}((k))^{n-i+1}}$.

PROOF. (i) Put $\ell = \ell(\alpha)$. Take any element $b \in W_\infty k$ which satisfies $k_{ab}^{[p]} = k(\wp^{-1}b)$. Then we have $\langle \rho_K(\alpha), b\rangle_\infty^{\Gamma K} = \langle \alpha, b\rangle_\infty^{M_n K} = S_F(\mathrm{Res}_\infty^K(\alpha, b)) = S_F(\ell b) = \langle F, \ell b\rangle_\infty^{\Gamma k} = \langle F^\ell, b\rangle_\infty^{\Gamma k}$ by Lemma 21. Thus $\rho_K(\alpha)\Big|_{k_{ab}^{[p]}} = F^\ell$.

(ii) By Lemma 25, (i), we have $\mathrm{Ker}\ \rho_{L/K} = \rho_K^{-1}(\mathrm{Gal}(K_{ab}^{[p]}/L))$. By Corollary to Lemma 25, the mapping: $L \mapsto \mathrm{Ker}\ \rho_{L/K}$ is bijective. Moreover, by Lemma 25, (iv), the homomorphism $\rho_{L/K} : M_n K \to \mathrm{Gal}(L/K)$ is surjective.

(iii) By Lemma 15, Lemma 21, Lemma 22 and Lemma 24, (iii), we have

$$\mathrm{Ann}^{M_n K}(U_K^{(0)}) = Wk ,$$

and hence $\overline{\rho_K(U_K^{(0)})} = \mathrm{Ann}^{\Gamma K}(\mathrm{Ann}^{M_n K}(U_K^{(0)})) = \mathrm{Ann}^{\Gamma K}(Wk) = \mathrm{Gal}(K_{ab}^{[p]}/K_{ur,ab}^{[p]})$ by Lemma 25, (iii). Therefore we obtain

$$L/K \text{ is unramified} \iff L \subset K_{ur,ab}^{[p]} \iff U_K^{(0)} \subset \mathrm{Ker}\ \rho_{L/K}$$

by Corollary to Theorem 1. □

COROLLARY. *$\Gamma K \cong \widehat{M_n K} = \text{proj.}\lim M_n K/A$, where A runs over all open subgroups of $M_n K$.*

Next we consider the relationship between the pairings $\langle\ ,\ \rangle^{M_n K}$ and $\langle\ ,\ \rangle^{M_n K'}$, where K'/K is a finite separable p-extension of fields.

LEMMA 26. *For a perfect field k of characteristic p and $n \geqq 1$, we put $K = ((k))^n$. Let K'/K be a finite separable extension of fields and k' the algebraic closure of k in K'.*

(i) *For any $\alpha \in M_n K$, $b \in W_\infty K$, we have*

$$\mathrm{Res}_\infty^{K'}(M_{nK|K'}\alpha, b) = e\mathrm{Res}_\infty^{K}(\alpha, b)\,.$$

Here $e = [K' : k'K]$.

(ii) *For any $\alpha \in M_n K$, $b' \in W_\infty K'$, we obtain*

$$T_{W_\infty k'/W_\infty k}\mathrm{Res}_\infty^{K'}(M_{nK|K'}\alpha, b') = \mathrm{Res}_\infty^{K}(\alpha, T_{W_\infty K'/W_\infty K}b')\,.$$

PROOF. (i) If we put $K = k((t_n))\cdots((t_1))$, then we can write $k'K = k'((t_n))\cdots((t_1))$ by [11, Corollary (i) to Theorem 2]. Therefore, by (R-6), we obtain $\mathrm{Res}_\infty^{K}(\alpha, b) = \mathrm{Res}_\infty^{k'K}(M_{nK|k'K}\alpha, b)$. If we apply (R-10) for $i : k'K \hookrightarrow K'$, then we have

$$\mathrm{Res}_\infty^{K'}(M_{nK|K'}\alpha, b) = \ell(t_1, \cdots, t_n)\mathrm{Res}_\infty^{k'K}(M_{nK|k'K}\alpha, b)\,.$$

Moreover, by [11, Theorem 2, (ii)] and [11, Corollary (i) to Theorem 2], we get $\ell(t_1, \cdots, t_n) = e = [K' : k'K]$. Thus $\mathrm{Res}_\infty^{K'}(M_{nK|K'}\alpha, b) = e\mathrm{Res}_\infty^{K}(\alpha, b)$.

(ii) It suffices to prove $T_{k'/k}\mathrm{Res}_\infty^{K'}(M_{nK|K'}\alpha, b') = \mathrm{Res}_\infty^{K}(\alpha, T_{K'/K}b')$ in the case when the extension K'/K is Galois. Put $G = \mathrm{Gal}(K'/K)$, and decompose $G = \bigcup_{i=1}^{f}\mathrm{Gal}(K'/k'K)\sigma_i$. Then $\mathrm{Gal}(k'/k) = \mathrm{Gal}(k'K/K) = \{\sigma_1, \cdots, \sigma_f\}$ by [11, Theorem 2, (ii)]. If we put $\alpha' = M_{nK|K'}\alpha$, $b = T_{K'/K}b'$, then we get $\mathrm{Res}_\infty^{K}(\alpha, b) = \frac{1}{e}\mathrm{Res}_\infty^{K'}(\alpha', b) = \frac{1}{e}\mathrm{Res}_\infty^{K'}(\alpha', \sum_{\sigma\in G}\sigma b') = \frac{1}{e}\sum_{\sigma\in G}\sigma\mathrm{Res}_\infty^{K'}(\alpha', b') = \sum_{i=1}^{f}\sigma_i\mathrm{Res}_\infty^{K'}(\alpha', b')$ $= T_{k'/k}\mathrm{Res}_\infty^{K'}(\alpha', b')$ by (i) and (R-6). □

LEMMA 27. *For a p-quasifinite field k and $n \geqq 1$, we put $K = ((k))^n$.*

(i) *Let $\sigma : K \to K'$ be an isomorphism of fields. Then for any $\alpha \in M_n K$, $\beta \in WK$, we have*

$$\langle\alpha, \beta\rangle^{M_n K} = \langle\sigma\alpha, \sigma\beta\rangle^{M_n K'}\,.$$

(ii) *Let K'/K be a finite separable p-extension of fields. Then for any $\alpha \in M_n K$, $\beta' \in WK'$, we obtain*

$$\langle M_{nK|K'}\alpha, \beta'\rangle^{M_n K'} = \langle\alpha, T_{K'/K}\beta'\rangle^{M_n K}\,,$$

and for any $\alpha' \in M_n K'$, $\beta \in WK$, *we obtain*

$$\langle N_{K'/K}\alpha', \beta\rangle^{M_n K} = \langle \alpha', W_{K|K'}\beta\rangle^{M_n K'} .$$

PROOF. The statment (i) is proved easily from (R-6).

(ii) $\langle M_{nK|K'}\alpha, \beta'\rangle^{M_n K'} = \langle \alpha, T_{K'/K}\beta'\rangle^{M_n K}$ is easy from Lemma 26, (ii). It suffices to prove $\langle N_{K'/K}\alpha', \beta\rangle^{M_n K} = \langle \alpha', W_{K|K'}\beta\rangle^{M_n K'}$ in the case when the extension K'/K is Galois. Since $T_{K'/K}$ is surjective, there exists $\beta' \in WK'$ such that $\beta = T_{K'/K}\beta'$. Then $T_G\beta' = W_{K|K'}\beta$. On the other hand, if we put $\alpha = N_{K'/K}\alpha'$, then $N_G\alpha' = M_{nK|K'}\alpha$. Thus $\langle \alpha, \beta\rangle^{M_n K} = \langle \alpha, T_{K'/K}\beta'\rangle^{M_n K} = \langle M_{nK|K'}\alpha, \beta'\rangle^{M_n K'} = \langle N_G\alpha', \beta'\rangle^{M_n K'} = \langle \alpha', T_G\beta'\rangle^{M_n K'} = \langle \alpha', W_{K|K'}\beta\rangle^{M_n K'}$. □

LEMMA 28. *Suppose that k is a p-quasifinite field, $n \geqq 1$ and $K = ((k))^n$. Then for any finite separable p-extension K'/K of fields, we obtain*

$$\mathrm{Ann}^{M_n K}(N_{K'/K} M_n K') = \mathrm{Ker}\ W_{K|K'}$$

and

$$\overline{N_{K'/K} M_n K'} = \mathrm{Ann}^{M_n K}(\mathrm{Ker}\ W_{K|K'}) .$$

Here "overline" means the closure of $M_n K$ with respect to the weak topology.

PROOF. $\mathrm{Ann}^{M_n K}(N_{K'/K} M_n K') = \mathrm{Ker}\ W_{K|K'}$ is induced from Lemma 27, (ii) and Lemma 24, (iii). $\overline{N_{K'/K} M_n K'} = \mathrm{Ann}^{M_n K}(\mathrm{Ker}\ W_{K|K'})$ is easy from the above equation and Lemma 25, (iii). □

Then the proof of **Main Theorem** is complete from Lemma 25, Lemma 28 and Theorem 2.

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Present Address:
FACULTY OF FUNDAMENTAL EDUCATION,
KOCHI UNIVERSITY OF TECHNOLOGY,
TOSAYAMADA-CHO, KOCHI, 782–8502 JAPAN.