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On the Unobstructedness of the Deformation Problems of Residual Modular Representations

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Abstract. Let f be a primitive form whose weight is greater than 2. Weston [23, Theorem 1] showed that the mod p representation $\bar{\rho}$ associated to f is irreducible and the deformation problem for $\bar{\rho}$ is unobstructed for almost all p. The aim of this article is to give a simpler proof of his result in some cases.

0. Introduction

Let N be a positive integer, $k \geq 2$ an integer and f a primitive form of level N, weight k and character ε. Here we mean by "a primitive form" that f is a normalized newform. Let

$$f(q) = \sum_{n \geq 1} a_n(f) q^n$$

be the q-expansion of f. We denote by $\mathbf{Q}(f)$ the finite extension of $\mathbf{Q}$ generated by $\{a_n(f)\}_{n\geq 1}$. We fix a prime number p and a prime ideal $\mathfrak{p}$ above p of $\mathbf{Q}(f)$. Then we denote by $\mathcal{O}$ the ring of integers of the completion $\mathbf{Q}(f)_{\mathfrak{p}}$ of $\mathbf{Q}(f)$ with respect to $\mathfrak{p}$ and by $\boldsymbol{k}$ the residue field of $\mathcal{O}$. We assume that p is prime to $2N$. Let $G_{\mathbf{Q}}$ be the absolute Galois group of $\mathbf{Q}$. It is known that there exists a Galois representation

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$$

associated to f satisfying the following conditions:

(i) ρ is unramified outside $S := \{\text{the prime divisors of } Np\} \cup \{\infty\}$;

(ii) for each prime number $l \notin S$,

$$\mathrm{Trace}(\rho(\mathrm{Frob}_l)) = a_l(f)\,, \quad \det(\rho(\mathrm{Frob}_l)) = \varepsilon(l) l^{k-1}\,,$$

where Frob_l is a Frobenius element at l.

By the condition (i), we know that ρ factors through the Galois group G_S of the maximal Galois extension of $\mathbf{Q}$ unramified outside S. Then we put

$$\bar{\rho} := \rho \ (\mathrm{mod}\ \mathfrak{p}) : G_S \to \mathrm{GL}_2(\boldsymbol{k})\,.$$

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We say that the deformation problem for $\bar{\rho}$ is *unobstructed* if we have

$$H^2(G_S, \mathrm{Ad}(\bar{\rho})) = 0\,,$$

where $\mathrm{Ad}(\bar{\rho})$ is the adjoint representation associated to $\bar{\rho}$, i.e., the matrix ring $M_2(\boldsymbol{k})$ of degree 2 over $\boldsymbol{k}$ on which G_S-action is given by

$$\sigma \cdot M := \bar{\rho}(\sigma) M \bar{\rho}(\sigma)^{-1} \quad (\sigma \in G_S,\ M \in M_2(\boldsymbol{k}))\,.$$

Weston [23, Theorem 1] showed that if $k > 2$, then $\bar{\rho}$ is absolutely irreducible and the deformation problem for $\bar{\rho}$ is unobstructed for almost all ρ by means of the theory of irreducible admissible automorphic representations and the theory of Dieudonné modules. The aim of this paper is to give a simpler proof than his method in some cases using elementary calculations of representation matrices of $\mathrm{Ad}(\bar{\rho})$. Namely, in this article, we shall give another proof of the following

MAIN THEOREM. *Let f be a primitive form of level N, weight $k > 2$ and character ε. We assume that there are only finitely many prime ideals $\mathfrak{p}$ of $\mathbf{Q}(f)$ for which the restriction of $\bar{\rho}$ to the inertia group at each prime number $q \in S \setminus \{p, \infty\}$ is irreducible. Then for almost all prime ideals $\mathfrak{p}$ of $\mathbf{Q}(f)$, $\bar{\rho}$ is absolutely irreducible and the deformation problem for $\bar{\rho}$ is unobstructed.*

We put $S^{\mathrm{fin}} := S \setminus \{\infty\}$. We denote by D_q (resp. I_q) the decomposition (resp. inertia) group at q in G_S. In Section 1, we define the *Selmer groups* $\mathrm{Sel}(\bar{M})$ for $\boldsymbol{k}[G_S]$-modules $\bar{M}$ and obtain the following exact sequence of the Galois cohomology groups:

$$\begin{aligned}\mathrm{Sel}(\mathrm{Ad}(\bar{\rho})(1))^{\vee} &\to H^2(G_S, \mathrm{Ad}(\bar{\rho})) \\ &\to \bigoplus_{q \in S^{\mathrm{fin}}} H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1))^{\vee}\,,\end{aligned}$$

where $\mathrm{Ad}(\bar{\rho})(1)$ is the Tate twist of $\mathrm{Ad}(\bar{\rho})$ by the mod p cyclotomic character $\bar{x}$ and the symbol $\vee$ stands for the dual space of $\boldsymbol{k}$-vector spaces (cf. Proposition 1.2). Then we apply a result of Diamond, Flach and Guo [4, Theorems 7.15 and 8.2] on the vanishing of the Selmer groups, which is based on the method of Wiles [24] completed with Taylor [21], and get a condition for $\mathrm{Sel}(\mathrm{Ad}(\bar{\rho})(1)) = 0$ (Theorem 1.4). In Section 2, we give some conditions for $H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0$ at $q = p$ and each $q \in S^{\mathrm{fin}} \setminus \{p\}$ for which $\bar{\rho}|_{I_q}$ is reducible. In this situaion, twisting $\bar{\rho}|_{D_q}$ by a suitable local character ψ_q at q enables us to obtain some conditions for $H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0$ by means of representation matrices of $\mathrm{Ad}(\bar{\rho})(1)$. Putting these conditions together, we obtain the Main Theorem.

REMARK 0.1. As to the case where $q \neq p$ and $\bar{\rho}|_{I_q}$ is irreducible, we can see calculations of Weston on the vanishing of $H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1))$ with supercuspidal automorphic representations in the proof of [23, Proposition 3.2]. Also, in [23, Section 5.4], we can see examples of the unobstructedness of deformation problems for $\bar{\rho}$ associated to the primitive forms of level 1 and weight 12, 16, 18, 20, 22 and 26.

REMARK 0.2. As to the case of weight 2, by a result of Flach [8, Theorem 2] on the unobstructedness of residual representations associated to elliptic curves, Mazur [16, Corollary 2] showed that if f is a newform having Fourier coefficients in $\mathbf{Q}$ of weight 2 with trivial character and not of CM-type, then the set of prime numbers p for which $\bar{\rho}$ is absolutely irreducible and the deformation problem for $\bar{\rho}$ is unobstructed is of Dirichlet density 1. This result has been generalized by Weston [23, Theorem 1] to the case of any type.

REMARK 0.3. For a newform f and an odd prime number p, Gouvêa [10, Question III.5] conjectured that if the mod p representation $\bar{\rho}$ associated to f is absolutely irreducible, any deformation of $\bar{\rho}$ to a complete Noetherian local ring with residue field $\boldsymbol{k}$ is associated to a Katz's p-adic eigenform. (For the details of deformation theory of residual representations, see [15], and for Gouvêa's conjecture, see [10, Chapter III].) The author [25, Main Theorem] proved that if the deformation problem for $\bar{\rho}$ is unobstructed, then Gouvêa's conjecture is true. (The proof of this theorem is based on the method of Gouvêa and Mazur [12]. See also Böckle's article [2].) Therefore, by his work combined with [23, Theorem 1], we see that Gouvêa's conjecture is true for almost all p when the weight of f is greater than 2.

We denote by $\mathbf{Q}$, $\mathbf{Q}_p$, $\mathbf{R}$ and $\mathbf{C}$ the fields of rational numbers, p-adic numbers, real numbers and complex numbers, respectively and by $\mathbf{Z}$ and $\mathbf{Z}_p$ the rings of integers and p-adic integers, respectively. We denote by $\mathbf{F}_q$ the finite field consisting of q elements. We fix once and for all an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_v$ for each rational place v.

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1. The Galois cohomology groups and the Selmer groups

Let p be an odd prime number, S a finite set of rational places containing p and ∞ and G_S the Galois group of the maximal Galois extension of $\mathbf{Q}$ unramified outside S. Let F be a finite extension of $\mathbf{Q}_p$, $\mathcal{O}$ the ring of integers of F with a prime element π and $\boldsymbol{k}$ the residue field of F. Let M be a finite flat $\mathcal{O}$-module with G_S-action. Then we put

$$V := M \otimes_{\mathcal{O}} F\,, \quad W := M \otimes_{\mathcal{O}} F/\mathcal{O}\,, \quad \bar{M} := M \otimes_{\mathcal{O}} \boldsymbol{k}\,.$$

We consider the Galois cohomology groups $H^i(G_S, A)$ for $i = 0, 1$ and 2 with $A = M, V, W$ or $\bar{M}$ (for the definition of the Galois cohomology groups, see [22, Section 1]). For each $v \in S$, we regard the absolute Galois group $G_{\mathbf{Q}_v}$ of $\mathbf{Q}_v$ as a decomposition group at v in G_S. We denote by $H^i(\mathbf{Q}_v, A)$ the local Galois cohomology group $H^i(G_{\mathbf{Q}_v}, A)$ and by

$$\mathrm{res}_v : H^i(G_S, A) \to H^i(\mathbf{Q}_v, A)$$

the natural restriction map.

For each $v \in S$, we define the subgroup $H^1_f(\mathbf{Q}_v, A)$ of $H^1(\mathbf{Q}_v, A)$ as follows: First following Bloch and Kato [1, Section 3], we define

$$H^1_f(\mathbf{Q}_v, V) := \begin{cases} \mathrm{Ker}(H^1(\mathbf{Q}_v, V) \xrightarrow{\mathrm{res}} H^1(I_v, V)) , & \text{if} \quad v \neq p , \\ \mathrm{Ker}(H^1(\mathbf{Q}_p, V) \to H^1(\mathbf{Q}_p, V \otimes B_{\mathrm{crys}})) , & \text{if} \quad v = p , \end{cases}$$

where I_v is the inertia group at v in $G_{\mathbf{Q}_v}$ and B_{crys} is the ring defined by Fontaine (see [9, Section I.2.1]). From the short exact sequence

$$0 \to M \to V \to W \to 0 ,$$

we have an exact sequence of the Galois cohomology groups

$$H^1(\mathbf{Q}_v, M) \to H^1(\mathbf{Q}_v, V) \to H^1(\mathbf{Q}_v, W) .$$

Then we define the subgroup $H^1_f(\mathbf{Q}_v, M)$ of $H^1(\mathbf{Q}_v, M)$ (resp. $H^1_f(\mathbf{Q}_v, W)$ of $H^1(\mathbf{Q}_v, W)$) as the inverse image (resp. the image) of $H^1_f(\mathbf{Q}_v, V)$ in the exact sequence above. Moreover, by the inclusion

$$\bar{M} = \mathrm{Ker}(W \xrightarrow{1\otimes\pi} W) \hookrightarrow W ,$$

we obtain a natural homomorphism

$$H^1(\mathbf{Q}_v, \bar{M}) \to H^1(\mathbf{Q}_v, W) .$$

Then we define the subgroup $H^1_f(\mathbf{Q}_v, \bar{M})$ of $H^1(\mathbf{Q}_v, \bar{M})$ as the inverse image of $H^1_f(\mathbf{Q}_v, W)$ under the homomorphism above.

DEFINITION 1.1 (the Selmer groups). We define for $A = M, V, W$ or $\bar{M}$

$$\mathrm{Sel}(A) := \mathrm{Ker}\left(\bigoplus_{v\in S} \mathrm{res}_v : H^1(G_S, A) \to \bigoplus_{v\in S} \frac{H^1(\mathbf{Q}_v, A)}{H^1_f(\mathbf{Q}_v, A)}\right) .$$

REMARK 1.1. Note that we have

$$H^1(\mathbf{R}, \bar{M}) = 0 \quad \text{and} \quad H^2(\mathbf{R}, \bar{M}) = 0 .$$

Because for $i = 1$ and 2,

$$\begin{aligned} 0 &= \sharp\,\mathrm{Gal}(\mathbf{C}/\mathbf{R}) \cdot \mathrm{Ker}(H^i(\mathbf{R}, \bar{M}) \xrightarrow{\mathrm{res}} H^i(\{1\}, \bar{M})) \\ &= 2 \cdot \mathrm{Ker}(H^i(\mathbf{R}, \bar{M}) \to 0) \\ &= 2 \cdot H^i(\mathbf{R}, \bar{M}) \end{aligned}$$

by [19, Chapter I, Proposition 9] and the assumption that p is odd.

DEFINITION 1.2 (the Tate-Shafarevich groups). We put $S^{\text{fin}} := S \setminus \{\infty\}$ and define

$$Ш^1(\bar{M}) := \operatorname{Ker}\left(\bigoplus_{q \in S^{\text{fin}}} \operatorname{res}_q : H^1(G_S, \bar{M}) \to \bigoplus_{q \in S^{\text{fin}}} H^1(\mathbf{Q}_q, \bar{M})\right),$$

$$Ш^2(\bar{M}) := \operatorname{Ker}\left(\bigoplus_{q \in S^{\text{fin}}} \operatorname{res}_q : H^2(G_S, \bar{M}) \to \bigoplus_{q \in S^{\text{fin}}} H^2(\mathbf{Q}_q, \bar{M})\right).$$

Note that

$$Ш^1(\bar{M}) \subset \operatorname{Sel}(\bar{M}).$$

We now recall duality theorems of the Galois cohomology groups without their proofs:

THEOREM 1.1. (1) (*Global Tate Duality. cf.* [13, Theorem 4.50(1)]) *There exists a non-degenerate pairing*

$$Ш^1(\bar{M}) \times Ш^2(\bar{M}^\vee(1)) \to \boldsymbol{k},$$

where $\bar{M}^\vee$ *is the dual space* $\operatorname{Hom}_{\boldsymbol{k}}(\bar{M}, \boldsymbol{k})$ *of* $\bar{M}$ *with* G_S*-action defined by*

$$(\sigma \cdot \varphi)(m) := \varphi(\sigma^{-1} m) \quad (\sigma \in G_S,\ \varphi \in \bar{M}^\vee,\ m \in \bar{M}),$$

and $\bar{M}^\vee(1)$ *is the Tate twist of* $\bar{M}^\vee$ *by the mod* p *cyclotomic character* $\bar{x}$.

(2) (*Local Tate Duality. cf.* [17, Theorem 1.4.1]) *For each* $q \in S^{\text{fin}}$, *there exists a non-degenerate pairing*

$$H^2(\mathbf{Q}_q, \bar{M}) \times H^0(\mathbf{Q}_q, \bar{M}^\vee(1)) \to \boldsymbol{k}.$$

By these duality theorems, we have an important exact sequence of the Galois cohomology groups:

PROPOSITION 1.2. *We have an exact sequence*

$$\operatorname{Sel}(\bar{M}^\vee(1))^\vee \to H^2(G_S, \bar{M}) \to \bigoplus_{q \in S^{\text{fin}}} H^0(\mathbf{Q}_q, \bar{M}^\vee(1))^\vee.$$

Démonstration. By the inclusion $Ш^1(\bar{M}^\vee(1)) \hookrightarrow \operatorname{Sel}(\bar{M}^\vee(1))$ and Theorem 1.1(1), we have an exact sequence

(i) $$\operatorname{Sel}(\bar{M}^\vee(1))^\vee \to Ш^2(\bar{M}) \to 0$$

because taking dual spaces is an exact contravariant functor and

$$(\bar{M}^\vee(1))^\vee(1) \cong \bar{M}$$

as G_S-modules. On the other hand, by the definition of $Ш^2(\bar{M})$ and Theorem 1.1(2), we have an exact sequence

(ii) $$0 \to Ш^2(\bar{M}) \to H^2(G_S, \bar{M}) \to \bigoplus_{q \in S^{\text{fin}}} H^0(\mathbf{Q}_q, \bar{M}^\vee(1))^\vee.$$

Then by the exact sequences (i) and (ii), we obtain

$$\mathrm{Sel}(\bar{M}^\vee(1))^\vee \to H^2(G_S, \bar{M}) \to \bigoplus_{q\in S^{\mathrm{fin}}} H^0(\mathbf{Q}_q, \bar{M}^\vee(1))^\vee .$$

□

In the following, we use the same notation as in the Introduction. We put $M := \mathrm{End}_{\mathcal{O}}(\mathcal{O}\times\mathcal{O})$ on which G_S acts via $\mathrm{Ad}(\rho)$. Since the adjoint representation $\mathrm{Ad}(\bar{\rho})$ associated to $\bar{\rho}$ is nothing but $\bar{M} = M \otimes \boldsymbol{k}$ and $\mathrm{Ad}(\bar{\rho})^\vee \cong \mathrm{Ad}(\bar{\rho})$ as G_S-modules, we have an exact sequence

$$(\star)\qquad \begin{aligned}\mathrm{Sel}(\mathrm{Ad}(\bar{\rho})(1))^\vee &\to H^2(G_S, \mathrm{Ad}(\bar{\rho}))\\ &\to \bigoplus_{q\in S^{\mathrm{fin}}} H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1))^\vee\end{aligned}$$

by Proposition 1.2.

By the natural identification $\left\{\left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right) | a \in \boldsymbol{k}\right\} = \boldsymbol{k}$, we have the decomposition

$$\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus \boldsymbol{k}$$

as G_S-modules, where $\mathrm{Ad}^0(\bar{\rho})$ is the subrepresentation of $\mathrm{Ad}(\bar{\rho})$ consisting of all elements having trace 0 in $\mathrm{Ad}(\bar{\rho})$. Note that G_S acts trivially on $\boldsymbol{k}$. Then we obtain

$$\mathrm{Ad}(\bar{\rho})(1) = \mathrm{Ad}^0(\bar{\rho})(1) \oplus \boldsymbol{k}(1) ,$$

and

$$\mathrm{Sel}(\mathrm{Ad}(\bar{\rho})(1)) = \mathrm{Sel}(\mathrm{Ad}^0(\bar{\rho})(1)) \oplus \mathrm{Sel}(\boldsymbol{k}(1)).$$

PROPOSITION 1.3. *We have*

$$\mathrm{Sel}(\boldsymbol{k}(1)) = 0 \quad \textit{and} \quad \mathrm{Sel}(\mathrm{Ad}(\bar{\rho})(1)) = \mathrm{Sel}(\mathrm{Ad}^0(\bar{\rho})(1)) .$$

Démonstration. Since $\boldsymbol{k}(1) = \mathbf{F}_p(1) \otimes_{\mathbf{F}_p} \boldsymbol{k}$, we have

$$\mathrm{Sel}(\boldsymbol{k}(1)) = \mathrm{Sel}(\mathbf{F}_p(1)) \otimes_{\mathbf{F}_p} \boldsymbol{k} .$$

So it suffices to show that $\mathrm{Sel}(\mathbf{F}_p(1)) = 0$. By Kummer Theory, we see that

$$\begin{aligned}\mathrm{Sel}(\mathbf{F}_p(1)) &= \mathrm{Sel}(\mathbf{Z}/p\mathbf{Z}(1))\\ &= \mathrm{Ker}\left(\bigoplus_{q\in S^{\mathrm{fin}}} \mathrm{res}_q : H^1(G_S, \mathbf{Z}/p\mathbf{Z}(1)) \to \bigoplus_{q\in S^{\mathrm{fin}}} \frac{H^1(\mathbf{Q}_q, \mathbf{Z}/p\mathbf{Z}(1))}{H^1_f(\mathbf{Q}_q, \mathbf{Z}/p\mathbf{Z}(1))}\right)\\ &= \mathrm{Ker}\left(\bigoplus_{q:\text{all primes}} \mathrm{ord}_q(\cdot) \pmod p : \mathbf{Q}^\times/(\mathbf{Q}^\times)^p \to \bigoplus_{q:\text{all primes}} \mathbf{Z}/p\mathbf{Z}\right)\\ &= 0 .\end{aligned}$$

Here ord_q is the q-adic valuation normalized by $\mathrm{ord}_q(q) = 1$. □

Applying the results of Diamond, Flach and Guo [4] on the vanishing of the Selmer groups, we have the following

THEOREM 1.4. *If a prime ideal* $\mathfrak{p}$ *of* $\mathbf{Q}(f)$ *satisfies the condition*

(C1) $$\bar{\rho} \text{ is irreducible}, \quad \mathfrak{p} \nmid \eta_f^{\Sigma} \quad \text{and} \quad p \nmid N(2k-1)(2k-3)k!,$$

then we have

$$\mathrm{Sel}(\mathrm{Ad}^0(\bar{\rho})(1)) = 0,$$

where η_f^{Σ} *is the congruence ideal defined in* [4, Section 6.4]. *Especially, for almost all* $\mathfrak{p}$, *the Selmer group* $\mathrm{Sel}(\mathrm{Ad}^0(\bar{\rho})(1))$ *vanishes.*

Démonstration. Let V be the G_S-module consisting of the trace 0 endomorphisms on the representation space of ρ over $\mathbf{Q}(f)_{\mathfrak{p}}$ and M its $\mathcal{O}$-lattice with G_S-action via $\mathrm{Ad}(\rho)$. Note that we have $\bar{M}(1) = \mathrm{Ad}^0(\bar{\rho})(1)$.

By the exact sequence

$$0 \to \bar{M}(1) \to W(1) \to W(1) \to 0$$

of G_S-modules, we have the exact sequence

$$H^0(G_S, W(1)) \to H^1(G_S, \bar{M}(1)) \to H^1(G_S, W(1))$$

of the Galois cohomology groups. Since $\bar{\rho}$ is irreducible, we have $H^0(G_S, W(1)) = 0$. Then we obtain an inclusion

$$\mathrm{Sel}(\bar{M}(1)) \hookrightarrow \mathrm{Sel}(W(1)).$$

So in order to prove the theorem, it suffices to show that $\mathrm{Sel}(W(1)) = 0$ under the condition (C1).

By the exact sequence

$$0 \to M(1) \to V(1) \to W(1) \to 0$$

of G_S-modules, we have the exact sequence

$$H^1(G_S, V(1)) \xrightarrow{\phi} H^1(G_S, W(1)) \xrightarrow{\psi} H^2(G_S, M(1))$$

of the Galois cohomology groups. Then we obtain the exact sequence

$$0 \to \phi(\mathrm{Sel}(V(1))) \to \mathrm{Sel}(W(1)) \to \psi(\mathrm{Sel}(W(1))) \to 0.$$

By [4, Theorem 8.2], we know that $\phi(\mathrm{Sel}(V(1))) = 0$ under the condition (C1). On the other hand, we also obtain another exact sequence

$$0 \to \phi'(\mathrm{Sel}(V)) \to \mathrm{Sel}(W) \to \psi'(\mathrm{Sel}(W)) \to 0$$

of the Selmer groups from the exact sequence

$$0 \to M \to V \to W \to 0$$

of G_S-modules with suitable homomorphisms ϕ' and ψ'. We note that the Selmer group $\mathrm{Sel}(W)$ is included in the $\mathcal{O}$-module $H^1_\Sigma(G_{\mathbf{Q}}, W)$ defined in [4, Section 7.1], which vanishes under the condition (C1) by [4, Theorem 7.15]. Then we have

$$\phi'(\mathrm{Sel}(V)) = 0\,.$$

Since M is $\mathcal{O}$-free, we see that the Pontryagin dual of W is isomorphic to $W(1)$ as G_S-modules. By [7, Theorem 1], we then have

$$\mathrm{Sel}(W(1)) = 0$$

as desired. By [4, Lemma 7.13], we know that $\bar{\rho}$ is irreducible for almost all prime ideals $\mathfrak{p}$. So the last assertion is verified. □

2. The vanishing of $H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1))$

In this section, we shall give some conditions for

$$H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0$$

for each $q \in S^{\mathrm{fin}}$ and prove the Main Theorem. We denote by D_q (resp. I_q) the decomposition (resp. inertia) group at q in G_S. First we consider the case where $q = p$. We denote by V the representation space of $\bar{\rho}$. Then we have

$$\begin{aligned}\mathrm{Ad}(\bar{\rho})(1) &\cong (V \otimes V) \otimes (\det \bar{\rho})^{-1}(1) \\ &\cong (V \otimes V)(2-k) \otimes \bar{\varepsilon}^{-1}\end{aligned}$$

as G_S-modules, where $(V \otimes V)(2-k)$ is the Tate twist of $(V \otimes V)$ by $\bar{\chi}^{2-k}$ and $\bar{\varepsilon}$ is the mod p reduction of ε. We recall some results on mod p modular representations restricted to D_p or I_p:

THEOREM 2.1 ([5, Theorems 2.5 and 2.6]). *We assume that* $2 \leqq k \leqq p+1$ *and* $p \nmid N$.
(1) *If* $a_p(f) \not\equiv 0 \pmod{\mathfrak{p}}$, *then we have*

$$\bar{\rho}|_{D_p} \sim \begin{pmatrix} \bar{\chi}^{k-1}\eta(\bar{\varepsilon}(p)a_p(f)^{-1}) & \xi' \\ 0 & \eta(a_p(f)) \end{pmatrix}$$

with a function $\xi' : D_q \to \bar{\mathbf{F}}_p$.
(2) *If* $a_p(f) \equiv 0 \pmod{\mathfrak{p}}$, *then we have*

$$\bar{\rho}|_{I_p} \sim \begin{pmatrix} \psi^{k-1} & 0 \\ 0 & \psi'^{k-1} \end{pmatrix},$$

and $\bar{\rho}|_{D_p}$ *is irreducible. Here* ψ *and* ψ' *are the fundamental characters of level* 2.

By means of the theorem above, we obtain the following

PROPOSITION 2.2. *We assume the condition* (C1) *and* $k > 2$.

(1) *If* $a_p(f) \not\equiv 0 \pmod{\mathfrak{p}}$ *and the following condition is satisfied*:

(C2) $$k \not\equiv 0 \ (\mathrm{mod}\ p-1)\,,$$

then we have

$$H^0(\mathbf{Q}_p, \mathrm{Ad}(\bar{\rho})(1)) = 0\,.$$

(2) *If* $a_p(f) \equiv 0 \pmod{\mathfrak{p}}$, *then*

$$H^0(\mathbf{Q}_p, \mathrm{Ad}(\bar{\rho})(1)) = 0\,.$$

Démonstration. By the condition (C1), we see that $p \geq k+1$ and $p \nmid N$. Therefore we can apply Theorem 2.1.

(1) By Theorem 2.1(1), the representation matrix of $(V \otimes V)(2-k) \otimes \bar{\varepsilon}^{-1}$ is equivalent to the matrix

$$\begin{pmatrix} \bar{\chi}^k & \xi' \cdot \bar{\chi} & \xi' \cdot \bar{\chi} & \xi'^2 \cdot \bar{\chi}^{2-k} \\ 0 & \bar{\chi} & 0 & \xi' \cdot \bar{\chi}^{2-k} \\ 0 & 0 & \bar{\chi} & \xi' \cdot \bar{\chi}^{2-k} \\ 0 & 0 & 0 & \bar{\chi}^{2-k} \end{pmatrix}$$

on D_p because $\bar{\varepsilon}$ and $\eta(\cdot)$ are unramified at p. By the conditions (C1) and (C2), we see that there exists an element $\sigma \in I_p$ such that

$$\bar{\chi}^k(\sigma) \neq 1 \quad \text{or} \quad \bar{\chi}^{2-k}(\sigma) \neq 1$$

because $k > 2$. Therefore we have

$$H^0(\mathbf{Q}_p, \mathrm{Ad}(\bar{\rho})(1)) = 0\,.$$

(2) By Theorem 2.1(2), the representation matrix of $(V \otimes V)(2-k) \otimes \bar{\varepsilon}^{-1}$ is equivalent to the diagonal matrix

$$\begin{pmatrix} \psi^{k-p(k-2)} & & & \\ & \bar{\chi} & & \\ & & \bar{\chi} & \\ & & & \psi^{(p-1)k+2} \end{pmatrix}$$

on I_p because $\psi\psi' = \bar{\chi}$ and $\psi' = \psi^p$. Since the fundamental character ψ is a surjection to $\mathbf{F}_{p^2}^{\times}$, we see that $\psi^{k-p(k-2)}$ and $\psi^{(p-1)k+2}$ are non-trivial under the condition (C1) which implies that $p \geq 5$. Therefore we have

$$H^0(\mathbf{Q}_p, \mathrm{Ad}(\bar{\rho})(1)) = 0.$$

□

Next we consider the case where $q \neq p$. By the assumption of the Main Theorem, we may assume that $\bar{\rho}|_{I_q}$ is reducible for any $q \in S^{\text{fin}} \setminus \{p\}$. We can see easily that there exists a primitive (p-adic) character ψ (of conductor d) for which we have either

$$\mathrm{ord}_q(N(\bar{\rho} \otimes \bar{\psi})) = \mathrm{ord}_q(C_{\varepsilon\psi^2}) \geq 1$$

or

$$\mathrm{ord}_q(N(\bar{\rho} \otimes \bar{\psi})) = 1 \quad \text{and} \quad \mathrm{ord}_q(C_{\varepsilon\psi^2}) = 0\,,$$

and the set of the prime divisors of the least common multiple N' of N, d^2 and dC_ε coincides with S. Here $\bar{\psi}$ is the mod p reduction of ψ and $N(\bar{\rho} \otimes \bar{\psi})$ and $C_{\varepsilon\psi^2}$ are the conductor of the residual representation $\bar{\rho} \otimes \bar{\psi}$ and the character $\varepsilon\psi^2$, respectively. (For the definition of the conductor of residual representations, see [18], [6].) By [20, Proposition 3.64], the twisted eigenform $f \otimes \psi$ to which $\bar{\rho} \otimes \bar{\psi}$ is associated is of level N' and weight k with character $\varepsilon\psi^2$. We assume that $p \geq 5$. Then by a result of Diamond [3, Corollary 1.2] on Serre's conjecture about residual modular representations combined with a result of Gouvêa [11, Lemma 7] on the level of primitive forms, we see that there exists a primitive form g of level $N(\bar{\rho} \otimes \bar{\psi})$ and weight $k(\bar{\rho} \otimes \bar{\psi}) \geq 2$ with character $\varepsilon(\bar{\rho} \otimes \bar{\psi})$ to which $\bar{\rho} \otimes \bar{\psi}$ is associated, where $k(\bar{\rho} \otimes \bar{\psi})$ and $\varepsilon(\bar{\rho} \otimes \bar{\psi})$ are the weight and the character defined by Serre in [18], respectively. Since we see that $C_{\varepsilon(\bar{\rho}\otimes\bar{\psi})} = C_{\varepsilon\psi^2}$ and $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}(\bar{\rho} \otimes \bar{\psi})$ as G_S-modules, it suffices to investigate the vanishing of $H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1))$ with the primitive form g having the following properties:

$$\mathrm{ord}_q(N(\bar{\rho} \otimes \bar{\psi})) = \mathrm{ord}_q(C_{\varepsilon(\bar{\rho}\otimes\bar{\psi})}) \geq 1$$

or

$$\mathrm{ord}_q(N(\bar{\rho} \otimes \bar{\psi})) = 1 \quad \text{and} \quad \mathrm{ord}_q(C_{\varepsilon(\bar{\rho}\otimes\bar{\psi})}) = 0\,.$$

REMARK 2.1. We will see later that the conditions for the vanishing of $H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1))$ for all $q \neq p$ are independent of the weight of eigenforms to which $\bar{\rho}$ is associated. This fact guarantees that the above argument works well in the proof of the Main Theorem, although the weight $k(\bar{\rho} \otimes \bar{\psi})$ can be equal to 2.

Now we recall some results on p-adic modular Galois representations ρ associated to a primitive form g of level N, weight $k \geq 2$ and character ε:

THEOREM 2.3 (Langlands [14], see [13, Theorem 3.26(3)]). *Let q be a prime divisor of N. We assume that $q \neq p$. Let χ be the p-adic cyclotomic character and $\eta(x)$ the unramified character on D_q such that $\eta(x)(\mathrm{Frob}_q) = x$.*

(1) *If* $\mathrm{ord}_q(N) = \mathrm{ord}_q(C_\varepsilon) \geq 1$, *then we have*

$$\rho|_{D_q} \sim \begin{pmatrix} \varepsilon\chi^{k-1}\eta(a_q(g))^{-1} & 0 \\ 0 & \eta(a_q(g)) \end{pmatrix}.$$

(2) *If* $\mathrm{ord}_q(N) = 1$ *and* $\mathrm{ord}_q(C_\varepsilon) = 0$, *then we have*

$$\rho|_{D_q} \sim \begin{pmatrix} \eta(a_q(g))\chi & * \\ 0 & \eta(a_q(g)) \end{pmatrix},$$

and $\rho|_{D_q}$ *is ramified.*

We put $\bar{\rho}' := \bar{\rho} \otimes \bar{\psi},\ k' := k(\bar{\rho}')$ and $\varepsilon' := \varepsilon(\bar{\rho}')$. We denote the representation space of $\bar{\rho}'$ by V'. Then we have

$$\mathrm{Ad}(\bar{\rho})(1) \cong (V' \otimes V')(2 - k') \otimes \bar{\varepsilon}'^{-1}$$

as G_S-modules, where $\bar{\varepsilon}'$ is the mod p reduction of ε'. We are going to give some conditions for $p \geq 5$ under which $(V' \otimes V')(2 - k') \otimes \bar{\varepsilon}'^{-1}$ has no $G_{\mathbf{Q}_q}$-invariant element by showing the following

PROPOSITION 2.4. (1) *In the case where* $\mathrm{ord}_q(N(\bar{\rho}')) = \mathrm{ord}_q(C_{\varepsilon'})$, *we have*

$$H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0.$$

(2) *In the case where* $\mathrm{ord}_q(N(\bar{\rho}')) = 1$ *and* $\mathrm{ord}_q(C_{\varepsilon'}) = 0$, *we assume the following condition:*

$$\text{(C3)} \qquad q \not\equiv 1 \pmod{p},$$

then we have

$$H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0.$$

Démonstration. (1) By Theorem 2.3(1), the representation matrix of $(V' \otimes V')(2 - k') \otimes \bar{\varepsilon}'^{-1}$ is equivalent to the diagonal matrix

$$\begin{pmatrix} \bar{\varepsilon}'\bar{\chi}^{k'}\eta(a_q(g))^{-2} & & & \\ & \bar{\chi} & & \\ & & \bar{\chi} & \\ & & & \bar{\varepsilon}'^{-1}\bar{\chi}^{2-k'}\eta(a_q(g))^2 \end{pmatrix}$$

on D_q. Since $\bar{\chi}$ is non-trivial and unramified by the condition (C3), $\bar{\varepsilon}'$ is ramified and $\eta(a_q(g))$ is unramified at q, we see that all diagonal components are non-trivial characters. We then have

$$H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0.$$

(2) We note that $\bar{\rho}'$ is ramified at q. By Theorem 2.3(2), we have

$$\bar{\rho}'|_{D_q} \sim \begin{pmatrix} \eta(a_q(g))\bar{\chi} & \xi \\ 0 & \eta(a_q(g)) \end{pmatrix}$$

with a function $\xi : D_q \to \bar{\mathbf{F}}_p$. Then we see that the representation matrix of $(V' \otimes V')(2-k') \otimes \bar{\varepsilon}'^{-1}$ is equivalent to the matrix

$$\begin{pmatrix} \bar{\chi}^2 & \eta'\bar{\chi} & \eta'\bar{\chi} & \eta'^2 \\ 0 & \bar{\chi} & 0 & \eta' \\ 0 & 0 & \bar{\chi} & \eta' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

on D_q, where $\eta' := \xi \cdot \eta(a_q(g))^{-1}$. If there exist elements $a, b, c, d \in \bar{\mathbf{F}}_p$ such that

$$\begin{pmatrix} \bar{\chi}^2(\sigma) & \eta'\bar{\chi}(\sigma) & \eta'\bar{\chi}(\sigma) & \eta'^2(\sigma) \\ 0 & \bar{\chi}(\sigma) & 0 & \eta'(\sigma) \\ 0 & 0 & \bar{\chi}(\sigma) & \eta'(\sigma) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad (\sigma \in D_q),$$

then we see that

$$\bar{\chi}^2(\sigma)a + \eta'\bar{\chi}(\sigma)b + \eta'\bar{\chi}(\sigma)c + \eta'^2(\sigma)d = a, \tag{1}$$

$$(\bar{\chi}(\sigma) - 1)(b - c) = 0, \tag{2}$$

$$(\bar{\chi}(\sigma) - 1)b + \eta'(\sigma)d = 0, \tag{3}$$

for any $\sigma \in D_q$. By the condition (C3), we see that $\bar{\chi}(\mathrm{Frob}_q) = q \pmod p \neq 1$ in $\mathbf{F}_p^\times$. Then $b = c$ by the equation (2). Since $\bar{\rho}$ is ramified at q, there exists an element $\sigma_0 \in I_q$ such that $\eta'(\sigma_0) \neq 0$. Taking σ_0 as σ in the equation (3), we have $d = 0$. Then, taking Frob_q as σ in the equation (3), we have $b = c = 0$. This implies $a = 0$ by the equation (1). Therefore we have

$$H^0(\mathbf{Q}_q, \mathrm{Ad}(\bar{\rho})(1)) = 0.$$

□

Note that in the case where $\mathfrak{p}$ does not divide 2, $\bar{\rho}$ is absolutely irreducible if and only if it is irreducible, because the residual modular representation $\bar{\rho}$ is *odd*, i.e., the image of complex conjugation under $\bar{\rho}$ has determinant -1. Then by putting Theorem 1.4, Propositions 1.3, 2.2 and 2.4 together, the Main Theorem is proven because of the exact sequence $(\star)$ in Section 1.

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