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# On the Complex WKB Analysis for a Second Order Linear O.D.E. with a Many-Segment Characteristic Polygon

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**Abstract.** Asymptotics of ODE appearing in the turning point problems can be characterized literally by its characteristic polygon. The Airy equation has a one-segment characteristic polygon. Fedoryuk ([4]), Nakano ([8], [12]), Nakano et al. ([13]), and Roos ([17], [18]) studied ODE's with a several-segement one. The more segments, the more complicated asymptotics. Here, we study an ODE with a many-segment one. Firstly, the ODE is reduced to the simpler ODE's in some subdomains, and then the reduced ODE's have the WKB solutions as their asymptotic solutions. Secondly, two sets of the WKB solutions in the neighboring subdomains are related by a matching matrix. In our analysis the stretching-matching method is applied and the Stokes curves play an important role. How to get the Stokes curve configuration for the reduced ODE's is analyzed precisely.

## 1. Introduction

**1.1.** We study the following one-dimensional Schrödinger equation:

$$(1.1)\qquad \begin{cases} \varepsilon^{2h}\,\dfrac{d^2y}{dx^2} = Q(x,\varepsilon)y, \quad Q(x,\varepsilon) := \displaystyle\sum_{j=0}^{h} a_j\varepsilon^j x^{m_j}\,; \\ m_j := \dfrac{(h-j+1)(h-j)}{2}\,, \quad \mathbf{C} \ni \forall a_j \neq 0\,; \\ h = 2,3,4,\cdots; \quad x, y \in \mathbf{C}\,; \quad 0 < \varepsilon \le \varepsilon_0\,; \quad D : 0 \le |x| \le x_0\,, \end{cases}$$

where $x_0$ and $\varepsilon_0$ are positive small constants. The zeros of $Q(x,0)$ $(= a_0x^{(h+1)h/2})$ are called *turning points* of (1.1), and so (1.1) has a turning point at $x = 0$ of order $(h+1)h/2$. Sometimes $\varepsilon$ can be considered as the Plank constant $\hbar$

Our aim is to analyze the asymptotics of solutions of (1.1) in $D$ by using the concept of *the characteristic polygon* (defined below) and by applying what we call *the stretching-matching method* (Nakano ([8]–[12]), Nakano et al. ([13]), Nishimoto ([14]), Wasow ([22])).

We set another $(X,Y)$-plane, on which we put points $P_j$'s according to the indices of $\varepsilon$ and $x$ of $Q(x,\varepsilon)$ and a point $R$ according to the index of $\varepsilon$ on the left hand side of (1.1) defined, respectively, by

$$(1.2)\qquad P_j := \left(\frac{j}{2}, \frac{m_j}{2}\right) \quad (j = 0,1,2,\cdots,h) \quad \text{and} \quad R := (h,-1)\,.$$

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The characteristic polygon for (1.1) is, by definition, a polygon consisting of segments connecting points $P_0, P_1, \cdots, P_h$ and $R$ in order (Iwano-Sibuya [7]). The point $P_0$ is on the $Y$-axis and corresponds to the first term $a_0 x^{(h+1)h/2}$ of $Q(x,\varepsilon)$. $P_j$ corresponds to the $(j+1)$-st term of $Q(x,\varepsilon)$. This characteristic polygon is convex downward, snaps at $P_j$'s, and consists of $h+1$ segments. For $j = 0, 1, \cdots, h-1$, $P_j$ and $P_{j+1}$ are on the $(j+1)$-st one, and $P_h$ and $R$ are on the $(h+1)$-st one, i.e., on the last one. In order to know the asymptotic property of the solutions, we need several steps of analysis (§2).

The differential equations with a two- or three-segment characteristic polygon are studied by Fedoryuk ([4]), Nakano ([8], [9]), Nakano et al. ([13]), Roos ([17], [18]). Nakano ([10], [11]) studied the $n$-th order O.D.E. Some of them are reviewed in §7.

**1.2.** The differential equation (1.1) can be represented in the matrix form by the standard transformation and it is a special case of

$$
\begin{aligned}
&\varepsilon^h \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ q(x,\varepsilon) & 0 \end{bmatrix} Y\,, \\
&q(x,\varepsilon) := \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{j,k}\, x^{m_{j,k}}\right)\varepsilon^j \quad (m_{j,k} \geq 0;\ 0 < \varepsilon \leq \varepsilon_0;\ 0 < |x| \leq x_0)\,.
\end{aligned}
\tag{1.3}
$$

Since the power of $\varepsilon$ of the left hand side of (1.3) is $h$, every term of $\varepsilon^j$ $(j \geq h+1)$ can be considered to be a regularly perturbed term and so it can be asymptotically neglected. The higher degree terms of $a_{j,k} x^{m_{j,k}}$ $(k \geq 1)$ can also be asymptotically neglected because their asymptotic contribution to the solutions is known to be small due to the theory of its characteristic polygon. Thus (1.1) can be regarded as a fairly general differential equation in this sense although it looks simple.

**1.3.** The contents of this paper are as follows. In §2, we reduce (1.1) asymptotically to the simpler differential equations in appropriate subdomains of $D$. In §3, the WKB approximations for the reduced differential equations are given. The WKB approximations are the trancated formal solutions. In §4, the Stokes curve configurations and the canonical domains for the reduced differential equations are constructed. Every canonical domain is bounded by several Stokes curves. The asymptotic expansions of the true solutions are their WKB approximations in a canonical domain. Thus, it is very essential for the analytic but not for the formal asymptotic theory to determine Stokes curves and canonical domains. In §5, two sets of the solutions of the reduced differential equaions are matched, i.e., they are connected linearly by the matching matrix by using their WKB approximations. In §6, we consider the differential equation of the same type as (1.1) in order to compute formally the matching matrices by assuming the existence of its canonical domain. Then, in §7, we compare the matching matrices in §6 with ones appeared in the papers cited in §1.1. The last section (§8) is the summary, in which the process of all the analysis is repeated concisely.

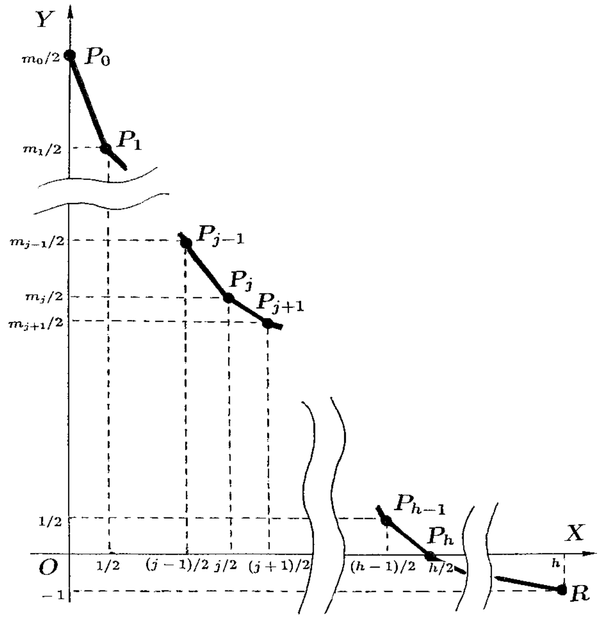


FIGURE 1.1. The characteristic polygon ($m_j := (h-j+1)(h-j)/2$).

## 2. The asymptotic reductions of (1.1)

**2.1.** Each term of $Q(x,\varepsilon)$ can be considered "the asymptotically dominant term" in some subdomain of $D$. In order to show the $(j+1)$-st term $(= a_j\varepsilon^j x^{m_j})$ to be dominant, we pick it up to the head and separate $Q(x,\varepsilon)$ into three parts as follows:

$$Q(x,\varepsilon) = a_j\varepsilon^j x^{m_j}\left\{\left(\sum_{k=0}^{j-1} + 1 + \sum_{k=j+1}^{h}\right)\frac{a_k}{a_j}\,\varepsilon^{k-j}x^{(h-k+1)(h-k)/2-(h-j+1)(h-j)/2}\right\}. \tag{2.1}$$

The first sum is represented as

$$\sum_{k=0}^{j-1}\frac{a_k}{a_j}\,\{(\varepsilon x^{j-h-1})^{-1}\}^{j-k}x^{(j-k-1)(j-k)/2}, \tag{2.2}$$

which is small for $x$ satisfying $|x| \le k_j\varepsilon^{1/(h-j+1)}$ with a sufficiently small constant $k_j$ because $(j-k-1)(j-k)/2$ is nonnegative and $x^{(j-k-1)(j-k)/2}$ is bounded in $D$.

The last sum is represented as

$$\sum_{k=j+1}^{h}\frac{a_k}{a_j}(\varepsilon x^{j-h})^{k-j}x^{(k-j)(k-j-1)/2}, \tag{2.3}$$

which is small for $x$ satisfying $|x| \geq K_{j+1}\varepsilon^{1/(h-j)}$ with a sufficiently large constant $K_{j+1}$ because $(k-j)(k-j-1)/2$ is nonnegative and $x^{(k-j)(k-j-1)/2}$ is bounded in $D$. Thus, we can regard that the $(j+1)$-st term is dominant to obtain the reduced differential equation

$$(2.4)_j \qquad \varepsilon^{2h-j}\frac{d^2y}{dx^2} = a_j x^{m_j} y \quad \left(m_j := \frac{(h-j+1)(h-j)}{2};\quad j = 0, 1, 2, \cdots, h\right)$$

as $\varepsilon \to 0$ for $x$ in the subdomain $D_{out,j}(\subset D)$ defined by

$$(2.5)_j \qquad \begin{gathered} D_{out,j} := \{(x, \varepsilon) :\ K_{j+1}\varepsilon^{1/(h-j)} \leq |x| \leq k_j\varepsilon^{1/(h-j+1)}, 0 < \varepsilon \leq \varepsilon_0\} \\ (j = 0, 1, 2, \cdots, h)\,, \end{gathered}$$

where we should read $K_{h+1}\varepsilon^{1/0} := 0$ for $j = h$ and $k_0\varepsilon^{1/(h+1)} := x_0$ for $j = 0$. Then, $(2.4)_j$ becomes especially for $j = h$

$$(2.4)_h \qquad \varepsilon^h\frac{d^2y}{dx^2} = a_h y \quad \text{in} \quad D_{out,h} := \{(x, \varepsilon) :\quad 0 \leq |x| \leq k_h\varepsilon, \quad 0 < \varepsilon \leq \varepsilon_0\}\,.$$

We notice that $(2.4)_h$ has a constant coefficient and it is easily solved near the turning point $x = 0$, that is to say, the equation $(2.4)_h$ gives the asymptotic property of (1.1) near the turning point.

**2.2.** In the intermediate domain, designated by $D_{j+1}$, between $D_{out,j}$ and $D_{out,j+1}$, (1.1) can be asymptotically reduced as follows. Applying the *stretching transformation*, which is the first step of *the stretching-matching method* (§5.1),

$$(2.6) \qquad x := t\varepsilon^{1/(h-j)} \quad (j = 0, 1, 2, \cdots, h-1)$$

to (1.1), we obtain the differential equation

$$(2.7)_{j+1} \qquad \begin{cases} \varepsilon_{j+1}^2\dfrac{d^2y}{dt^2} = Q_{j+1}(t)y\,, \\ \varepsilon_{j+1} := \varepsilon^{(3h-1-j)/4-1/(h-j)}, \quad (3h-1-j)/4 - 1/(h-j) > 0\,, \\ Q_{j+1}(t) := a_j t^{m_j} + a_{j+1}t^{m_{j+1}} \\ \equiv a_j t^{(h-j-1)(h-j)/2}\left(t^{h-j} + \dfrac{a_{j+1}}{a_j}\right) \quad (j = 0, 1, 2, \cdots, h-1) \end{cases}$$

in the domain

$$(2.8)_{j+1} \qquad D_{j+1} := \{t :\ k_{j+1} \leq |t| \leq K_{j+1},\ t := x\,\varepsilon^{-1/(h-j)}\}\,.$$

Here, we supposed the inequality between $h$ and $j$ of $(2.7)_{j+1}$, which is called *a singular perturbation condition* (cf. $(6.1)'$ in [10]). When this inequality holds, $(2.7)_{j+1}$ is a differential equation of singular perturbation type and it possesses its own turning points at zeros of $Q_{j+1}(t)$ which are called *secondary turning points* of (1.1). Thus, analyzing differential equations with secondary turning points is called *a secondary turning point problem.* The secondary turning problem was firstly studied in [13].

We show how to get $(2.7)_{j+1}$ in the following. After substituting (2.6) for $x$ in $Q(x,\varepsilon)$ and separating it into two parts, we see that

$$
\begin{aligned}
Q(x,\varepsilon) &\equiv Q(t\,\varepsilon^{1/(h-j)},\varepsilon)\\
&= \sum_{k=0}^{h} a_k t^{(h-k+1)(h-k)/2} \varepsilon^{k+(h-k+1)(h-k)/(2(h-j))}\\
&= \varepsilon^{(h+1+j)/2}\Bigg(\sum_{k=j}^{j+1} + \sum_{k\neq j,j+1}\Bigg) a_k t^{(h-k+1)(h-k)/2}\,\varepsilon^{(j-k+1)(j-k)/(2(h-j))}\,.
\end{aligned}
\tag{2.9}
$$

The first sum in the last expression of (2.9) contains two terms representing

$$
a_j t^{(h-j+1)(h-j)/2} + a_{j+1}\,t^{(h-j)(h-j-1)/2} \equiv a_j t^{m_j} + a_{j+1}\,t^{m_{j+1}}\,. \tag{2.10}
$$

The general term in the second sum is represented as

$$
a_k t^{(h-k+1)(h-k)/2}(\varepsilon^{1/(h-j)})^{(j-k+1)(j-k)/2}\,, \tag{2.11}
$$

where the exponent $(h-k+1)(h-k)/2$ is nonnegative and $1/(h-j)$ and $(j-k+1)(j-k)/2$ are also positive. Since (2.11) tends to zero as $\varepsilon \to 0$ for $t \in D_{j+1}$, we obtain

$$
Q(x,\varepsilon) \sim \varepsilon^{(h+1+j)/2} a_j t^{(h-j-1)(h-j)/2}(t^{h-j} + a_{j+1}/a_j) \quad (\varepsilon \to 0,\quad t \in D_{j+1})\,,
$$

which gives $(2.7)_{j+1}$. Especially, $(2.7)_h$ is

$$
\varepsilon^{h-2}\,\frac{d^2y}{dt^2} = a_{h-1}\left(t + \frac{a_h}{a_{h-1}}\right)y \quad \big(k_h \le |t| \le K_h\big), \tag*{$(2.7)_h$}
$$

which has its own turning point $t = -a_h/a_{h-1}$, namely, a secondary turning point of (1.1) at $t \neq 0$. Solutions of $(2.7)_h$ can be represented by the Airy functions. Since $x = 0$ corresponds to $t = 0$, $(2.4)_h$ may be unnecessary if solutions of $(2.7)_h$ are adopted. However, we take account of all the reduced differential equations in this paper. We call $(2.4)_j$ an *outer equation* for (1.1) and $(2.7)_{j+1}$ an *inner equation* for (1.1) according to [13].

**2.3.** Summing up the above consideration, we get the following result.

THEOREM 2.1. *The differential equation* (1.1) *is asymptotically reduced to* $(2.4)_j$ *in* $(2.5)_j$, *and* $(2.4)_j$ *corresponds to the point* $P_j$ *on the characteristic polygon. Especially,* $(2.4)_h$ *gives the asymptotic property of* (1.1) *near the turning point.*

*The differential equation* (1.1) *is also asymptotically reduced to* $(2.7)_{j+1}$ *in* $(2.8)_{j+1}$, *and* $(2.7)_{j+1}$ *corresponds to the points* $P_j$ *and* $P_{j+1}$ *on the* $(j+1)$*-st segment of the characteristic polygon.*

We should notice that the domain $(2.8)_{j+1}$ is bounded but it must be extended to an unbounded domain

$$(2.12)_{j+1} \qquad D^{\infty}_{in,j+1} := \{t\,:\ 0 < |t| < \infty\}$$

in order to *match*, i.e., to connect a set of solutions of an inner equation and that of an outer equation. It is the second step of *the stretching-matching method* (§3.3 and §5). We call $(2.5)_j$ and $(2.12)_{j+1}$ the *outer domain* and the *inner domain* of (1.1), respectively.

### 3. The WKB approximations

**3.1.** Both $(2.4)_j$ and $(2.7)_{j+1}$ have the form common to the singular perturbation. Then, we here study

$$(3.1) \qquad \varepsilon^2 \frac{d^2 y}{dx^2} = Q(x)y \quad (x, y \in \mathbf{C};\ 0 \le |x| < \infty;\ \ 0 < \varepsilon \le \varepsilon_1)\,,$$

where $\varepsilon_1$ is a small constant and $Q(x)$ is a polynomial. $x = \infty$ is an irregular singular point of (3.1). A point $x = a$ is called a *turning point* of (3.1) if $Q(a) = 0$.

*WKB approximations* $\tilde{y}^{\pm}(x, \varepsilon)$ for (3.1) are, by definition, given by

$$(3.2) \qquad \tilde{y}^{\pm}(x, \varepsilon) := \frac{C_{\pm}}{\sqrt[4]{Q(x)}} e^{\pm \frac{1}{\varepsilon}\xi(a,x)} \quad (C_{\pm}\,:\ \text{constants})\,,$$

where

$$(3.3) \qquad \xi(a, x) := \int_a^x \sqrt{Q(x)}\, dx\,.$$

(3.2) is sometimes called *a formal WKB solution.* A curve on the $x$-plane defined by the equation

$$(3.4) \qquad \Re\, \xi(a, x) = C \quad (Q(a) = 0)$$

is called a *level curve* of level $C$ and it is called a *Stokes curve* for (3.1) when $C = 0$, and a curve defined by the equation

$$(3.5) \qquad \Im\, \xi(a, x) = C \quad (Q(a) = 0)$$

is also called a *level curve* of level $C$ and it is called an *anti-Stokes curve* for (3.1) when $C = 0$.

The map $\xi := \xi(a, x)$ $(Q(a) = 0)$ defined by (3.3) is a confomal mapping from the $x$-plane to the $\xi$-plane except for the turning points because $d\xi/dx \neq 0$ at $x \neq a$. Then the level curves defined by (3.4) and defined by (3.5) are mapped perpendicular on the $\xi$-plane by $\xi := \xi(a, x)$.

**3.2.** A Stokes curve configuration for (3.1) plays an important role for asymptotics of (3.1) as shown, for example, in [1], [5] and [8]–[12]. We give here a brief sketch of main properties of the Stokes curve configuration for (3.1) (cf. [2], [4], [10], [12]).

(i) *Stokes curves and anti-Stokes curves from a turning point tend to other turning points or to the irregular singular point* $x = \infty$.

(ii) *A Stokes curve cannot intersect itself. An anti-Stokes curve cannot intersect itself, either.*

(iii) *There are no* (*sums of several*) *Stokes curves or anti-Stokes curves homotopic to a circle.*

We prepare terminology for the Stokes curve configuration. A *Stokes domain* is, by definition, a simply connected domain bounded by several Stokes curves without any Stokes curves in its interior. There are two types of the Stokes domains. One is of *half-plane type*, which is mapped onto a half plane ($\Re\xi > C$ or $\Re\xi < C$) in the $\xi$-plane by $\xi := \xi(a, x)$. The other is of *strip type*, which is mapped onto a strip domain ($C_1 < \Re\xi < C_2$) in the $\xi$-plane.

A *canonical domain* for (3.1) is, by definition, a domain consisting of any two adjoining Stokes domains of half-plane type, or a sum of Stokes domains of half-plane type and one or more Stokes domains of strip type. A canonical domain must be mapped conformally onto the whole $\xi$-plane except for one or several slits which are images of boundaries of Stokes domains. For example, consider $Q = Q_3(x) := x^3(x^3 - 1)$ in (3.3). In Fig. 4.1 the solid lines are the Stokes curves and the broken lines are the anti-Stokes curves. $\mathcal{S}_1$ is bounded by three Stokes curves $l_1, l_2, l_3$, $\mathcal{S}_3$ by two $l_4, l_5$, and $\mathcal{S}_2$ by four $l_3, l_4, l_6, l_7$. Every $\mathcal{S}_j$ is a Stokes domain. $\mathcal{S}_1$ and $\mathcal{S}_3$ are of half-plane type and mapped conformally onto the half-planes on the $\xi$-plane depicted in Fig. 4.1′. (Note that the same letters are used for the images.) $\mathcal{S}_2$ is of strip type and mapped conformally onto a strip plane on the $\xi$-plane depicted in Fig. 4.1′. The union of the three Stokes domains $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ together with the two Stokes curves ($l_3, l_4$) makes up a canonical domain because it is conformally mapped onto the whole $\xi$-plane except for two slits, which emerge from the images of 0 and $\omega$ and go upward (Figs. 4.1 and 4.1′). Also, $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ together with two Stokes curves make up a canonical domain for $Q_4(t)$ and its image is shown in Fig. 4.2′.

In Figs. 4.3 and 4.4, there exist six Stokes domains, only one of which is of strip type. In Fig. 4.5, there are 12 Stokes domains of half-plane type, four of which are bounded by two Stokes curves and the rest by three. Any two adjoining Stokes domains together with a Stokes curve between them make up a canonical domain. There is no strip type. In Fig. 4.6, there are four Stokes domains of strip type.

When $Q(x)$ is a rational function, the Stokes curve configuration is very different from the case of a polynomial. Several examples can be seen in Nakano [9], one of which will be cited in §7.5.

The WKB approximations $\tilde{y}^{\pm}(x, \varepsilon)$ possess the *double asymptotic property* stated as follows (Fedoryuk [4]).

LEMMA 3.1. *Let* $\tilde{y}^{\pm}(x, \varepsilon)$ *and* $\mathcal{D}^{can}$ *be the WKB approximations and a canonical domain for* (3.1) *respectively. Then, there exist the true solutions* $y^{\pm}(x, \varepsilon)$ *of* (3.1) *having*

$\tilde{y}^{\pm}(x,\varepsilon)$ *as their asymptotic expansions* :

$$y^{\pm}(x,\varepsilon) \sim \tilde{y}^{\pm}(x,\varepsilon) \quad \text{as} \quad \begin{cases} x \to \infty & \text{in } \mathcal{D}^{can} \quad (0 < \varepsilon \le \varepsilon_1)\,, \\ \varepsilon \to 0 & \text{for } x \in \mathcal{D}^{can} \quad (0 < \varepsilon \le \varepsilon_1)\,, \end{cases} \tag{3.6}$$

*where* $\varepsilon_1$ *is a small constant.*

Note that the first relation is vacant if $\mathcal{D}^{can}$ is bounded from $x = \infty$.

**3.3.** The WKB approximations $\tilde{y}^{\pm}_{out,j}(x,\varepsilon)$ for $(2.4)_j$ and $\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon)$ for $(2.7)_{j+1}$ are respectively given by

$$(3.7)_j \qquad \begin{aligned} \tilde{y}^{\pm}_{out,j}(x,\varepsilon) &:= a_j^{-1/4}\, x^{-(h-j+1)(h-j)/8} \\ &\quad \times \exp\left( \pm \frac{4\,a_j^{1/2}}{(h-j+1)(h-j)+4} \frac{x^{\{(h-j+1)(h-j)+4\}/4}}{\varepsilon^{h-j/2}} \right) \\ &\qquad\qquad (j = 0, 1, 2, \cdots, h)\,, \end{aligned}$$

$$(3.8)_{j+1} \qquad \begin{aligned} \tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) &:= \frac{1}{\sqrt[4]{Q_{j+1}(t)}} \\ &\quad \times \exp\left( \pm \frac{1}{\varepsilon_{j+1}} \int_0^t \sqrt{Q_{j+1}(t)}\, dt \right) \quad (j = 0, 1, 2, \cdots, h-1)\,. \end{aligned}$$

Then, from Lemma 3.1, we obtain

THEOREM 3.2. *There exist true solutions* $y^{\pm}_{out,j}(x,\varepsilon)$ (*resp.* $y^{\pm}_{in,j+1}(t,\varepsilon)$) *of* $(2.4)_j$ (*resp.* $(2.7)_{j+1}$) *such that the following asymptotic properties are valid*:

$$(3.9)_j \qquad y^{\pm}_{out,j}(x,\varepsilon) \sim \tilde{y}^{\pm}_{out,j}(x,\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad x \in \mathcal{D}^{can}_{out,j} \quad (0 < \varepsilon \le \varepsilon_0)\,,$$

*where* $\mathcal{D}^{can}_{out,j}$ *is a canonical domain for* $(2.4)_j$, *and*

$$(3.10)_{j+1} \qquad y^{\pm}_{in,j+1}(t,\varepsilon) \sim \tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) \quad \text{as} \quad \begin{cases} t \to \infty & \text{in } \mathcal{D}^{can}_{in,j+1} \quad (0 < \varepsilon \le \varepsilon_0)\,, \\ \varepsilon \to 0 & \text{for } t \in \mathcal{D}^{can}_{in,j+1} \quad (0 < \varepsilon \le \varepsilon_0)\,, \end{cases}$$

*where* $\mathcal{D}^{can}_{in,j+1}$ *is a canonical domain for* $(2.7)_{j+1}$.

We should notice that the WKB approximations $\tilde{y}^{\pm}_{out,j}(x,\varepsilon)$ have a single asymptotic property as shown in Lemma 3.1, because the $x$-domain $\mathcal{D}^{can}_{out,j}$ is bounded.

## 4. The Stokes curves and the canonical domains

**4.1.** In this section, we will construct or define all the necessary canonical domains. In order to get exact form of canonical domains we need to know first the Stokes curve configuration. Since ambiguity of $a_j$'s can determine neither positions of turning points nor the geometry of Stokes curves, we must specify coefficients $a_j$'s of $Q(x,\varepsilon)$ in (1.1). Thus we put, for the sake of simplicity,

$$(4.1) \qquad a_j := (-1)^j \quad (j = 0, 1, 2, \cdots, h)\,.$$

Then we analyze the following two equations:

For $j = 0, 1, 2, \cdots, h$, the outer differential equation

$$(4.2)_j \qquad \varepsilon^{2h-j}\,\frac{d^2y}{dx^2} = (-1)^j\,x^{(l+1)l/2}y\,, \quad l := h-j\ (= 0, 1, 2, \cdots, h)$$

in the corresponding outer domain $(2.5)_j$.

For $j = 0, 1, 2, \cdots, h-1$, the inner differential equation

$(4.3)_{j+1}$

$$\begin{cases} \varepsilon_{j+1}^2\,\dfrac{d^2y}{dt^2} = Q_{j+1}(t)y\,, \quad \varepsilon_{j+1} := \varepsilon^{(3l-1+2j)/4-1/l} \quad ((3l-1+2j)/4 - 1/l > 0) \\ Q_{j+1}(t) := (-1)^j\,t^{\,l\,(l-1)/2}(t^l\ -\ 1), \quad l := h-j\ (= 1, 2, 3, \cdots, h) \end{cases}$$

in the corresponding inner domain $(2.12)_{j+1}$ (but not in $(2.8)_{j+1}$ (§2.3)). Here we supposed the singular perturbation condition $(4.3)_{j+1}$ (cf. $(2.7)_{j+1}$).

The WKB approximations $(3.7)_j$ and $(3.8)_{j+1}$ induce respectively

$$(4.4)_j \qquad \begin{aligned} \tilde{y}^{\pm}_{out,j}(x,\varepsilon) := (-1)^{-j/4}\,x^{-(l+1)l/8}\exp\left(\pm\,\frac{4\,(-1)^{j/2}}{(l+1)l+4}\,\frac{x^{\{(l+1)l+4\}/4}}{\varepsilon^{h-j/2}}\right) \\ (j = 0, 1, 2, \cdots, h) \end{aligned}$$

and

$$(4.5)_{j+1} \qquad \begin{aligned} \tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) := Q_{j+1}(t)^{-1/4}\,\exp\left(\pm\,\frac{1}{\varepsilon_{j+1}}\int_0^t \sqrt{Q_{j+1}(t)}\,dt\right) \\ (j = 0, 1, 2, \cdots, h-1)\,. \end{aligned}$$

The origin $x = 0$ is only one turning point for $(4.2)_j$. The point $t = 1$ is a turning point for any $(4.3)_{j+1}$. If $l = h - j \neq 0$, then $l$-th roots of 1 are also turning points for $(4.3)_{j+1}$, which are secondary turning points for (1.1) (§2.2). The origin $t = 0$ also is a turning point of $(4.3)_{j+1}$ if $l(l-1) \neq 0$, i.e., if $j = 0, 1, 2, \cdots, h-2$.

**4.2.** First, we study the Stokes curve configuration for $(4.3)_{j+1}$. Since $\Re \int_a^t \sqrt{q(t)}dt = \Im \int_a^t \sqrt{-q(t)}dt (q(a) = 0)$ for any polynomial $q(t)$, Stokes curves for $-q(t)$ coincide with anti-Stokes curves for $q(t)$. Therefore, it is sufficient to study the Stokes curve configuration defined by the integral $\int_a^t \sqrt{q(t)}dt$. Putting

$$(4.6) \qquad q(t) := t^{l\,(l-1)/2}(t^l - 1)\ \ (= (-1)^j Q_{j+1}(t),\ \ l \in \mathbf{N})\,,$$

we choose a branch of square roots of $q(t)$ such that $\sqrt{q(t)} > 0$ for a large $t$ $(t > 1)$ on the positive real axis. The interval between $t = 0$ and $t = 1$ is a (part of) Stokes curve for any $l \in \mathbf{N}$.

We investigate into the Stokes curve configurations for every $l = 1, 2, 3, \cdots$.

(i) The case $l = 1$.

Since $q(t) = t - 1$, the turning point $t = 1$ is of order one and the Stokes curve configuration is essentially the same as the Airy equation: $\varepsilon^2 d^2 y/dt^2 = ty$. The straight line $t \leq 1$ on the real axis is a Stokes curve.

(ii) The case $l = 2$.

Since $q(t) = t(t^2 - 1)$, the points $t = 0, \ \pm 1$ are the turning points of order one and the Stokes curve configuration is simple.

(iii) The case $l = 3$.

Since $q(t) = t^3(t^3 - 1)$, the turning points $t = 1, \omega, \omega^2$ $(\omega := e^{i2\pi/3})$ are order one and the turning point $t = 0$ is of order three. From $t = 0$ five Stokes curves and five anti-Stokes curves emerge. This Stokes curve configuration is rather complicated.

(iv) The case $l = 4$

Since $q(t) = t^6(t^4 - 1)$, there exist five turning points $t = 0, \ t = e^{i m\pi/2}$ $(m = 0, 1, 2, 3)$ and we see that

$$\int_0^t \sqrt{q(t)}\, dt = \int_0^{\tau e^{i m\pi/2}} \sqrt{q(\tau)}\, d\tau \quad (t := \tau\, e^{i m\pi/2})\,.$$

Then, the Stokes curve configuration is symmetric with respect to the origin $t = 0$, and its form is not changed by rotation of $\pi/2$.

(v) The case $l \geq 5$.

There exist much more numbers of Stokes curves emerging from the origin than a number of turning points. Then, at least two Stokes curves emerging from $t = 0$ pass between $t = 1$ and its neighboring turning point $t = e^{i2\pi/l}$ and tend to $t = \infty$. Thus we can get one Stokes domain of half-plane type bounded by two Stokes curves emerging from $t = 0$, and there is another Stokes domain of half-plane type bounded by two unbounded Stokes curves emerging from $t = 0, \ t = 1$.

We illustrate rather complicated examples cited in (iii) (Figs. 4.1, 4.2 ).

EXAMPLE 1. $Q_3(t) := t^3\ (t^3 - 1)$ for $h = 5, \quad j = 2$.

EXAMPLE 2. $Q_4(t) := -\, t^3\ (t^3 - 1)$ for $h = 6, \quad j = 3$.

In Figs. 4.1 and 4.2, the shadow zones represent the canonical domains $\mathcal{D}_{in,3}^{even,can}$ (for $h = 5, \ j = 2$) and $\mathcal{D}_{in,4}^{odd,can}$ (for $h = 6, \ j = 3$). The Stokes curves for $Q_3(t)$ are the anti-Stokes curves for $Q_4(t)$, and the anti-Stokes curves for $Q_3(t)$ are the Stokes curves for $Q_4(t)$. The thick lines and the broken lines designate the Stokes curves and the anti-Stokes curves, respectively. $\mathcal{S}_1$ and $\mathcal{S}_3$ are Stokes domains of half-plane type and $\mathcal{S}_2$ is of strip-type. The real part $\Re\ \xi_j(t)$ of the function $\xi_j(t) := \int_0^t \sqrt{Q_j(\tau)}\, d\tau\, (j = 3, 4)$ takes positive values

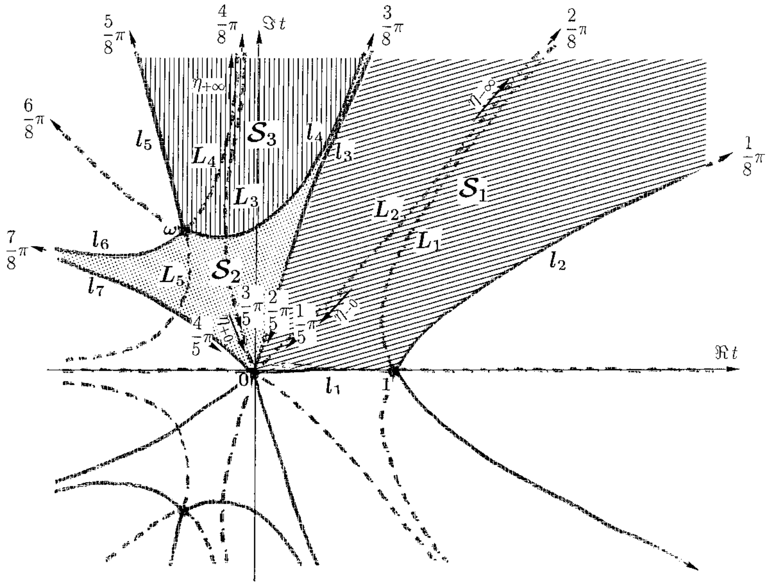


FIGURE 4.1. $\mathcal{D}_{in,3}^{even,can}$ ($h = 5$, $j = 2$. $Q_3(t) = t^3(t^3 - 1)$).

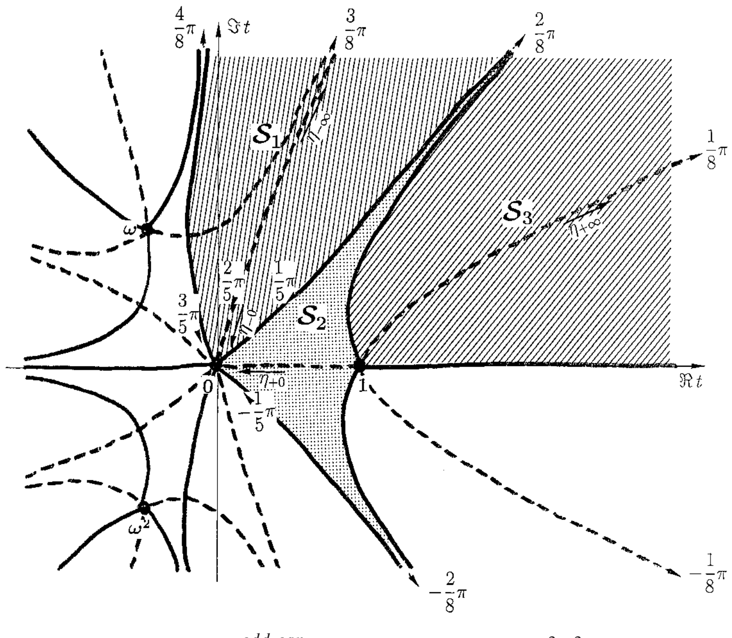


FIGURE 4.2. $\mathcal{D}_{in,4}^{odd,can}$ ($h = 6$, $j = 3$. $Q_4(t) = -t^3(t^3 - 1)$).

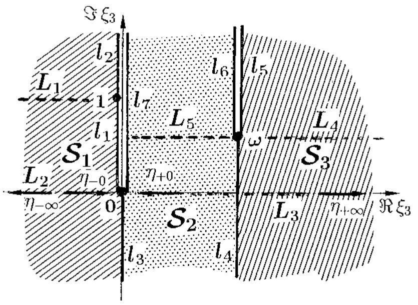


FIGURE 4.1′. The image of $\mathcal{D}^{even,can}_{in,3}$ by $\xi := \xi_3$. Same letters are used for images.

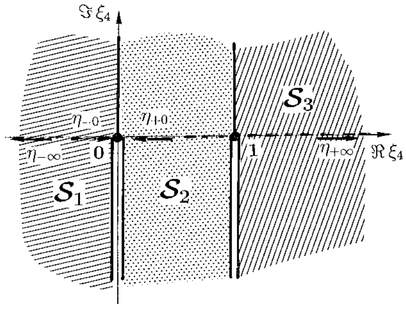


FIGURE 4.2′. The image of $\mathcal{D}^{odd,can}_{in,4}$ by $\xi := \xi_4$. Same letters are used for images.

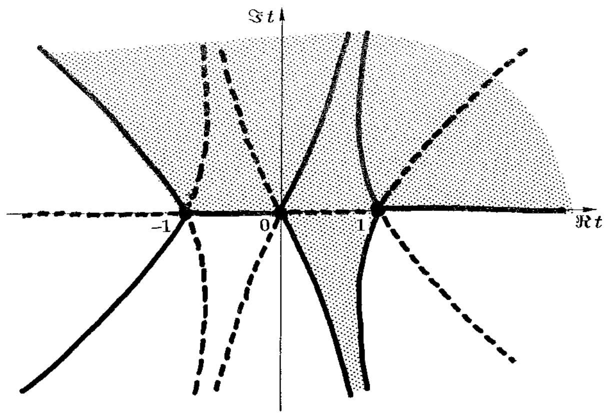


FIGURE 4.3. $\mathcal{D}^{odd,can}_{in,2}$ ($h = 3$, $j = 1$. $Q_2(t) = -t(t^2-1)$).

in $\mathcal{S}_2 \cup \mathcal{S}_3$. Figs. 4.1′ and 4.2′ show the images of the canonical domains by the mapping $\xi := \xi_j(t)$. (Notice that the same letters are used for images for simplicity.) Other canonical domains for various pairs of $h$ and $j$ are illustrated by the shadow zones in Figs. 4.3–4.6.

**4.3.** If "$j$ is even", we can choose $\mathcal{D}^{even,can}_{in,j+1}$ as a canonical domain for $(4.3)_{j+1}$, which is composed of two Stokes domains of half-plane type with or without a Stokes domain of strip-type. Though the Stokes curves are not straight lines, $\mathcal{D}^{even,can}_{in,j+1}$ has a sector-like shape near $t = \infty$ such that

$$(4.7)_\infty \qquad \mathcal{D}^{even,can}_{in,j+1}: \quad \frac{2\pi}{(l+1)l+4} < \arg t < \frac{10\pi}{(l+1)l+4} \quad (t \sim \infty),$$

and it has also a sector-like shape near $t = 0$ such that

$$(4.7)_0 \qquad \mathcal{D}^{even,can}_{in,j+1}: \quad 0 < \arg t < \frac{8\pi}{(l-1)l+4} \quad (t \sim 0).$$

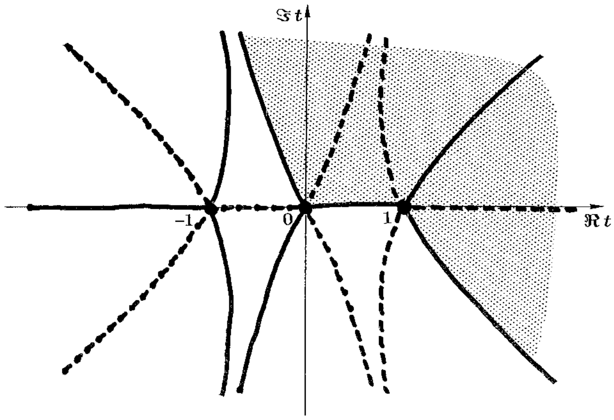


FIGURE 4.4. $\mathcal{D}^{even,can}_{in,9}$ ($h = 10$, $j = 8$. $Q_9(t) = t(t^2 - 1)$).

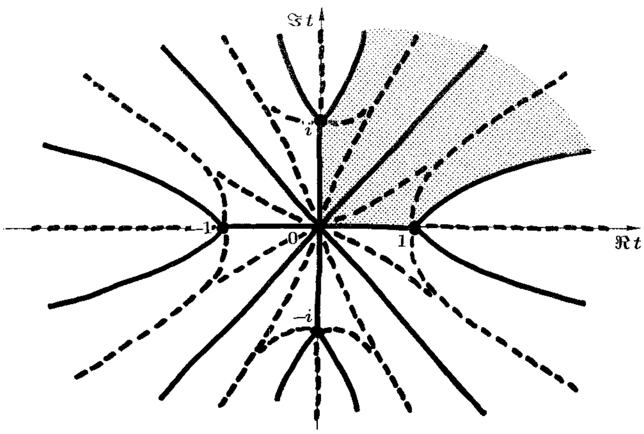


FIGURE 4.5. $\mathcal{D}^{odd,can}_{in,2}$ ($h = 5$, $j = 1$. $Q_2(t) = -t^6(t^4 - 1)$).

We notice, as shown later in §4.5, two arguments $2\pi/((l+1)l+4)$, $10\pi/((l+1)l+4)$ of $(4.7)_\infty$ correspond to ones of the canonical domain $\mathcal{D}^{even,can}_{out,j}$ for $(4.2)_j$, and two arguments 0, $8\pi/((l-1)l+4))$ of $(4.7)_0$ correspond to ones of the canonical domain $\mathcal{D}^{can,even}_{out,j+1}$ for $(4.2)_{j+1}$.

If "$j$ is odd", we can obtain $\mathcal{D}^{odd,can}_{in,j+1}$ as a canonical domain for $(4.3)_{j+1}$, which is composed of two Stokes domains of half-plane type and one Stokes domain of strip type. The canonical domain $\mathcal{D}^{odd,can}_{in,j+1}$ is bounded by several "curves" but it has a sector-like shape near

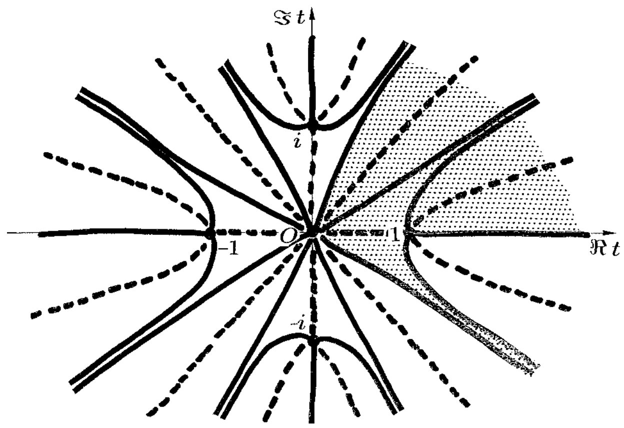


FIGURE 4.6. $\mathcal{D}^{even,can}_{in,7}$ ($h = 10$, $j = 6$. $Q_7(t) = t^6(t^4 - 1)$).

$t = \infty$ such that

$$(4.8)_\infty \qquad \mathcal{D}^{odd,can}_{in,j+1}: \quad 0 < \arg t < \frac{8\pi}{l\,(l+1)+4} \quad (t \sim \infty)\,,$$

and it has also a sector-like shape near $t = 0$ such that

$$(4.8)_0 \qquad \mathcal{D}^{odd,can}_{in,j+1}: \quad -\frac{2\pi}{l(l-1)+4} < \arg t < \frac{6\pi}{l(l-1)+4} \quad (t \sim 0)\,.$$

We notice that these arguments correspond to ones of the canonical domains $\mathcal{D}^{odd,can}_{out,j}$ for $(4.2)_j$ and $\mathcal{D}^{odd,can}_{out,j+1}$ for $(4.2)_{j+1}$, respectively (§4.5).

**4.4.** Before we construct canonical domains for $(4.2)_j$, we study (3.1) with $Q(x) := x^{p/2}$ $(p/2 \in \mathbf{N})$, i.e.,

$$(4.9) \qquad \varepsilon^2 \frac{d^2 y}{dx^2} = x^{p/2} y \quad (K\varepsilon^\beta \le |x| \le k\varepsilon^\alpha)$$

with positive constants $k$, $K$, $\alpha$ and $\beta$. First we analyze the Stokes curve configuration for

$$(4.9)_\infty \qquad \varepsilon^2 \frac{d^2 y}{dx^2} = x^{p/2} y \quad (0 < |x| < \infty)\,.$$

It is clear that the Stokes domains for (4.9) are derived from ones for $(4.9)_\infty$ if they are limited in the region $K\varepsilon^\beta \le |x| \le k\varepsilon^\alpha$. We choose a branch of square roots of $x^{p/2}$ such that $(x^{p/2})^{1/2} > 0$ for large $x > 0$, and put

$$(4.10) \qquad \hat{\xi}(0, x) := \int_0^x x^{p/4}\, dx\,.$$

As easily seen from the equation $\Re\hat{\xi} = 0$, there exist $p/2 + 2$ Stokes curves emerging from $x = 0$ with arguments

$$\pm\frac{2\pi}{p+4}, \quad \pm\frac{6\pi}{p+4}, \quad \pm\frac{10\pi}{p+4}, \quad \pm\frac{14\pi}{p+4}, \cdots \tag{4.11}$$

and they tend to $x = \infty$ with the same arguments, respectively. Similarly, from the equation $\Im\hat{\xi} = 0$, there exist $p/2 + 2$ anti-Stokes curves emerging from $x = 0$ with arguments

$$0, \quad \pm\frac{4\pi}{p+4}, \quad \pm\frac{8\pi}{p+4}, \quad \pm\frac{12\pi}{p+4}, \cdots \tag{4.12}$$

and they tend to $x = \infty$ with the same arguments, respectively. That is to say, Stokes and anti-Stokes curves are straight lines connecting $x = 0$ and $x = \infty$. Any sector bounded by neighboring two Stokes curves produces a Stokes domain of half-plane type for $(4.9)_\infty$. There exist no Stokes domains of strip type .

Then, through the above consideration, we can get several canonical domains for (4.9) such as, say,

$$\left\{x : \frac{2\pi}{p+4} < \arg x < \frac{10\pi}{p+4};\ K\varepsilon^\beta \le |x| \le k\varepsilon^\alpha\right\}, \tag{4.13}$$

$$\left\{x : -\frac{2\pi}{p+4} < \arg x < \frac{6\pi}{p+4};\ K\varepsilon^\beta \le |x| \le k\varepsilon^\alpha\right\}. \tag{4.13$'$}$$

The function $\Re\hat{\xi}$ takes positive values for $x$ satisfying $6\pi/(p+4) < \arg x < 10\pi/(p+4)$ or $-2\pi/(p+4) < \arg x < 2\pi/(p+4)$, and negative values for $x$ satisfying $2\pi/(p+4) < \arg x < 6\pi/(p+4)$.

Stokes curves for the differential equation

$$\varepsilon^2\frac{d^2y}{dx^2} = -x^{p/2}\, y \quad (K\varepsilon^\beta \le |x| \le k\varepsilon^\alpha) \tag{4.14}$$

are anti-Stokes curves for (4.9). Thus, we can obtain a canonical domain for (4.14) such as

$$\left\{x : 0 < \arg x < \frac{8\pi}{p+4},\ K\varepsilon^\beta \le |x| \le k\varepsilon^\alpha\right\}. \tag{4.15}$$

**4.5.** Suppose that "$j$ is even" in $(4.2)_j$. Then, from §4.4, we obtain a canonical domain for $(4.2)_j$ such as

(4.16)

$$\mathcal{D}^{even,can}_{out,j} := \left\{x : \frac{2\pi}{(l+1)l+4} < \arg x < \frac{10\pi}{l(l+1)+4},\ K_{j+1}\varepsilon^{1/l} \le |x| \le k_j\varepsilon^{1/(l+1)}\right\}$$

corresponding to $(4.7)_\infty$, and a canonical domain for $(4.2)_{j+1}$ such as

$$(4.17)\qquad \mathcal{D}^{even,can}_{out,j+1} := \left\{x :\ 0 < \arg x < \frac{8\pi}{l(l-1)+4},\ K_{j+2}\varepsilon^{1/(l-1)} \le |x| \le k_{j+1}\varepsilon^{1/l}\right\}$$

corresponding to $(4.7)_0$.

Suppose that "$j$ is odd" in $(4.2)_j$. Then, we obtain a canonical domain for $(4.2)_j$ such as

$$(4.18)\qquad \mathcal{D}^{odd,can}_{out,j} := \left\{x :\ 0 < \arg x < \frac{8\pi}{(l+1)l+4},\ K_{j+1}\varepsilon^{1/l} \le |x| \le k_j\varepsilon^{1/(l+1)}\right\}$$

corresponding to $(4.8)_\infty$, and a canonical domain for $(4.2)_{j+1}$ such as

(4.19)

$$\mathcal{D}^{odd,can}_{out,j+1} := \left\{x :\ -\frac{2\pi}{l(l-1)+4} < \arg x < \frac{6\pi}{l(l-1)+4},\ K_{j+2}\varepsilon^{1/(l-1)} \le |x| \le k_{j+1}\varepsilon^{1/l}\right\}$$

corresponding to $(4.8)_0$.

## 5. Matching matrices

**5.1.** Since the reduced differential equations $(4.2)_j$ and $(4.3)_{j+1}$ are asymptotically derived from (1.1) (with (4.1)), their solutions have linear relations, which can be represented by matrices. These matrices are called *matching matrices* (the second step of *the stretching-matching method* (§2.2)).

Let $y^{\pm}_{out,j}(x,\varepsilon)$ and $y^{\pm}_{in,j+1}(t,\varepsilon)$ be the true solutions of $(4.2)_j$ and $(4.3)_{j+1}$, respectively. Their corresponding WKB approximations are $(4.4)_j$ and $(4.5)_{j+1}$. The matching matrix $M[O_j, I_{j+1}]$ connecting two sets of solutions is, by definition, given by the relation

$$(5.1)\qquad M[O_j, I_{j+1}]\begin{bmatrix} y^{+}_{out,j}(x,\varepsilon) \\ y^{-}_{out,j}(x,\varepsilon) \end{bmatrix} = \begin{bmatrix} y^{+}_{in,j+1}(t,\varepsilon) \\ y^{-}_{in,j+1}(t,\varepsilon) \end{bmatrix} \quad (j = 0, 1, 2, \cdots, h-1)\,.$$

Then, $M[O_j, I_{j+1}]$'s are asymptotically given in the following

THEOREM 5.1. *For $j = 0, 1, 2, \cdots, h-1$,*

$$(5.2)\qquad M[O_j, I_{j+1}] \sim \varepsilon^{(h-j+1)/8}E \quad (\varepsilon \to 0)\,,$$

*where $E$ is the $2\times 2$ identity matrix.*

PROOF. Substituting the corresponding WKB approximations for the true solutions in (5.1), we get the asymptotic relation

$$(5.3)\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \tilde{y}^{+}_{out,j}(x,\varepsilon) \\ \tilde{y}^{-}_{out,j}(x,\varepsilon) \end{bmatrix} \sim \begin{bmatrix} \tilde{y}^{+}_{in,j+1}(t,\varepsilon) \\ \tilde{y}^{-}_{in,j+1}(t,\varepsilon) \end{bmatrix} \quad (\varepsilon \to 0)\,,$$

where we put

$$M[O_j, I_{j+1}] := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then, each element of (5.3) reads as

$$\text{(5.4)} \qquad \begin{cases} a\,\dfrac{\tilde{y}^+_{out,j}(x,\varepsilon)}{\tilde{y}^+_{in,j+1}(t,\varepsilon)} + b\,\dfrac{\tilde{y}^-_{out,j}(x,\varepsilon)}{\tilde{y}^+_{in,j+1}(t,\varepsilon)} \sim 1 \\ c\,\dfrac{\tilde{y}^+_{out,j}(x,\varepsilon)}{\tilde{y}^-_{in,j+1}(t,\varepsilon)} + d\,\dfrac{\tilde{y}^-_{out,j}(x,\varepsilon)}{\tilde{y}^-_{in,j+1}(t,\varepsilon)} \sim 1 \end{cases} \qquad (\varepsilon \to 0)\,.$$

Since $x$ and $t$ are related by the equality $x = t\,\varepsilon^{1/l}$, we split the exponent $1/l$ of $\varepsilon$ such that $x = t\varepsilon^{\{1/l-1/(l+1)\}/2} \cdot \varepsilon^{\{1/l+1/(l+1)\}/2}$ and put

$$\text{(5.5)} \qquad x := \eta\,\varepsilon^{\{1/l+1/(l+1)\}/2}\,, \quad t := \eta\,\varepsilon^{\{1/(l+1)-1/l\}/2}$$

with a new parameter $\eta$ ($|\eta| = 1$) determined later. Then, $x$ belongs to the outer domain and $t$ to the inner domain, and we see that $x \to 0,\ t \to \infty$ as $\varepsilon \to 0$. We get, by substituting (5.5) in $(4.4)_j$,

(5.6)
$$\begin{cases} \tilde{y}^{\pm}_{out,j}(x,\varepsilon) := (-1)^{-j/4}(\eta\,\varepsilon^{\{1/l+1/(l+1)\}/2})^{-(l+1)\,l/8} \\ \qquad\qquad \times \exp\left( \pm\, \dfrac{4\,(-1)^{j/2}}{(l+1)l+4}\, \eta^{\{(l+1)l+4\}/4}\, \varepsilon^{(-6\,l^3-5\,l^2+9\,l+4)/(8\,l\,(l+1))-j/2} \right), \end{cases}$$

where the power of $\varepsilon$ in the exponential part is negative.

On the other hand, noticing that $Q_{j+1}(t) \sim (-1)^j\, t^{(l+1)\,l/2}$ as $\varepsilon \to 0$ or $t \to \infty$, integrating it and substituting (5.5) in $(4.5)_{j+1}$, we can see that

(5.7)
$$\begin{cases} \tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) \sim (-1)^{-j/4}(\eta\,\varepsilon^{\{1/(l+1)-1/l\}/2})^{-(l+1)\,l/8} \\ \qquad\qquad \times \exp\left( \pm\, \dfrac{4\,(-1)^{j/2}}{(l+1)l+4}\, \eta^{\{l\,(l+1)+4\}/4}\, \varepsilon^{(-6\,l^3-5\,l^2+9\,l+4)/(8\,l\,(l+1))-j/2} \right). \end{cases}$$

Since the exponential parts of (5.6) and (5.7) equal each other, we get the relations

$$\text{(5.8)} \qquad \frac{\tilde{y}^+_{out,j}(x,\varepsilon)}{\tilde{y}^+_{in,j+1}(t,\varepsilon)} \sim \left(\frac{t}{x}\right)^{l\,(l+1)/8} \cdot \exp(0) = \varepsilon^{-(l+1)/8} \quad (\varepsilon \to 0)$$

and

(5.9)
$$\begin{cases} \dfrac{\tilde{y}^{-}_{out,j}(x,\varepsilon)}{\tilde{y}^{+}_{in,j+1}(t,\varepsilon)} \ \sim \ \varepsilon^{-(l+1)/8} \\ \quad \times \exp\left( - \ \dfrac{8\,(-1)^{j/2}}{(l+1)\,l+4}\ \eta^{\{(l+1)\,l+4\}/4}\ \varepsilon^{(-6\,l^3-5\,l^2+9\,l+4)/(8\,l\,(l+1))-j/2}\right) \to \infty \quad (\varepsilon \to 0) \end{cases}$$

with an appropriate parameter $\eta$ which can be chosen in the canonical domain such as, say, $\eta := \eta_{+\infty}$ in Fig. 4.1 or 4.2. Thus, the first relation of (5.4) becomes

$$\varepsilon^{-(l+1)/8}(a \ + \ b \cdot \infty) \sim 1 \quad (\varepsilon \to 0)\,, \tag{5.10}$$

which induces

$$a \ \sim \ \varepsilon^{(l+1)/8}\,, \quad b \sim 0 \quad (\varepsilon \to 0)\,. \tag{5.11}$$

In the similar way, by choosing an appropriate $\eta$, say, $\eta := \eta_{-\infty}$ in Fig. 4.1 or 4.2, we can see that

$$c \sim 0\,, \quad d \ \sim \ \varepsilon^{(l+1)/8} \quad (\varepsilon \to 0)\,. \tag{5.12}$$

Thus, the matching matrix is given by (5.2). Q.E.D.

**5.2.** Let $y^{\pm}_{in,j+1}(t,\varepsilon)$ and $y^{\pm}_{out,j+1}(x,\varepsilon)$ be the true solutions of $(4.3)_{j+1}$ and $(4.2)_{j+1}$, respectively. Then, the matching matrix $M[I_{j+1}, O_{j+1}]$ connecting two sets of solutions is, by definition, given by the relation

$$M[I_{j+1}, O_{j+1}]\begin{bmatrix} y^{+}_{in,j+1}(t,\varepsilon) \\ y^{-}_{in,j+1}(t,\varepsilon) \end{bmatrix} = \begin{bmatrix} y^{+}_{out,j+1}(x,\varepsilon) \\ y^{-}_{out,j+1}(x,\varepsilon) \end{bmatrix}. \tag{5.13}$$

Then $M[I_{j+1}, O_{j+1}]$'s are asymptotically given as follows.

THEOREM 5.2. *For* $j = 0, 1, 2, \cdots, h-1$,

$$M[I_{j+1}, O_{j+1}] \sim \varepsilon^{-(h-j-1)/8}E \quad (\varepsilon \to 0)\,. \tag{5.14}$$

PROOF. By substituting the corresponding WKB approximations for the true solutions in (5.13), we get the asymptotic relation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \tilde{y}^{+}_{in,j+1}(t,\varepsilon) \\ \tilde{y}^{-}_{in,j+1}(t,\varepsilon) \end{bmatrix} \sim \begin{bmatrix} \tilde{y}^{+}_{out,j+1}(x,\varepsilon) \\ \tilde{y}^{-}_{out,j+1}(x,\varepsilon) \end{bmatrix} \quad (\varepsilon \to 0)\,, \tag{5.15}$$

where we put again

$$M[I_{j+1}, O_{j+1}] := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(We hope there is no confusion even though we use the same letters as the matching matrix in §5.1.) The elements of (5.15) read as

$$\text{(5.16)}\qquad \begin{cases} a\,\dfrac{\tilde{y}^{+}_{in,j+1}(t,\varepsilon)}{\tilde{y}^{+}_{out,j+1}(x,\varepsilon)} \;+\; b\,\dfrac{\tilde{y}^{-}_{in,j+1}(t,\varepsilon)}{\tilde{y}^{+}_{out,j+1}(x,\varepsilon)} \;\sim\; 1 \\[2ex] c\,\dfrac{\tilde{y}^{+}_{in,j+1}(t,\varepsilon)}{\tilde{y}^{-}_{out,j+1}(x,\varepsilon)} \;+\; d\,\dfrac{\tilde{y}^{-}_{in,j+1}(t,\varepsilon)}{\tilde{y}^{-}_{out,j+1}(x,\varepsilon)} \;\sim\; 1 \end{cases} \qquad (\varepsilon \to 0)\,.$$

Although two variables $x$ and $t$ are again related by the same equality $x = t\varepsilon^{1/l}$, we split, in this case, the power $1/l$ of $\varepsilon$ such that $x = t\varepsilon^{\{1/l-1/(l-1)\}/2} \cdot \varepsilon^{\{1/l+1/(l-1)\}/2}$ and put

$$\text{(5.17)}\qquad x := \eta\,\varepsilon^{\{1/l+1/(l-1)\}/2}\,, \quad t := \eta\,\varepsilon^{\{1/(l-1)-1/l\}/2}$$

with another new parameter $\eta$ ($|\eta| = 1$) determined later. Then, $x$ belongs to the outer domain and $t$ to the inner domain, and we see that $x \to 0,\ t \to 0$ as $\varepsilon \to 0$.

By noticing that $Q_{j+1}(t) \sim (-1)^{j+1}\, t^{(l-1)\,l/2}$ as $\varepsilon \to 0$ or $t \to 0$, by integrating it, and by substituting (5.17) in $\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon)$, we can see that

$$\text{(5.18)}\qquad \begin{cases} \tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) \;\sim\; (-1)^{-(j+1)/4}(\eta\,\varepsilon^{\{1/(l-1)-1/l\}/2})^{-l\,(l-1)/8} \\[1ex] \qquad\qquad \times \exp\left( \pm\, \dfrac{4(-1)^{(j+1)/2}}{l(l-1)+4}\; \eta^{\{l\,(l-1)+4\}/4} \varepsilon^{(-6\,l^3+9\,l^2+5\,l-4)/(8\,l\,(l-1))-j/2} \right), \end{cases}$$

where the power of $\varepsilon$ in the exponential part is negative. We also get

$$\text{(5.19)}\qquad \begin{cases} \tilde{y}^{\pm}_{out,j+1}(x,\varepsilon) := (-1)^{-(j+1)/4}(\eta\,\varepsilon^{\{1/l+1/(l-1)\}/2})^{-l(l-1)/8} \\[1ex] \qquad\qquad \times \exp\left( \pm\, \dfrac{4\,(-1)^{(j+1)/2}}{l(l-1)+4}\; \eta^{\{l\,(l-1)+4\}/4} \varepsilon^{(-6\,l^3+9\,l^2+5\,l-4)/(8\,l\,(l-1))-j/2} \right). \end{cases}$$

Since the exponential parts of (5.18) and (5.19) equal each other, we get the relations

$$\text{(5.20)}\qquad \frac{\tilde{y}^{+}_{in,j+1}(t,\varepsilon)}{\tilde{y}^{+}_{out,j+1}(x,\varepsilon)} \;\sim\; \left(\frac{x}{t}\right)^{l\,(l-1)/8} \cdot \exp(0) = \varepsilon^{(l-1)/8} \quad (\varepsilon \to 0)$$

and

$$\text{(5.21)}\qquad \begin{cases} \dfrac{\tilde{y}^{-}_{in,j+1}(t,\varepsilon)}{\tilde{y}^{+}_{out,j+1}(x,\varepsilon)} \;\sim\; \varepsilon^{(l-1)/8} \\[2ex] \qquad\qquad \times \exp\left( -\, \dfrac{8\,(-1)^{(j+1)/2}}{l(l-1)+4}\; \eta^{\{l(l-1)+4\}/4} \varepsilon^{(-6\,l^3+9\,l^2+5\,l-4)/(8\,l\,(l-1))-j/2} \right) \\[2ex] \qquad\qquad\qquad\qquad \to \infty \quad (\varepsilon \to 0) \end{cases}$$

with an appropriate parameter $\eta$ which can be chosen in the canonical domain such as, say, $\eta := \eta_{+0}$ in Fig. 4.1 or 4.2. Thus, the first relation of (5.16) becomes

$$\varepsilon^{(l-1)/8}(a + b \cdot \infty) \sim 1 \quad (\varepsilon \to 0)\,,$$

which induces

$$a \sim \varepsilon^{-(l-1)/8}, \quad b \sim 0 \quad (\varepsilon \to 0)\,. \tag{5.22}$$

In the similar way, by choosing an appropriate $\eta$, say, $\eta := \eta_{-0}$ in Fig. 4.1 or 4.2, we can see that

$$c \sim 0\,, \quad d \sim \varepsilon^{-(l-1)/8} \quad (\varepsilon \to 0)\,. \tag{5.23}$$

Thus the desired matching matrix is given by (5.14). Q.E.D.

## 6. The formal computation of matching matrices

**6.1.** Since we specified the values of $a_j$'s of $Q(x, \varepsilon)$ in §4, we could construct the exact form of several canonical domains for the reduced differentail equations and could compute the matching matrices in §5. Here, we try to compute formally the matching matrices without specifying the concrete values of $a_j$'s. We can properly suppose the existence of canonical domains because we can construct them for, e.g., (3.1) with a polynomial coefficient $Q(x)$.

Theorem 8.5-2 of Wasow [22] gives a general form of the matching matrix for an $n$-th order differential equation. It points out that the diagonal entries are asymptotically important and the off-diagonal entries equal asymptotically to zero. However, it does not give concrete values of entries of the matching matrix since the differential equation is not concrete.

Let us consider the rather concrete differential equation

$$\varepsilon^{2h}\,\frac{d^2y}{dx^2} = \hat{Q}(x, \varepsilon)y\,, \quad \hat{Q}(x, \varepsilon) := \sum_{j=0}^{h} a_j \varepsilon^j x^{m_j}\,, \quad \mathbf{C} \ni \forall a_j \neq 0\,, \tag{6.1}$$

$$\begin{cases} m_0 > \cdots > m_{j-1} > m_j > m_{j+1} > \cdots > m_h \geq 0\,, \\ m_{j-1} + m_{j+1} > 2m_j \qquad (j = 1, 2, \cdots, h-1)\,, \\ h > \{j + \alpha(m_j + 2)\}/2 \quad \big(\alpha := 1/(m_j - m_{j+1})\big) \quad (j = 0, 1, 2, \cdots, h-1)\,, \end{cases} \tag{6.1$'$}$$

where $h$ is a positive integer and $m_j$ may be, in general, different from that of (1.1). The inequalities among $m_j$'s are important, especially the second means that the characteristic polygon of (6.1) is convex downward and all the points $P_j$'s are situated at the snapping points of the characteristic polygon. The last one is *a singular perturbation condition* (cf. $(2.7)_{j+1}$, $(4.3)_{j+1}$), which means that an inclination of the segment $\overline{P_{j+1}R}$ is bigger than that of the segment $\overline{P_j P_{j+1}}$.

By the same method of the reduction in §2, we can get the following two outer and one inner asymptotically reduced differential equations:

$$(6.2)_j \qquad \varepsilon^{2h-j}\,\frac{d^2y}{dx^2} = a_j x^{m_j} y \quad \left( K\varepsilon^{\alpha} \le |x| \le k'\varepsilon^{\alpha'},\ \alpha := \frac{1}{m_j - m_{j+1}} > \alpha' \right),$$

$$(6.3)_{j+1} \qquad \begin{cases} \varepsilon_{j+1}^2\,\dfrac{d^2y}{dt^2} = \hat{Q}_{j+1}(t)y, \quad \varepsilon_{j+1} := \varepsilon^{h-j/2-\alpha(m_j+2)/2}, \\ \hat{Q}_{j+1}(t) := a_j t^{m_j} + a_{j+1}t^{m_{j+1}} \quad (k \le |t| \le K,\ \ t := x\,\varepsilon^{-\alpha}), \end{cases}$$

$$(6.2)_{j+1} \qquad \varepsilon^{2h-j-1}\,\frac{d^2y}{dx^2} = a_{j+1}x^{m_{j+1}} y \quad (K'\varepsilon^{\alpha''} \le |x| \le k\varepsilon^{\alpha},\ \alpha'' > \alpha),$$

where $K$ and $K'$ are sufficiently large constants, $k$ and $k'$ are sufficiently small constants, and $\alpha'$ and $\alpha''$ are positive constants. When we consider $(6.3)_{j+1}$ in the domain $\{t\ :\ 0 \le |t| < \infty\}$, the origin $t = 0$ and $(m_j - m_{j+1})$-th roots of $-a_{j+1}/a_j$ are secondary turning points of (6.1) and $t = \infty$ is always an irregular singular point of $(6.3)_{j+1}$.

As shown in §3, their WKB approximations are, respectively, given by

$$(6.4)_j \qquad \tilde{y}^{\pm}_{out,j}(x,\varepsilon) := a_j^{-1/4}\,x^{-m_j/4}\exp\left( \pm\,\frac{2\sqrt{a_j}}{m_j+2}\,\frac{x^{(m_j+2)/2}}{\varepsilon^{h-j/2}} \right),$$

$(6.5)_{j+1}$

$$\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) := \hat{Q}_{j+1}^{-1/4}(t)\,\exp\left( \pm\,\frac{1}{\varepsilon_{j+1}}\xi_{j+1}(t) \right), \quad \xi_{j+1}(t) := \int_0^t \sqrt{\hat{Q}_{j+1}(t)}\,dt\,,$$

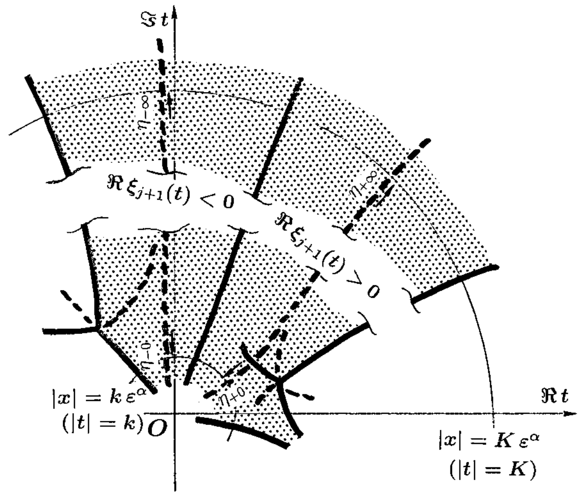


FIGURE 6.1. The assumed canonical domain.

$$(6.4)_{j+1} \qquad \tilde{y}^{\pm}_{out,j+1}(x,\varepsilon) := a_{j+1}^{-1/4} x^{-m_{j+1}/4} \exp\left( \pm \frac{2\sqrt{a_{j+1}}}{m_{j+1}+2} \frac{x^{(m_{j+1}+2)/2}}{\varepsilon^{h-(j+1)/2}} \right),$$

and we see that

$(6.5)'_{j+1}$

$$\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon) \sim \begin{cases} (a_j t^{m_j})^{-1/4} \exp\left( \pm \dfrac{1}{\varepsilon_{j+1}} \displaystyle\int_0^t \sqrt{a_j t^{m_j}}\, dt \right) & (t \to \infty) \\ (a_{j+1} t^{m_{j+1}})^{-1/4} \exp\left( \pm \dfrac{1}{\varepsilon_{j+1}} \displaystyle\int_0^t \sqrt{a_{j+1} t^{m_{j+1}}}\, dt \right) & (t \to 0) \end{cases}.$$

Though we cannot determine the concrete form of the Stokes curve configuration due to the ambiguity of $a_j$, we assume here the existence of a canonical domain illustrated in Fig. 6.1. In about a half of it $\Re\xi_{j+1}(t) > 0$ and $\Re\xi_{j+1}(t) < 0$ in the rest. In the region $\{t : \Re\xi_{j+1}(t) > 0 \text{ (resp. } < 0)\}$ there exist anti-Stokes curves along which $\Re\xi_{j+1}(t)$ tends to $+\infty$ (resp. $-\infty$) as $t \to \infty$ and $\Re\xi_{j+1}(t)$ tends to $+0$ (resp. $-0$) as $t \to 0$.

**6.2.** We compute two particular matching matrices similar to ones given in §5. The first is as follows.

THEOREM 6.1. *Let $M[O_j, I_{j+1}]$ be the matching matrix connecting a set of two WKB approximations $\tilde{y}^{\pm}_{out,j}(x,\varepsilon)$ to a set of two WKB approximations $\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon)$:*

$$(6.5) \qquad M[O_j, I_{j+1}] \begin{bmatrix} \tilde{y}^{+}_{out,j}(x,\varepsilon) \\ \tilde{y}^{-}_{out,j}(x,\varepsilon) \end{bmatrix} \sim \begin{bmatrix} \tilde{y}^{+}_{in,j+1}(t,\varepsilon) \\ \tilde{y}^{-}_{in,j+1}(t,\varepsilon) \end{bmatrix} \quad (\varepsilon \to 0).$$

*Then, $M[O_j, I_{j+1}]$ is given by*

$$(6.6) \qquad M[O_j, I_{j+1}] \sim \varepsilon^{m_j/(4(m_j - m_{j+1}))} E \quad (\varepsilon \to 0).$$

PROOF. The proof of this theorem is very similar to that of Theorem 5.1. Putting

$$M[O_j, I_{j+1}] := \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we get just the same relation as (5.4). By choosing some constant $\beta$ $(0 < \beta < \alpha$ $(:= 1/(m_j - m_{j+1}))$, which is a very different choice from Theorem 5.1, we put

$$x := \eta\, \varepsilon^{\alpha-\beta} \to 0\,, \quad t := \eta\, \varepsilon^{-\beta} \to \infty \quad (\varepsilon \to 0)$$

with a new parameter $\eta$ $(|\eta| = 1)$. Then, $x$ belongs to the outer domain (i.e., $K\varepsilon^{\alpha} \le |x| \le k'\varepsilon^{\alpha'}$) and $t$ belongs to the inner domain $(0 \le |t| < \infty)$, so that

$$\frac{\tilde{y}^{+}_{out,j}(x,\varepsilon)}{\tilde{y}^{+}_{in,j+1}(t,\varepsilon)} \sim \varepsilon^{-\alpha\, m_j/4} \quad (\varepsilon \to 0)\,,$$

$$\frac{\tilde{y}^-_{out,j}(x,\varepsilon)}{\tilde{y}^+_{in,j+1}(t,\varepsilon)} \sim \varepsilon^{-\alpha m_j/4} \ \exp\left( -\frac{4\sqrt{a_j}}{m_j+2}\,\eta^{(m_j+2)/2}\varepsilon^{-h+j/2+(\alpha-\beta)(m_j+2)/2} \right) \to \infty \quad (\varepsilon \to 0)$$

with a parameter $\eta := \eta_{+\infty}$, which can be chosen on one of the anti-Stokes curves in the canonical domain (Fig. 6.1). The exponent of $\varepsilon$ in the exponential part is negative because of the singular perturbation condition. Thus, we can see

$$a \cdot \varepsilon^{-\alpha\, m_j/4} + b \cdot \infty \sim 1 \quad (\varepsilon \to 0)\,,$$

which induces

$$a \sim \varepsilon^{\alpha\, m_j/4}, \quad b \sim 0 \quad (\varepsilon \to 0)\,.$$

In the similar way, by choosing $\eta := \eta_{-\infty}$ on another anti-Stokes curve in the canonical domain, we see that $c \sim 0$, $d \sim \varepsilon^{\alpha\, m_j/4} \quad (\varepsilon \to 0)$. Q.E.D.

We compare this general result to the matching matrix given in §5.1. Putting $m_j := l(l+1)/2$, $m_{j+1} := l(l-1)/2 \quad (l := h-j)$ for (6.6), we see that $m_j/4(m_j - m_{j+1}) = (l+1)/8 = (h-j+1)/8$, which coincides with (5.2).

In Theorem 6.1, the inverse matching matrix $M[O_j, I_{j+1}]^{-1}$ connects a set of two WKB approximations $\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon)$ to a set of two WKB approximations $\tilde{y}^{\pm}_{out,j}(x,\varepsilon)$. Hence the relation $M[O_j, I_{j+1}]^{-1} = M[I_{j+1}, O_j]$ holds. Thus, we get

COROLLARY.

$$(6.6)' \qquad M[I_{j+1}, O_j] = M[O_j, I_{j+1}]^{-1} \sim \varepsilon^{-m_j/(4(m_j-m_{j+1}))}E \quad (\varepsilon \to 0)\,.$$

**6.3.** We compute another matching matrix.

THEOREM 6.2. *Let $M[I_{j+1}, O_{j+1}]$ be the matching matrix connecting a set of two WKB approximations $\tilde{y}^{\pm}_{in,j+1}(t,\varepsilon)$ to a set of two WKB approximations $\tilde{y}^{\pm}_{out,j+1}(x,\varepsilon)$:*

$$(6.7) \qquad M[I_{j+1}, O_{j+1}]\begin{bmatrix} \tilde{y}^+_{in,j+1}(t,\varepsilon) \\ \tilde{y}^-_{in,j+1}(t,\varepsilon) \end{bmatrix} \sim \begin{bmatrix} \tilde{y}^+_{out,j+1}(x,\varepsilon) \\ \tilde{y}^-_{out,j+1}(x,\varepsilon) \end{bmatrix} \quad (\varepsilon \to 0)\,.$$

*Then, $M[I_{j+1}, O_{j+1}]$ is given by*

$$(6.8) \qquad M[I_{j+1}, O_{j+1}] \sim \varepsilon^{-m_{j+1}/4(m_j-m_{j+1})}E \quad (\varepsilon \to 0)\,.$$

We remark that (6.8) has a similar form to (6.6)$'$. The proof of this theorem is similar to that of Theorem 5.2 and we omit it here. We compare this general result got above to the matching matrix given in §5.2. Putting $m_j := l\,(l+1)/2$, $m_{j+1} := l\,(l-1)/2 \quad (l := h-j)$ for (6.8), we see that $-m_{j+1}/4(m_j - m_{j+1}) = -(l-1)/8 = -(h-j-1)/8$, which coincides with (5.14).

## 7. Reviewing the known matching matrices

**7.1** We review some known results about the matching matrices for the differential equations with a several-segment characteristic polygon, and we compare them with our formulas (6.6) and (6.16).

The first example is the simplest case of (6.1) given in [13],

$$\varepsilon^2 \frac{d^2 y}{dx^2} = (x^3 - \varepsilon) y \quad (h = 1, a_0 = 1, a_1 = -1, m_0 = 3, m_1 = 0)\,. \tag{7.1}$$

The reduced differential equations and their WKB approximations are as follows:

$$(O_0) \qquad \begin{cases} \varepsilon^2 \dfrac{d^2 y}{dx^2} = x^3 y & (K\varepsilon^{1/3} \leq |x| \leq x_0)\,, \\ \tilde{y}^{\pm}_{out,0}(x,\varepsilon) := \dfrac{1}{\sqrt[4]{x^3}} \exp\left( \pm \dfrac{2}{5\varepsilon} x^{5/2} \right), \end{cases}$$

$$(I_1) \qquad \begin{cases} \varepsilon^{1/3} \dfrac{d^2 y}{dt^2} = (t^3 - 1) y & (k \leq |t| \leq K,\ t := x\varepsilon^{-1/3})\,, \\ \tilde{y}^{\pm}_{in,1}(t,\varepsilon) := \dfrac{1}{\sqrt[4]{t^3 - 1}} \exp\left( \pm \dfrac{1}{\varepsilon^{1/6}} \displaystyle\int_1^t \sqrt{t^3 - 1}\, dt \right), \end{cases}$$

$$(O_1) \qquad \begin{cases} \varepsilon \dfrac{d^2 y}{dx^2} = -\, y & (0 \leq |x| \leq k\varepsilon^{1/3})\,, \\ \tilde{y}^{\pm}_{out,1}(x,\varepsilon) := \exp\left( \pm \dfrac{i}{\sqrt{\varepsilon}}\, x \right), \end{cases}$$

where $K$ (resp. $k$) is a sufficiently large (resp. small) constant.

The WKB approximations in $(O_1)$ are the true solutions of the equation in $(O_1)$. Let us denote the set of two WKB approximations in $(O_j)$ (resp. $(I_1)$) by the same symbol $O_j$ (resp. $I_1$). Our formulas (6.6) and (6.16) give us the matching matrices

$$M[O_0, I_1] \sim \varepsilon^{m_0/(4(m_0 - m_1))} E = \varepsilon^{1/4} E\,,$$

$$M[I_1, O_1] \sim \varepsilon^{-m_1/(4(m_0 - m_1))} E = \varepsilon^0 E = E\,.$$

$M[O_0, I_1]$ coincides with the matching matrix $\mathfrak{M}_1$ of Theorem G in [13]. $\mathfrak{M}_1$ is defined to connect two fundamental matrices of solutions so that it looks different from $M[O_0, I_1]$. In analysing $(I_1)$ for $t$ in $\{t : 0 \leq |t| < \infty\}$, $M[I_1, O_1]$ is not necessary because the information around $t = 0$, i.e., $x = 0$, can be obtained from $(I_1)$. In fact, it was not computed in [13].

(7.1) possesses three secondary turning points: $1, \omega\ (:= e^{i2\pi/3}), \omega^2\ (= \omega^{-1})$. The Stokes curve configuration for $(I_1)$ is determined by the integral $\int \sqrt{t^3 - 1} dt$ and shown in

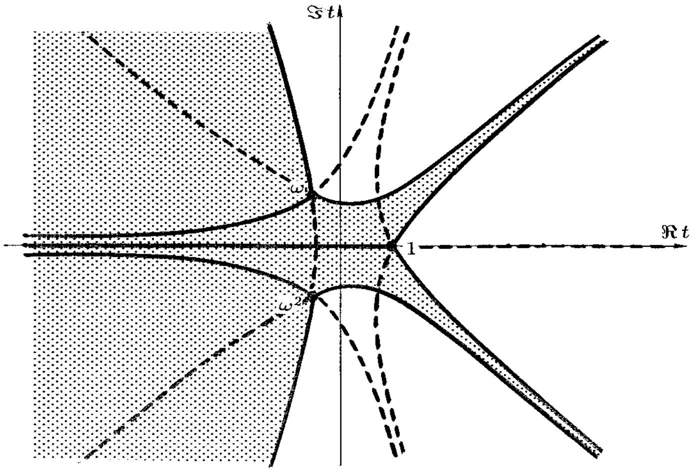


FIGURE 7.1. The Stokes curve configuration for $Q(t) = t^3 - 1$.

Fig. 7.1. The solid lines and the broken lines show the Stokes curves and the anti-Stokes curves, respectively, and the shaded region is one of the canonical domains.

We remark that there is a slight mistake about anti-Stokes curves of Fig. 2 in [13]. That is two anti-Stokes curves linking 1 and $\omega$, 1 and $\omega^2$. Really, these anti-Stokes curves do not exist. Here, we correct them. We see that two anti-Stokes curves emerging from $t = 1$ do not tend to the other turning points, because the value of the integral

$$\int_1^{\Omega} \sqrt{t^3 - 1}\, dt = \left( \int_1^0 + \int_0^{\Omega} \right) \sqrt{t^3 - 1}\, dt = (\Omega - 1) \int_0^1 \sqrt{r^3 - 1}\, dr \quad (\Omega := \omega^{\pm 1})$$

is neither real nor pure imaginary. Then, they must tend to $\infty$ in the first and the fourth quadrants respectively, because they cannot cross other anti-Stokes curves from $t = \omega$ or $t = \omega^2$. Also, we can show the existence of an anti-Stokes curve linking $\omega$ and $\omega^2$. In fact, the integral

$$\int_{\omega}^{\omega^2} \sqrt{t^3 - 1}\, dt = \left( \int_{\omega}^0 + \int_0^{\omega^2} \right) \sqrt{t^3 - 1}\, dt = \sqrt{3} \int_0^1 \sqrt{1 - r^3}\, dr$$

takes real values only, and hence this definite integral suggests the existence of the anti-Stokes curve linking $\omega$ and $\omega^2$. There does not exist an anti-Stokes curve linking $\omega$ and $\omega^2$ via $\infty$ because of the symmetry of the Stokes curve configuration with respect to the real axis and the general property of the anti-Stokes curves (§3.2). This Stokes curve configuration appeared in [1], too.

**7.2.** Fedoryuk studied the generalized differential equation of (7.1) such as

$$\varepsilon^2 \frac{d^2 y}{dx^2} = (x^n - \varepsilon) y \quad (h = 1,\ m_0 = n \geq 3,\ m_1 = 0) \tag{7.2}$$

in §8.4 of [4]. The reduced differential equations and their WKB approximations are as follows:

$$(\tilde{O}_0)\qquad \begin{cases} \varepsilon^2 \dfrac{d^2y}{dx^2} = x^n y \quad (K\varepsilon^{1/n} \le |x| \le x_0)\,, \\ \tilde{y}^{\pm}_{out,0}(x,\varepsilon) := x^{-n/4} \exp\left( \pm \dfrac{2}{(n+2)\varepsilon}\, x^{(n+2)/2} \right), \end{cases}$$

$$(\tilde{I}_1)\qquad \begin{cases} \varepsilon^{(n-2)/n} \dfrac{d^2y}{dt^2} = (t^n - 1)y \qquad (0 \le |t| < \infty,\ t := x\varepsilon^{-1/n})\,, \\ \tilde{y}^{\pm}_{in,1}(t,\varepsilon) := (t^n - 1)^{-1/4} \exp\left( \pm \dfrac{1}{\varepsilon^{(n-2)/2n}} \displaystyle\int_1^t \sqrt{t^n - 1}\, dt \right), \end{cases}$$

where $K$ is a sufficiently large constant. We denote the solution of the equation $(\tilde{O}_j)$ (resp. $(\tilde{I}_j)$) by the same symbol $\tilde{O}_j$ (resp. $\tilde{I}_j$).

Then, formula (6.6) gives us the matching matrix

$$M[\tilde{O}_0, \tilde{I}_1] \sim \varepsilon^{m_0/(4(m_0-m_1))} E = \varepsilon^{1/4} E\,,$$

which coincides essentially with $\Omega(\varepsilon)$ in p.226 of [4].

**7.3.** Other cases were studied in [8] and [17], and Roos also studied in [18] the following equation

$$(7.3)\quad \varepsilon^4 \frac{d^2y}{dx^2} = (x^5 + \varepsilon x^2 + \varepsilon^2)y \quad (h = 2, a_0 = a_1 = a_2 = 1, m_0 = 5, m_1 = 2, m_2 = 0)\,.$$

Our reduction method yields five reduced equations:

$$(\hat{O}_0)\qquad \varepsilon^4 \frac{d^2y}{dx^2} = x^5 y\,,$$

$$(\hat{I}_1)\qquad \varepsilon^{5/3} \frac{d^2y}{dt^2} = t^2(t^3+1)y \quad (t := x\varepsilon^{-1/3})\,,$$

$$(\hat{O}_1)\qquad \varepsilon^3 \frac{d^2y}{dx^2} = x^2 y\,,$$

$$(\hat{I}_2)\qquad \varepsilon \frac{d^2y}{dt^2} = (t^2+1)y \quad (t := x\varepsilon^{-1/2})\,,$$

$$(\hat{O}_2)\qquad \varepsilon^2 \frac{d^2y}{dx^2} = y\,.$$

We denote the solutions of the equations $(\hat{O}_j)$ and $(\hat{I}_j)$ by the same symbols $\hat{O}_j$ and $\hat{I}_j$, respectively. Now, let us check matching matrices. Applying the formulas (6.6) and (6.16), we can get the following matching matrices.

$$\begin{aligned}
M[\hat{O}_0, \hat{I}_1] &\sim \varepsilon^{m_0/4(m_0-m_1)} E = \varepsilon^{5/12} E \,, \\
M[\hat{I}_1, \hat{O}_1] &\sim \varepsilon^{-m_1/4(m_0-m_1)} E = \varepsilon^{-1/6} E \,, \\
M[\hat{O}_1, \hat{I}_2] &\sim \varepsilon^{m_1/4(m_1-m_2)} E = \varepsilon^{1/4} E \,, \\
M[\hat{I}_2, \hat{O}_2] &\sim \varepsilon^{-m_2/4(m_1-m_2)} E = \varepsilon^{0} E = E \,.
\end{aligned}$$

Our reduction of the equations is different from Roos's, and we can see that

$$M[\hat{I}_1, \hat{O}_0] = M[\hat{O}_0, \hat{I}_1]^{-1} \sim C_{21}(\varepsilon) \,, \quad M[\hat{O}_2, \hat{I}_2] = M[\hat{I}_2, \hat{O}_2]^{-1} \sim C_{32}(\varepsilon) \,,$$

where $C_{jk}(\varepsilon)$'s are matching matrices given in Satz 2 of [18].

**7.4.** The next example was given in [12]

$$\varepsilon^{2h} \frac{d^2 y}{dx^2} = \left( \sum_{j=0}^{h} a_j \varepsilon^j x^{m_j} \right) y \,, \tag{7.4}$$

where

$$a_j := \begin{cases} {}_kC_j & (j = 0, 1, 2, \cdots, k) \,, \\ {}_{h-k}C_{j-k} & (j = k+1, k+2, \cdots, h) \,, \end{cases}$$

$$m_j := \begin{cases} h + m + k - 2j & (j = 0, 1, 2, \cdots, k) \,, \\ h + m - j & (j = k+1, k+2, \cdots, h) \,, \end{cases}$$

and points $P_j$'s and $R$ are defined in (1.2). (7.4) is out of a category of (1.1) because it has a three-segment characteristic polygon on each segment of which there exist many points, that is, the points $P_0, \cdots, P_k$ are on the first segment defined by the equation $Y = -X + (h+m)/2$, the points $P_k, \cdots, P_h$ are on the second one defined by the equation $Y = -(m+2)/h\, X + m + 1$, and the points $P_h$ and $R$ are on the third one defined by the equation $Y = -2X + (h + m + k)/2$. We dare, however, to apply our formulas (6.6) and (6.16) to (7.4).

First, before applying directly our formulas, we must recognize that $P_j$ is the adjacent point to $P_{j\pm 1}$. By our reduction method, we have five reduced equations:

$$\varepsilon^{2h} \frac{d^2 y}{dx^2} = x^{h+m+k} y \,, \tag{$\check{O}_0$}$$

$$\varepsilon^{(3h-m-k-2)/2} \frac{d^2 y}{dt^2} = t^{h+m-k} (t^2 + 1)^k \, y \quad (t := x\varepsilon^{-1/2}) \,, \tag{$\check{I}_1$}$$

$$(\check{O}_1) \qquad \varepsilon^{2h-k}\,\frac{d^2y}{dx^2} = x^{h+m-k}\; y,$$

$$(\check{I}_2) \qquad \varepsilon^{h-m-2}\,\frac{d^2y}{dt^2} = t^m(t+1)^{h-k}\; y \quad (t := x\varepsilon^{-1})\,,$$

$$(\check{O}_2) \qquad \varepsilon^{h}\,\frac{d^2y}{dx^2} = x^m y\,.$$

We denote the solution of the equation ($\check{O}_j$) (resp. ($\check{I}_j$)) by the same symbol $\check{O}_j$ (resp. $\check{I}_j$) as before. Now, let us check matching matrices.

(i) The first point is $P_0$ and $m_0 := h+m+k$. Since the adjacent point to $P_0$ is $P_1$ and $m_1 := h+m+k-2$, we get

$$M[\check{O}_0, \check{I}_1] \sim \varepsilon^{m_0/(4(m_0-m_1))}E = \varepsilon^{(h+m+k)/8}E\,,$$

which coincides with $M_{1,2}$ of Theorem 6.1 in [12].

(ii) The point $P_{k-1}$ is nearest to the left of $P_k$ and we put $m_{k-1} := h+m-k+2$ and $m_k := h+m-k$. Then, we get

$$M[\check{I}_1, \check{O}_1] \sim \varepsilon^{-m_k/4(m_{k-1}-m_k)}E = \varepsilon^{-(h+m-k)/8}E\,,$$

which coincides with $M_{2,3}$.

(iii) Another nearest point to $P_k$ is $P_{k+1}$ and we put $m_k := h+m-k$ and $m_{k+1} := h+m-k-1$. Then, we get

$$M[\check{O}_1, \check{I}_2] \sim \varepsilon^{m_k/(4(m_k-m_{k+1}))}E = \varepsilon^{(h+m-k)/4}E\,,$$

which coincides with $M_{3,4}$.

(iv) The point $P_{h-1}$ is adjacent to $P_h$ and we put $m_{h-1} := m+1$ and $m_h := m$. Then, we get

$$M[\check{I}_2, \check{O}_2] \sim \varepsilon^{-m_h/(4(m_{h-1}-m_h))}E = \varepsilon^{-m/4}E\,,$$

which coincides with $M_{4,5}$.

Thus we see that formulas (6.6) and (6.16) can be applied to the differential equation of different type from (1.1). On each segment of the characteristic polygon of (1.1) are only two points. Therefore, we understand that formal matching matrices can be calculated from the snapping points, say, $P_{j_0}$ and its adjacent points $P_{j_0\pm1}$. As shown above, we could calculate matching matrices from the coordinates of snapping points, so that we should notice that they can be got even without the (formal) solutions. In fact, the snapping points designate the asymptotic character of the differential equation.

**7.5.** Next, we review an equation having a singular point

$$(7.5) \qquad \varepsilon^2\,\frac{d^2y}{dx^2} = \left(x^\nu - \frac{\varepsilon}{x^2}\right)y \quad (\nu \in \mathbf{N};\, 0 < |x| \le x_0)\,,$$

which was studied in the part II of [9]. Apparently this is out of a category of (1.1). The origin $x = 0$ is a turning point and a regular singular point as well. Since $h = 1, m_0 = \nu$ and $m_1 = -2$, its characteristic polygon consists of two segments connecting $P_0 := (0, \nu/2)$ and $P_1 := (1/2, -1)$, and $P_1$ and $R := (1, -1)$. Since the second segment is parallel to the $X$-axis, the characteristic polygon does not have the general property given in [7]. However, the matching matrix connecting the inner and the outer solutions was computed in [9].

We see that two reduced differential equations and their WKB approximations are

$$(\check{O}_0) \qquad \begin{cases} \varepsilon^2 \dfrac{d^2y}{dx^2} = x^\nu y \qquad (K\varepsilon^{1/(\nu+2)} \le |x| \le x_0), \\ \tilde{y}^{\pm}_{out,0}(x,\varepsilon) := \dfrac{1}{x^{\nu/4}} \exp\left( \pm \dfrac{1}{\varepsilon} \dfrac{2}{\nu+2} x^{(\nu+2)/2} \right), \end{cases}$$

$$(\check{I}_1) \qquad \begin{cases} \varepsilon \dfrac{d^2y}{dt^2} = \left( t^\nu - \dfrac{1}{t^2} \right) y \quad (t := x\varepsilon^{-1/(\nu+2)};\ \ 0 < |t| \le K), \\ \tilde{y}^{\pm}_{in,1}(t,\varepsilon) := \dfrac{1}{(t^\nu - t^{-2})^{1/4}} \exp\left( \pm \dfrac{1}{\sqrt{\varepsilon}} \displaystyle\int_1^t \sqrt{t^\nu - t^{-2}} dt \right), \end{cases}$$

where $K$ is a sufficiently large constant. The WKB approximations come from the asymptotic approximations of the solution

$$\frac{1}{x^{\nu/4}} \exp\left( \pm \frac{1}{\varepsilon} \frac{2}{\nu+2} x^{(\nu+2)/2} + \frac{1}{\nu+2} x^{-(\nu+2)/2} \right)$$

of (5.2) in [9] as $\varepsilon x^{-(\nu+2)} \to 0$. Thus, applying the formula (6.6) to $(\check{O}_0)$ and $(\check{I}_1)$, we get

$$M[\check{O}_0, \check{I}_1] \sim \varepsilon^{m_0/4(m_0-m_1)} E = \varepsilon^{\nu/4(\nu+2)} E ,$$

which coincides with $M$ in §6 of [9]. Thus, again we understand that the formula (6.6) can be applied to a differential equation of different type from (1.1).

**7.6.** As the last example, we review another equation with a singular point

$$(7.6) \qquad \varepsilon^{2h} \frac{d^2y}{dx^2} = \left( x^m - \frac{\varepsilon^l}{x^r} \right)^2 y \qquad \left( m, l \in \mathbf{N};\ r \ge 0;\ h > \frac{m+1}{m+r} l;\ 0 < |x| \le x_0 \right),$$

which is a special case ($n = 2,\ p_1 = 0,\ p_2 = 1$) of [10]. This is also apparently out of a category of (1.1). $x = 0$ is a turning point and a singular point as well. Since $(x^m - \varepsilon^l/x^r)^2 = x^{2m} - 2\varepsilon^l x^{m-r} + \varepsilon^{2l} x^{-2r}$, its characteristic polygon consists of two segments connecting $P_0 := (0, m)$ and $P_{2l} := (l, -r)$, and $P_{2l}$ and $R := (h, -1)$. The point $P_l := (l/2, (m-r)/2)$ is on the first segment.

We see that two reduced differential equations and their WKB approximations are

$$(\bar{O}_0)\qquad \begin{cases} \varepsilon^{2h}\,\dfrac{d^2y}{dx^2} = x^{2m}y \quad (K\varepsilon^{l/(m+r)} \le |x| \le x_0) \\ \tilde{y}^{\pm}_{out,0}(x,\varepsilon) := \dfrac{1}{x^{m/2}}\ \exp\left(\pm\dfrac{1}{\varepsilon^h}\,\dfrac{x^{m+1}}{m+1}\right) \end{cases},$$

$$(\bar{I}_1)\qquad \begin{cases} \varepsilon^{2h'}\dfrac{d^2y}{dt^2} = p(t)^2 y \left(t := x\varepsilon^{-l/(m+r)};\ 0 < |t| \le K;\right. \\ \qquad\qquad \left. h' := h - \dfrac{m+1}{m+r}l > 0;\ p(t) := t^m - \dfrac{1}{t^r}\right), \\ \tilde{y}^{\pm}_{in,1}(t,\varepsilon) := \dfrac{1}{p(t)^{1/2}}\ \exp\left(\pm\dfrac{1}{\varepsilon^{h'}}\displaystyle\int_1^t p(t)\,dt\right), \end{cases}$$

where $K$ is a sufficiently large constant.

Since $m_0 = 2m$, $m_1 = m - r$ when $l = 1$, we can apply formula (6.6) to get the matching matrix connecting two sets of solutions $(\bar{O}_0)$ and $(\bar{I}_1)$ such that

$$M[\bar{O}_0,\bar{I}_1] \sim \varepsilon^{m_0/(4(m_0-m_1))}E = \varepsilon^{m/(2(m+r))}E\,.$$

This result coincides with $M$ of (7.4) in [10], because we can see that

$$M \sim \varepsilon^{m/(m+r)\ \mathrm{diag}[\mu_1,\mu_2]} = \varepsilon^{m/(m+r)\ \mathrm{diag}[1/2,1/2]} = \varepsilon^{m/(2(m+r))}E\,.$$

We remark that the characteristic equation is $\lambda^2 - x^{2m} = (\lambda - a_1x^m)(\lambda - a_2x^m) = 0$ and so we can put $a_1 := -1$, $a_2 := 1$ $(a_1 < a_2)$, and $\mu_1 := a_1/(a_1-a_2) = 1/2$, $\mu_2 := a_2/(a_2-a_1) = 1/2$ in the computation of $M$ (cf. (1.4), (1.5), (3.6) in [10]).

When $l > 1$, we cannot apply our formula directly because there are no points between $P_0$ and $P_l$, namely, there exist no points $P_1$, $P_{l\pm1}$ and $P_{2l-1}$ adjacent to $P_0$, $P_l$ and $P_{2l}$, respectively. We should insert, for example, an additional point $P_1 := (1/2, m_1/2)$ $(m_1 := 2m - (m+r)/l)$ adjacent to $P_0$ on the first segment. Then, the formula (6.6) gives us a matching matrix connecting two sets of solutions $(\bar{O}_0)$ and $(\bar{I}_1)$ such that

$$M[\bar{O}_0,\bar{I}_1] \sim \varepsilon^{m_0/(4(m_0-m_1))}E = \varepsilon^{ml/(2(m+r))}E\,,$$

which again coincides with $M$ of (7.4) in [10].

From the above consideration we can see that the matching matrix $M[\bar{O}_0,\bar{I}_1]$ above coincides with one for the differential equation having more terms

$$(7.7)\qquad \varepsilon^{2h}\,\frac{d^2y}{dx^2} = \left(\sum_{j=0}^{2l} b_j\varepsilon^j x^{m_j}\right)y \quad \left(b_j \in \mathbf{C};\ \ m_j := 2m - \frac{m+r}{l}j\right).$$

The equation (7.6) with $r = 0$ (for $n = 2$) was studied in [11] and the corresponding matching matrix coincides with (7.4) in [11].

## 8. Summary

**8.1.** We studied the one-dimensional Schrödinger equation (1.1) which has a turning point at $x = 0$ and secondary turning points at zeros of $Q_{j+1}(t)$ in $(2.7)_{j+1}$. Its characteristic polygon consists of $h + 1$ segments and (1.1) is reduced asymptotically to the $h + 1$ outer differential equations $(2.4)_j$ in the outer domain $(2.5)_j$ and to the $h$ inner differential equations $(2.7)_{j+1}$ in the inner domain $(2.12)_{j+1}$. The equation $(2.4)_h$ has a constant coefficient and its true solution gives the asymptotic information at the turning point. The inner domain $(2.12)_{j+1}$ is adopted as the *true* inner domain instead of the *formal* inner domain $(2.8)_{j+1}$, because there exist common interior points in $(2.5)_j$ and $(2.12)_{j+1}$. The common interior points are necessary for computing matching matrices. The outer and the inner WKB approximations are given by $(4.4)_j$ and $(4.5)_{j+1}$, respectively. Here we specified coefficients of (1.1) like (4.1) to get the exact Stokes curve configurations and the canonical domains for the reduced differential equations. The Stokes curve configurations for several examples are shown in §4. The linear relations between the outer and the inner solutions, called the matching matrices, are got in (5.2) and (5.14). Thus we can know every value of the asymptotic solution of (1.1) in some sector containing the turing point.

For the concrete analysis of the matching matrix we need the canonical domains given in §4, but we show in §6 that the matching matrices can be calculated formally without specifying the coefficients of the equation. The differential equation to be studied is (6.1) and the formal matching matrices are (6.6) and (6.16), which are compared with some known results for the concrete equations (7.1) - (7.7). The matching matrices (5.2) and (5.14) are special cases of (6.6) and (6.16), respectively.

**8.2.** It is well-known that the asymptotic theory of differential equations consists of two aspects.

(i) *Formal Theory*: To determine the form of solutions (the formal solutions),

(ii) *Analytic Theory*: To determine their asymptotic property, that is, to determine in which regions (or sets) they increase or decrease exponentially or in which regions they oscillate.

The WKB approximations and formal matching matrices belong to (i), and the canonical domains belong to (ii). We considered (i), (ii) in §§4,5, and (i) in §§6, 7.

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