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On Totally Real Cubic Orders Whose Unit Groups are of Type $\langle a\theta + b, c\theta + d\rangle$

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1. Introduction

Let $\phi(x)$ be a cubic, monic and irreducible polynomial in x with rational integer coefficients and three real roots. We fix one of these roots and denote it by θ. Set $K = \mathbf{Q}(\theta)$, and let E_K be the unit group of K and E_K^+ the subgroup of E_K which consists of units with norm $+1$. By Dirichlet's unit theorem, E_K^+ is generated by two units and so is $\mathbf{Z}[\theta] \cap E_K^+$. Hereafter we denote the latter by E_θ^+. It is difficult to determine the generators of E_θ^+ even though that problem is important for number theory. In this paper, for given $a, b, c, d \in \mathbf{Z}$, we shall find conditions under which $E_\theta^+ = \langle a\theta + b, c\theta + d\rangle$. As a result, we shall obtain new infinite families of $\mathbf{Z}[\theta]$ with explicit generators of E_θ^+, which will give useful examples for further study.

In 1972, Stender[6] found families of $\phi(x)$ such that $E_\theta^+ = \langle \theta + b, \theta + d\rangle$ for rational integers b, d with $2 \le b \le d - 3$ by using Berwick's theorem[1]. In 1979, Thomas [7] found families of $\phi(x)$ such that $E_\theta^+ = \langle a\theta + 1, \theta + d\rangle$ and $\langle a\theta + 1, c\theta + 1\rangle$ for rational integers a, c, d with $a \ge 4$ and some other conditions by using the continued fraction expansion of a certain conjugate of θ. In 1995, Grundman [3] modified Thomas's technique for determining fundamental systems of units, and determined all a with $|a| > 1$ such that $E_\theta^+ = \langle a\theta + 1, 2\theta + 3\rangle$ for some totally real number θ of degree 3, and found families of $\phi(x)$ for each a. We shall further utilize this method under a more general condition that $a\theta + b, c\theta + d \in E_\theta^+$.

THEOREM 1. *For rational integers a, b, c and d, assume the following conditions*:

1. $|ad - bc| > \max\{|a|, |c|\}, 2 \le |a| < |b|$ *and* $2 \le |c| < |d|$,
2. *there exist rational integers e, f and g such that*

$$b^3 - eab^2 + fa^2b - ga^3 = 1\,, \quad d^3 - ecd^2 + fc^2d - gc^3 = 1\,, \tag{1}$$

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3.

$$\left|\phi'\left(-\frac{b}{a}\right)\right| > \max\left\{\frac{\left|\phi''\left(-\frac{b}{a}\right)\right|}{6a^2|g|} + \left(\frac{1}{3a^2|g|}\right)^2 + \frac{3|g|}{|a|}, \frac{\left|\phi''\left(-\frac{b}{a}\right)\right|}{2} + 1 + 2|a|\right\}, \tag{2}$$

$$\left|\phi'\left(-\frac{d}{c}\right)\right| > \max\left\{\frac{\left|\phi''\left(-\frac{d}{c}\right)\right|}{6c^2|g|} + \left(\frac{1}{3c^2|g|}\right)^2 + \frac{3|g|}{|c|}, \frac{\left|\phi''\left(-\frac{d}{c}\right)\right|}{2} + 1 + 2|c|\right\}, \tag{3}$$

$$\left|e - \frac{d}{c} - 2\frac{b}{a}\right| > 4\max\{|a|, |c|\}, \tag{4}$$

$$\left|e - \frac{b}{a} - 2\frac{d}{c}\right| > 4\max\{|a|, |c|\}, \tag{5}$$

where we put $\phi(x) = x^3 + ex^2 + fx + g$.

Then $\phi(x)$ *is irreducible and has three real roots. Let* θ *be a root of* $\phi(x)$. *Then* E_θ^+ *is generated by* $a\theta + b$ *and* $c\theta + d$.

If $d = 0$, then we can get the following theorem.

THEOREM 2. *For rational integers* a, b *and* c, *assume the following conditions*:

1. $2 \le |a| < |b|$ *and* $|c| = 1$,
2. *there exist rational integers* e *and* f *such that*

$$b^3 - eab^2 + fa^2b + ca^3 = 1, \tag{6}$$

and

$$\pm e + f \ne 1, \tag{7}$$

3.

$$\left|\phi'\left(-\frac{b}{a}\right)\right| > \frac{\left|\phi''\left(-\frac{b}{a}\right)\right|}{2} + 1 + 2|a|, \tag{8}$$

$$\left|e - \frac{b}{a}\right| > 4|a|, \tag{9}$$

and

$$\left|e - 2\frac{b}{a}\right| > \frac{5}{2}, \tag{10}$$

where we put $\phi(x) = x^3 + ex^2 + fx - c$.

Then $\phi(x)$ *is irreducible and has three real roots. Let* θ *be a root of* $\phi(x)$. *Then* E_θ^+ *is generated by* $a\theta + b$ *and* $c\theta$.

THEOREM 3. *For rational integers a, b, c and d, assume that*

$$|ad-bc| > \max\left\{|ac|, \left|\frac{3bd}{a}\right|, \left|\frac{3bd}{c}\right|\right\}, \quad |ac| \geq 2$$

and there exist rational integers e, f and g which satisfy (1). *Then we can explicitly construct infinitely many cubic irreducible polynomials $\phi(x)$ such that $E_\theta^+ = \langle a\theta+b, c\theta+d\rangle$, where θ is a root of $\phi(x)$.*

REMARK 1. When $D := ac(ad-bc) \neq 0$, we see that the simultaneous diophantine equations (1) is solvable if and only if $D\,\mathrm{gcd}(ac, ad+bc) \mid a^3(d^3-1) - c^3(b^3-1)$, $D\,\mathrm{gcd}(ac, bd) \mid a^2b(d^3-1) - c^2d(b^3-1)$ and $D\,\mathrm{gcd}(ac, bd, ad+bc) \mid ab^2(d^3-1) - cd^2(b^3-1)$. Then, the simultaneous congruences

$$D(ad+bc)e \equiv a^3(d^3-1) - c^3(b^3-1)\,,\ Dbde \equiv a^2b(d^3-1) - c^2d(b^3-1) \pmod{Dac}$$

have a solution $e \in \mathbf{Z}$, and we may put

$$f = \frac{a^3(d^3-1) - c^3(b^3-1) - D(ad+bc)e}{Dac}\,,\ g = \frac{a^2b(d^3-1) - c^2d(b^3-1) - Dbde}{Dac}\,.$$

Moreover, all solutions of (1) are given by

$$e + t\frac{ac}{\mathrm{gcd}(ac, bd, ad+bc)}\,, \quad f + t\frac{ad+bc}{\mathrm{gcd}(ac, bd, ad+bc)}\,, \quad g + t\frac{bd}{\mathrm{gcd}(ac, bd, ad+bc)}$$

with $t \in \mathbf{Z}$.

REMARK 2. When $G := \mathrm{gcd}(ac, bd, ad+bc)$, if rational integers e, f and g satisfy (1), then for any rational integer t,

$$e' = e + \frac{ac}{G}t\,, \quad f' = f + \frac{ad+bc}{G}t\,, \quad g' = g + \frac{bd}{G}t$$

also satisfy (1) by Remark 1. For these rational integers, we define $\phi(x) = x^3 + e'x^2 + f'x + g'$. Then we have

$$\left|\phi'\left(-\frac{b}{a}\right)\right| = \left|3\left(-\frac{b}{a}\right)^2 + 2\left(e + \frac{ac}{G}t\right)\left(-\frac{b}{a}\right) + f + \frac{ad+bc}{G}t\right| = \left|\frac{ad-bc}{G}t\right| + O(1)\,,$$

$$\left|\phi''\left(-\frac{b}{a}\right)\right| = \left|6\left(-\frac{b}{a}\right) + 2\left(e + \frac{ac}{G}t\right)\right| = 2\left|\frac{ac}{G}t\right| + O(1)\,,$$

$$\left|\phi'\left(-\frac{d}{c}\right)\right| = \left|3\left(-\frac{d}{c}\right)^2 + 2\left(e + \frac{ac}{G}t\right)\left(-\frac{d}{c}\right) + f + \frac{ad+bc}{G}t\right| = \left|\frac{ad-bc}{G}t\right| + O(1)\,,$$

$$\left|\phi''\left(-\frac{d}{c}\right)\right| = \left|6\left(-\frac{d}{c}\right) + 2\left(e + \frac{ac}{G}t\right)\right| = 2\left|\frac{ac}{G}t\right| + O(1)\,.$$

Hence if $|ad - bc| > \max\left\{|ac|, \left|\frac{3bd}{a}\right|, \left|\frac{3bd}{c}\right|\right\}$, then we can find infinitely many rational integers t for which e', f' and g' satisfy (2)–(5) or (7)–(10). Therefore, we can obtain infinitely many polynomials $\phi(x)$ such that $E_\theta^+ = \langle a\theta + b, c\theta + d\rangle$ (See Examples 1 and 2 below).

REMARK 3. If the discriminant of $\phi(x)$ is square-free, then $\mathbf{Z}[\theta]$ coincides with the ring of integers of K (cf. [2] chap. 4 Corollary 4.4.7) and $E_\theta^+ = E_K^+$. If the discriminant of $\phi(x)$ is perfect square, then $K/\mathbf{Q}$ is a Galois extension.

REMARK 4. Thomas [7] studied on $\phi(x)$ such that $E_\theta^+ = \langle a\theta + 1, \theta + d\rangle$ with some conditions. In other words, he investigated $\langle \theta, a\theta + b\rangle$ for $b \equiv 1 \pmod a$. Therefore Theorem 2 is an extension of Thomas's work (see Example 2 below).

REMARK 5. To prove Theorem 2, we use Theorem T (See section 2), in which the case $e + f = 1$ is excluded. In this case we are not sure whether $E_\theta^+ = \langle a\theta + b, \theta\rangle$ or not. But Thomas [7] gave families of $\phi(x)$ such that $E_\theta^+ = \langle -\theta + 1, \theta\rangle$, which are examples for the case $e + f = 1$.

REMARK 6. Stender [6], Watabe [8] and Minemura [5] studied in the case $|a| = 1, |c| = 1$.

We give examples for $b \not\equiv 1 \pmod a$, in which there has been no example until now. The following is an example of Theorem 1.

EXAMPLE 1. Put $a = 7$, $b = 11$, $c = 7$ and $d = 43$. Then for each $t \in \mathbf{Z}$, the integers $e = 49t + 39$, $f = 378t + 251$ and $g = 473t + 302$ satisfy (1). And if

$$t \neq 0, -1$$

hold, then $\phi(x) = x^3 + ex^2 + fx + g$ is irreducible and has three real roots. Let θ be a root of $\phi(x)$. Then $E_\theta^+ = \langle 7\theta + 11, 7\theta + 43\rangle$ holds.

The following is an example of Theorem 2.

EXAMPLE 2. For $r \neq -1, 0$, put $a = r^2 + r + 1$, $b = (a^2 + a + 1)r$, $c = 1$ and $d = 0$. Then for each $t \in \mathbf{Z}$, the integers $e = r - 1 + at$, $f = -a^2r^2 - a^2 - r^2 + bt$ and $g = -1$ satisfy (6). And if

$$|t - 2r| \geq 5\,, \quad |t - r| \geq 3$$

hold, then $\phi(x) = x^3 + ex^2 + fx - 1$ is irreducible and has three real roots. Let θ be a root of $\phi(x)$. Then $E_\theta^+ = \langle a\theta + b, \theta\rangle$ holds.

2. Preliminaries

Before we prove our theorems, we give some notations which are used throughout this paper. For a cubic irreducible polynomial $\phi(x) = x^3 + ex^2 + fx + g \in \mathbf{Z}[x]$ which has three real roots, set $K = \mathbf{Q}(\theta)$ where θ is one of the three roots of $\phi(x)$. Let E_K be the unit group of

K and E_K^+ the subgroup of E_K which consists of units of norm $+1$ and set $E_\theta^+ = \mathbf{Z}[\theta] \cap E_K^+$. Let $\theta^{(i)}$ $(i = 0, 1, 2)$ be the conjugates of θ over $\mathbf{Q}$. And let

$$\theta^{(0)} > \theta^{(1)} > \theta^{(2)},$$

which is also assumed in Theorem G and Theorem T below which are the bases of our theorems. For $i, i', i'' \in \{0, 1, 2\}$, $i \neq i' \neq i'' \neq i$, $m, n \in \mathbf{Z}$, $m > 0$, $n \geq 0$, let $-\theta^{(i)} = [k_{i,0}, k_{i,1}, \cdots]$ be the continued fraction expansions of $-\theta^{(i)}$ and $\frac{p_{i,n}}{q_{i,n}}$ the nth principal convergents of $-\theta^{(i)}$, and define

$$\begin{aligned}
\lambda_i &:= \frac{1}{|\theta^{(i')} - \theta^{(i'')}|}, \\
\delta_i &:= \lambda_i(\lambda_{i'} + \lambda_{i''}), \\
M_{i,n} &:= \lceil k_{i,n+1} - 2\lambda_i q_{i,n+1} \rceil, \\
N_i &:= \lceil \lambda_i(|\theta^{(i')}| + |\theta^{(i'')}|) \rceil, \\
\eta_{i,m,n} &:= mq_{i,n}\theta^2 + m(q_{i,n}e - p_{i,n})\theta - \left\lfloor \frac{mgq_{i,n}}{\theta^{(i)}} \right\rfloor, \\
C_i &:= \{\, \eta \in \mathbf{Z}[\theta] \ : \ \eta^{(i)} > 1,\ |\eta^{(i')}| < 1 \text{ and } |\eta^{(i'')}| < 1 \},
\end{aligned}$$

and if $M_{i,n} \geq 1$, we define

$$\begin{aligned}
S_{i,n} := \{\, \gamma \in C_i \cap E_K \ &: \ \gamma = (-1)^i(\eta_{i,m,n} + l) \\
&\text{with } 1 \leq m \leq M_{i,n},\ -N_i \leq l < N_i,\ m, l \in \mathbf{Z} \},
\end{aligned}$$

where $\lceil \alpha \rceil$ means the least integer which is greater than or equal to α, and $\lfloor \alpha \rfloor$ means the greatest integer which is less than or equal to α.

The following three theorems are the bases of our theorems.

THEOREM B (Berwick [1]). 1. *There exists a unit in each* C_i $(i = 0, 1, 2)$.

2. *There exists a unit* $\varepsilon_i \in C_i$ *such that* $\varepsilon_i^{(i)} \leq \eta^{(i)}$ *for every unit* $\eta \in C_i$. *Moreover, any two of the three units* $\varepsilon_0, \varepsilon_1, \varepsilon_2$ *form a fundamental system of units for* $\mathbf{Z}[\theta]$.

We call ε_i in Theorem B the fundamental C_i unit.

THEOREM G (Grundman [3]). *Let* $\theta^{(0)} > \theta^{(1)} > \theta^{(2)}$. *Suppose* $\delta_i < \frac{1}{2}$. *If there exists an integer* n_i *such that*

$$k_{i,n_i+1} \leq \frac{1}{2} q_{i,n_i+1} \quad \text{and} \quad S_{i,n_i} \neq \emptyset,$$

then $(-1)^i(\eta_{i,m_i,n_i} + l_i)$ *is the fundamental* C_i *unit, where*

$$\begin{aligned}
m_i &:= \min\{\, m \ : \ (-1)^i(\eta_{i,m,n_i} + l) \in S_{i,n_i} \text{ for some } l \,\}, \\
l_i &:= \min\{\, l \ : \ (-1)^i(\eta_{i,m_i,n_i} + l) \in S_{i,n_i} \,\}.
\end{aligned}$$

REMARK 7. Grundman [3] stated the theorem only for $i = 1, 2$, but the proof still goes through for $i = 0$.

THEOREM T (Thomas [7]). *Let* $\theta^{(0)} > \theta^{(1)} > \theta^{(2)}$. *Suppose* $g = \pm 1$, $(e + f, g) \neq (1, -1)$.

(a) *If* $1 < \theta^{(1)} < \theta^{(0)}$ *and* $(\theta^{(0)} - \theta^{(1)})(1 + g\theta^{(2)}) > 2$, *then* $-g\theta^{-1}$ *is the fundamental* C_2 *unit.*

(b) *If* $\theta^{(2)} < -1,\ 1 < \theta^{(0)}$ *and* $\theta^{(0)} > |\theta^{(2)}|$, *then* $g\theta^{-1}$ *is the fundamental* C_1 *unit.*

3. Proof of Theorems 1, 2 and 3

In this section, we shall prove Theorems 1, 2 and 3. First, we shall show that if the assumptions in Theorem 1 or 2 hold, then $\phi(x)$ is irreducible and has three real roots. We use the following elementary lemma all over our proofs.

LEMMA 1. *For real numbers* α, β *and* γ, *if* $\alpha = \beta + \gamma$ *and* $|\beta| > |\gamma|$, *then* $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta)$.

Now by $\phi\big(-\frac{b}{a}\big) = -\frac{1}{a^3}$, if $|a|$ and $\big|\phi'\big(-\frac{b}{a}\big)\big|$ are sufficiently large, then we have a real root of $\phi(x)$ nearby $-\frac{b}{a}$. Indeed we have the following lemma.

LEMMA 2. *If* $2 \leq |a| < |b|$, (1) *and* (2) *hold, then there exists at least one root of* $\phi(x)$ *in*

$$\left(-\frac{b}{a} - \frac{1}{3a^2|g|}, -\frac{b}{a} + \frac{1}{3a^2|g|}\right).$$

PROOF. Let y be an indeterminate. Then we have

$$\begin{aligned}\phi\left(-\frac{b}{a} + y\right) &= \phi\left(-\frac{b}{a}\right) + \phi'\left(-\frac{b}{a}\right)y + \frac{\phi''\big(-\frac{b}{a}\big)}{2}y^2 + y^3 \\ &= \phi'\left(-\frac{b}{a}\right)y + \frac{\phi''\big(-\frac{b}{a}\big)}{2}y^2 + y^3 - \frac{1}{a^3}.\end{aligned}$$

Let $|y| = \frac{1}{3a^2|g|}$ and let β and γ be the first term and the remains of the above respectively. By (2), we have

$$\begin{aligned}|\beta| - |\gamma| &\geq \frac{1}{3a^2|g|}\left\{\left|\phi'\left(-\frac{b}{a}\right)\right| - \left(\frac{1}{3a^2|g|}\right)\frac{\big|\phi''\big(-\frac{b}{a}\big)\big|}{2} - \left(\frac{1}{3a^2|g|}\right)^2 - \frac{3|g|}{|a|}\right\} \\ &> 0.\end{aligned}$$

Hence by Lemma 1 the signs of $\phi\big(-\frac{b}{a} \pm \frac{1}{3a^2|g|}\big)$ are equal to those of $\pm\phi'\big(-\frac{b}{a}\big)$ respectively.

So we have

$$\phi\Big(-\frac{b}{a}+\frac{1}{3a^2|g|}\Big)\phi\Big(-\frac{b}{a}-\frac{1}{3a^2|g|}\Big)<0\,,$$

and this completes the proof of Lemma 2. $\square$

Hereafter, let θ_1 be a root of $\phi(x)$ which satisfies the condition of Lemma 2, and fix it. Then Lemma 2 means

$$\left|\theta_1+\frac{b}{a}\right|<\frac{1}{3a^2|g|}\,. \tag{11}$$

Similarly, we shall obtain the second real root θ_2 nearby $-\frac{d}{c}$. Indeed, if $2\le|c|<|d|$ and (3) hold, then there exists a real root θ_2 of $\phi(x)$ such that

$$\left|\theta_2+\frac{d}{c}\right|<\frac{1}{3c^2|g|} \tag{12}$$

by Lemma 2. On the other hand, if $d=0$, then we have the following lemma.

LEMMA 3. *If* $2\le|a|<|b|$, (6) *and* (9) *hold, then there exists a real root* θ_2 *of* $\phi(x)$ *such that*

$$|\theta_2|<\frac{1}{4|a|}\,.$$

PROOF. By (6) we have $f=\frac{b}{a}\left(e-\frac{b}{a}\right)+g\frac{a}{b}+\frac{1}{a^2b}$. Therefore we have

$$\begin{aligned}\phi(x)&=x^3+ex^2+\left(\frac{b}{a}\left(e-\frac{b}{a}\right)+g\frac{a}{b}+\frac{1}{a^2b}\right)x+g\\&=\left(e-\frac{b}{a}\right)x\left(x+\frac{b}{a}\right)+x^3+\frac{b}{a}x^2+\left(g\frac{a}{b}+\frac{1}{a^2b}\right)x+g\,.\end{aligned}$$

Hence we have

$$\begin{aligned}\phi\left(\pm\frac{1}{4|a|}\right)=&\pm\left(e-\frac{b}{a}\right)\frac{1}{4|a|}\left(\pm\frac{1}{4|a|}+\frac{b}{a}\right)\\&\pm\frac{1}{64|a^3|}+\frac{b}{16a^3}\pm\left(g\frac{a}{b}+\frac{1}{a^2b}\right)\frac{1}{4|a|}+g\end{aligned}$$

respectively. Let β and γ be the first term and the remains of the right-hand side respectively. By (9), $|g|=1$ and $2\le|a|<|b|$, we have

$$\begin{aligned}|\beta|-|\gamma|&>4|a|\frac{1}{4|a|}\left(-\frac{1}{4|a|}+\frac{|b|}{|a|}\right)-\frac{1}{64|a^3|}-\frac{|b|}{16|a^3|}-\frac{1}{4|b|}-\frac{1}{4|a^3b|}-1\\&>\frac{|b|}{|a|}-\frac{1}{4|a|}-\frac{|b|}{16|a^3|}-\frac{1}{2|a|}-1\\&>0\,.\end{aligned}$$

Hence by Lemma 1 the sign of $\phi\big(\pm\frac{1}{4|a|}\big)$ is equal to that of β. Therefore we have $\phi\big(+\frac{1}{4|a|}\big)\phi\big(-\frac{1}{4|a|}\big) < 0$, and hence we obtain Lemma 3. □

From the above, we can obtain the third real root θ_3 of $\phi(x)$, and fix it. Next we shall show the roots θ_1, θ_2 and θ_3 are sufficiently far from each other.

LEMMA 4. *If the assumptions in Theorem* 1 *hold*, *then we have*

$$\begin{cases} |\theta_1 - \theta_2| > \dfrac{2}{3\min\{|a|, |c|\}}, \\ |\theta_2 - \theta_3| > 4\max\{|a|, |c|\} - \dfrac{1}{2}, \\ |\theta_3 - \theta_1| > 4\max\{|a|, |c|\} - \dfrac{1}{2}. \end{cases}$$

PROOF. By Lemma 2, (12), $|ad - bc| > \max\{|a|, |c|\}$, $2 \le |a|$ and $2 \le |c|$, we have

$$\begin{aligned} |\theta_1 - \theta_2| &= \left|\left(\theta_1 + \frac{b}{a}\right) - \left(\theta_2 + \frac{d}{c}\right) - \left(\frac{b}{a} - \frac{d}{c}\right)\right| \\ &> \left|\frac{b}{a} - \frac{d}{c}\right| - \frac{1}{3a^2|g|} - \frac{1}{3c^2|g|} \\ &> \frac{\max\{|a|, |c|\}}{|ac|} - \frac{1}{3a^2} - \frac{1}{3c^2} \\ &\ge \frac{2}{3\min\{|a|, |c|\}}. \end{aligned}$$

By $\theta_1 + \theta_2 + \theta_3 = -e$ and (5), we have

$$\begin{aligned} |\theta_2 - \theta_3| &= |e + \theta_1 + 2\theta_2| \\ &= \left|e + \left(\theta_1 + \frac{b}{a}\right) + 2\left(\theta_2 + \frac{d}{c}\right) - \frac{b}{a} - 2\frac{d}{c}\right| \\ &> \left|e - \frac{b}{a} - 2\frac{d}{c}\right| - \frac{1}{3a^2|g|} - \frac{2}{3c^2|g|} \\ &> 4\max\{|a|, |c|\} - \frac{1}{2}. \end{aligned}$$

Similarly by (4), we have

$$|\theta_3 - \theta_1| > 4\max\{|a|, |c|\} - \frac{1}{2}.$$

Hence we obtain Lemma 4. □

LEMMA 5. *If the assumptions in Theorem 2 hold, then we have*

$$\begin{cases} |\theta_1 - \theta_2| > 1\,, \\ |\theta_2 - \theta_3| > 4|a| - \dfrac{1}{2}\,, \\ |\theta_3 - \theta_1| > \dfrac{53}{24}\,. \end{cases}$$

PROOF. We have

$$\begin{aligned} |\theta_1 - \theta_2| &= \left|\left(\theta_1 + \frac{b}{a}\right) - \theta_2 - \frac{b}{a}\right| \\ &> \left|-\frac{b}{a}\right| - \frac{1}{3a^2} - \frac{1}{4|a|} \\ &> \frac{|b|-1}{|a|} \\ &\geq 1 \end{aligned}$$

by Lemma 2, Lemma 3 and $2 \leq |a| < |b|$,

$$\begin{aligned} |\theta_2 - \theta_3| &= |e + \theta_1 + 2\theta_2| \\ &= \left|e + \left(\theta_1 + \frac{b}{a}\right) + 2\theta_2 - \frac{b}{a}\right| \\ &> \left|e - \frac{b}{a}\right| - \frac{1}{3a^2} - \frac{1}{2|a|} \\ &> 4|a| - \frac{1}{2} \end{aligned}$$

by $\theta_1 + \theta_2 + \theta_3 = -e$ and (9), and

$$\begin{aligned} |\theta_1 - \theta_3| &= |e + 2\theta_1 + \theta_2| \\ &= \left|e + 2\left(\theta_1 + \frac{b}{a}\right) + \theta_2 - 2\frac{b}{a}\right| \\ &> \left|e - 2\frac{b}{a}\right| - \frac{2}{3a^2} - \frac{1}{4|a|} \\ &> \frac{53}{24} \end{aligned}$$

by (10). Hence we obtain Lemma 5. □

By (11), (12) or Lemma 3, θ_1 and θ_2 are not rational integers. On the other hand, by (1), we have

$$(a\theta_1 + b)(a\theta_2 + b)(a\theta_3 + b) = b^3 - eab^2 + fa^2b - ga^3 = 1\,.$$

If θ_3 is a rational integer, then we have $a\theta_3 + b = \pm 1$. Hence we have

$$\frac{1}{2} \geq \frac{1}{|a|} = \left|\theta_3 + \frac{b}{a}\right| = \left|(\theta_3 - \theta_1) + \left(\theta_1 + \frac{b}{a}\right)\right| > |\theta_3 - \theta_1| - \frac{1}{3a^2|g|},$$

which contradicts to Lemma 4 or 5. Thus θ_3 is not a rational integer as well as θ_1 and θ_2. These imply $\phi(x)$ is irreducible and has three real roots. Therefore $K = \mathrm{Q}(\theta)$ is a totally real cubic field. Hence by (1) we have

$$N_{K/\mathbf{Q}}(a\theta + b) = 1 \quad \text{and} \quad N_{K/\mathbf{Q}}(c\theta + d) = 1,$$

i.e., $a\theta + b, c\theta + d \in E_\theta^+$.

Next we shall show that $a\theta+b$ and $c\theta+d$ generate E_θ^+. First we recall $\theta^{(0)} > \theta^{(1)} > \theta^{(2)}$. Using this, we define integers i, i' and i'' by $\theta_1 = \theta^{(i)}$, $\theta_2 = \theta^{(i')}$ and $\theta_3 = \theta^{(i'')}$ respectively. In order to prove that they generate E_θ^+, we shall show that $(-1)^i(a\theta+b)^{-1}$ is the fundamental C_i unit by using Theorem G. To prove this, at first, we shall determine n_i in Theorem G (see (13) below), next check the conditions in Theorem G (see (14), Lemmas 7 and 9 below), and finally determine m_i, l_i in Theorem G (see Lemma 8 below). If $2 \leq |c| < |d|$, then the above argument implies that $(-1)^{i'}(c\theta + d)^{-1}$ is also the fundamental $C_{i'}$ unit. If $d = 0$, then we shall also get a same result by using Theorem T.

We assume $2 \leq |a| < |b|$, (1) and (2) hold. For i defined above, by Lemma 2, we have $\left|-\theta^{(i)} - \frac{b}{a}\right| < \frac{1}{3a^2|g|} < \frac{1}{2a^2}$. Hence there exists a natural number n_i such that

$$p_{i,n_i} = \mathrm{sgn}(-\theta^{(i)})|b| = \mathrm{sgn}(a)b \quad \text{and} \quad q_{i,n_i} = |a|. \tag{13}$$

by the well known fact on the continued fraction (cf. [4] chap.X Theorem 184). And we have

$$k_{i,n_i+1} < \frac{1}{2}q_{i,n_i+1} \tag{14}$$

by $q_{i,n_i+1} = q_{i,n_i}k_{i,n_i+1} + q_{i,n_i-1}$ and $q_{i,n_i} = |a| \geq 2$.

LEMMA 6. *If the assumptions in Theorem 1 or 2 hold, then we have*

$$k_{i,n_i+1} > 3|a|.$$

PROOF. Note that the minimal polynomial of $-\theta^{(i)}$ is $-\phi(-x)$. By (11), Lemma 4 or 5, $-\phi(-x)$ is a monotone function between $-\theta^{(i)}$ and $\frac{b}{a}$, where we use the following elementary fact: *if $u < v$ are two consecutive real roots of an equation of degree 3 with real coefficients and w is the extreme point between them, then we have*

$$\frac{2u + v}{3} \leq w \leq \frac{u + 2v}{3}.$$

Hence we have

$$
\begin{aligned}
q_{i,n_i}p_{i,n_i-1}-p_{i,n_i}q_{i,n_i-1} &= |a|p_{i,n_i-1}-\operatorname{sgn}(a)bq_{i,n_i-1}\\
&= \operatorname{sgn}\left(-\theta^{(i)}-\frac{b}{a}\right)\\
&= \operatorname{sgn}\left(\phi\left(-\frac{b}{a}\right)\right)\operatorname{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right)\\
&= \operatorname{sgn}(-a)\operatorname{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right). \qquad (15)
\end{aligned}
$$

We put $S=\left|a\phi'\left(-\frac{b}{a}\right)\right|-\frac{3q_{i,n_i-1}}{|a|}$. Then S is a rational integer. Indeed, by $|a|\phi'\left(-\frac{b}{a}\right)=|a|\left(3\frac{b^2}{a^2}-2e\frac{b}{a}+f\right)\equiv\frac{3b^2}{|a|} \pmod 1$, we have

$$
\begin{aligned}
S &= \operatorname{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right)|a|\phi'\left(-\frac{b}{a}\right)-\frac{3q_{i,n_i-1}}{|a|}\\
&\equiv \operatorname{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right)\frac{3b^2}{|a|}-\frac{3q_{i,n_i-1}}{|a|} \pmod 1\\
&\equiv 3\frac{\operatorname{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)b^2-q_{i,n_i-1}}{|a|} \pmod 1.
\end{aligned}
$$

Hence it is sufficient to show $\operatorname{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)b^2\equiv q_{i,n_i-1} \pmod a$. This is equivalent to $q_{i,n_i-1}b\equiv\operatorname{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big) \pmod a$ by (1), and holds by (15). Therefore S is a rational integer. Moreover by (2), we have $S>3|a|$. By (15), the following holds for an indeterminate T:

$$
\begin{aligned}
-\frac{\operatorname{sgn}(a)bT+p_{i,n_i-1}}{|a|T+q_{i,n_i-1}} &= -\frac{b}{a}-\frac{|a|p_{i,n_i-1}-\operatorname{sgn}(a)bq_{i,n_i-1}}{|a|(|a|T+q_{i,n_i-1})}\\
&= -\frac{b}{a}+\frac{\operatorname{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right)}{a(|a|T+q_{i,n_i-1})}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&-\phi\left(-\frac{\operatorname{sgn}(a)bT+p_{i,n_i-1}}{|a|T+q_{i,n_i-1}}\right)\\
&\quad= -\phi\left(-\frac{b}{a}\right)-\phi'\left(-\frac{b}{a}\right)\left(\frac{\operatorname{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)}{a(|a|T+q_{i,n_i-1})}\right)-\frac{\phi''\left(-\frac{b}{a}\right)}{2}\left(\frac{\operatorname{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)}{a(|a|T+q_{i,n_i-1})}\right)^2\\
&\qquad-\left(\frac{\operatorname{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)}{a(|a|T+q_{i,n_i-1})}\right)^3\\
&\quad= \frac{1}{a^3(|a|T+q_{i,n_i-1})^2}\left\{(|a|T+q_{i,n_i-1})^2-a^2\left|\phi'\left(-\frac{b}{a}\right)\right|(|a|T+q_{i,n_i-1})\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad - \frac{a\phi''\big(-\frac{b}{a}\big)}{2} - \frac{\mathrm{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)}{|a|T + q_{i,n_i-1}}\Bigg\} \\
&= \frac{1}{a^3(|a|(S+\tau)+q_{i,n_i-1})^2}\Bigg\{(|a|\tau - 2q_{i,n_i-1})(|a|(S+\tau)+q_{i,n_i-1}) \\
&\qquad\qquad - \frac{a\phi''\big(-\frac{b}{a}\big)}{2} - \frac{\mathrm{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right)}{|a|(S+\tau)+q_{i,n_i-1}}\Bigg\}
\end{aligned}
$$

where we put $T = S + \tau$. Now let τ be either 0 or 2 and put

$$
\beta = (|a|\tau - 2q_{i,n_i-1})(|a|(S+\tau)+q_{i,n_i-1}) \quad \text{and} \quad \gamma = -\frac{a\phi''\left(-\frac{b}{a}\right)}{2} - \frac{\mathrm{sgn}\big(\phi'\big(-\frac{b}{a}\big)\big)}{|a|(S+\tau)+q_{i,n_i-1}}\,.
$$

By $||a|\tau - 2q_{i,n_i-1}| > 1$ and (2), we have

$$
|\beta| > |a|S > |a|\left(\left|a\phi'\left(-\frac{b}{a}\right)\right| - 3\right) > |a|\left|\phi'\left(-\frac{b}{a}\right)\right|\,,
$$

hence we have

$$
|\beta| - |\gamma| > |a|\left(\left|\phi'\left(-\frac{b}{a}\right)\right| - \frac{\left|\phi''\left(-\frac{b}{a}\right)\right|}{2} - 1\right) > 0\,.
$$

By Lemma 1, we have

$$
\begin{aligned}
\mathrm{sgn}\left(-\phi\left(-\frac{\mathrm{sgn}(a)b(S+\tau)+p_{i,n_i-1}}{|a|(S+\tau)+q_{i,n_i-1}}\right)\right) &= \mathrm{sgn}\left(a(|a|\tau - 2q_{i,n_i-1})\right) \\
&= \begin{cases} \mathrm{sgn}(a) & \text{if } \tau = 2\,, \\ -\mathrm{sgn}(a) & \text{if } \tau = 0\,. \end{cases}
\end{aligned}
$$

Thus we have

$$
\phi\left(-\frac{\mathrm{sgn}(a)bS + p_{i,n_i-1}}{|a|S + q_{i,n_i-1}}\right)\phi\left(-\frac{\mathrm{sgn}(a)b(S+2) + p_{i,n_i-1}}{|a|(S+2) + q_{i,n_i-1}}\right) < 0\,.
$$

Hence $-\phi(-x)$ has a root between $\frac{\mathrm{sgn}(a)bS+p_{i,n_i-1}}{|a|S+q_{i,n_i-1}}$ and $\frac{\mathrm{sgn}(a)b(S+2)+p_{i,n_i-1}}{|a|(S+2)+q_{i,n_i-1}}$. It coincides with $-\theta^{(i)}$ by (11), Lemma 4 or 5. This means $S + 2 \geq k_{i,n_i+1} \geq S$. Hence we have $k_{i,n_i+1} > 3|a|$. □

LEMMA 7. *If the assumptions in Theorem* 1 *or* 2 *hold, then we have*

$$
(-1)^i(a\theta + b)^{-1} \in S_{i,n_i}\,.
$$

PROOF. It is sufficient to show that $(-1)^i(a\theta+b)^{-1} \in C_i$ and it can be expressed as $(-1)^i(\eta_{i,|a|,n_i} + l)$ such that $|a| < M_{i,n_i}$ and $-N_i \leq l < N_i$. By the proof of Lemma 6 and the definition of i, we have

$$\operatorname{sgn}(a\theta^{(i)}+b)=\operatorname{sgn}\left(\phi'\left(-\frac{b}{a}\right)\right)=(-1)^i\,.$$

Hence by (11), we have $0<(-1)^i(a\theta^{(i)}+b)<1$, and by Lemma 4 or 5, we have $|a\theta^{(i')}+b|>1$, $|a\theta^{(i'')}+b|>1$, i.e., $(-1)^i(a\theta+b)^{-1}\in C_i$. Next, we shall show $|a|<M_{i,n_i}$. We have $q_{i,n_i+1}=k_{i,n_i+1}|a|+q_{i,n_i-1}<|a|(k_{i,n_i+1}+1)$ by (13) and $q_{i,n_i-1}<|a|$, and $\lambda_i=\frac{1}{|\theta^{(i')}-\theta^{(i'')}|}=\frac{1}{|\theta_2-\theta_3|}<\frac{1}{4\max\{|a|,|c|\}-\frac{1}{2}}<\frac{2}{7|a|}$ by Lemma 4 or 5. Hence we have

$$\begin{aligned}M_{i,n_i}&=\lceil k_{i,n_i+1}-2\lambda_i q_{i,n_i+1}\rceil\\&\geq\left\lceil k_{i,n_i+1}-\frac{4}{7}(k_{i,n_i+1}+1)\right\rceil\\&=\left\lceil\frac{3k_{i,n_i+1}-4}{7}\right\rceil\\&\geq\left\lceil\frac{9|a|-1}{7}\right\rceil\\&>|a|\end{aligned}$$

by Lemma 6. Finally, we shall show $-N_i\leq l<N_i$. By elementary calculation and (1), we have $(a\theta+b)^{-1}=a^2\theta^2+(a^2e-ab)\theta+\frac{a^3g+1}{b}$. On the other hand, by (13) we have

$$\begin{aligned}\eta_{i,|a|,n_i}&=|a|q_{i,n_i}\theta^2+|a|(q_{i,n_i}e-p_{i,n_i})\theta-\left\lfloor\frac{|a|gq_{i,n_i}}{\theta^{(i)}}\right\rfloor\\&=a^2\theta^2+(a^2e-ab)\theta-\left\lfloor\frac{ga^2}{\theta^{(i)}}\right\rfloor.\end{aligned}$$

Hence we have

$$\begin{aligned}l&=(a\theta+b)^{-1}-\eta_{i,|a|,n_i}\\&=\frac{a^3g+1}{b}+\left\lfloor\frac{a^2g}{\theta^{(i)}}\right\rfloor\\&=\left\lfloor a^2g\left(\frac{a}{b}+\frac{1}{\theta^{(i)}}\right)+\frac{1}{b}\right\rfloor.\end{aligned}$$

Now by (11), we have

$$\left|a^2g\left(\frac{a}{b}+\frac{1}{\theta^{(i)}}\right)\right|=|a^2g|\frac{|\frac{b}{a}+\theta^{(i)}|}{|\frac{b}{a}||\theta^{(i)}|}<\frac{1}{2}\,.$$

Hence by $\left|\frac{1}{b}\right|<\frac{1}{2}$, we have

$$l\;=\;0\quad\text{or}\quad-1\,.$$

By the definition of N_i, we have $1\leq N_i$. Hence we have $-N_i\leq l<N_i$. This completes the

proof of Lemma 7. □

Let us determine m_i and l_i in Theorem G.

LEMMA 8. *If the assumptions in Theorem 1 or 2 hold, then we have*

$$m_i = |a| \quad \text{and} \quad l_i = (a\theta + b)^{-1} - \eta_{i,|a|,n_i}\,.$$

PROOF. By Lemma 7, $m = |a|$ and $l = (a\theta+b)^{-1} - \eta_{i,|a|,n_i}$ imply $(-1)^i(\eta_{i,m,n_i}+l) \in S_{i,n_i}$. Hence it is sufficient to prove that there exists no other pair (m, l) with $1 \le m \le |a|$ such that $(-1)^i(\eta_{i,m,n_i} + l) \in S_{i,n_i}$. By (13), for any m and l we have

$$\begin{aligned}
&(a\theta + b)(\eta_{i,m,n_i} + l)\\
&\quad = (a\theta + b)\left(m|a|\theta^2 + m(|a|e - \operatorname{sgn}(a)b)\theta - \left\lfloor \frac{mg|a|}{\theta^{(i)}} \right\rfloor + l\right)\\
&\quad = \left(m|a|\frac{-ga^3-1}{ab} + a\left(-\left\lfloor \frac{mg|a|}{\theta^{(i)}} \right\rfloor + l\right)\right)\theta - gma|a| + b\left(-\left\lfloor \frac{mg|a|}{\theta^{(i)}} \right\rfloor + l\right)\\
&\quad = A\theta + \frac{m}{|a|} + \frac{bA}{a}
\end{aligned}$$

where $A = m|a|\frac{-ga^3-1}{ab} + a\big(-\big\lfloor \frac{mg|a|}{\theta^{(i)}} \big\rfloor + l\big)$. We note that $A \in \mathbf{Z}$ by (1). If $A \neq 0$, then we have

$$N_{K/\mathbf{Q}}\left(A\theta + \frac{m}{|a|} + \frac{bA}{a}\right) = A^3 N_{K/\mathbf{Q}}\left(\theta - \left(-\frac{b}{a} - \frac{m}{A|a|}\right)\right) = -A^3\phi\left(-\frac{b}{a} - \frac{m}{A|a|}\right),$$

and hence we have

$$A^3 + m|a|a\phi'\left(-\frac{b}{a}\right)A^2 - \frac{m^2a}{2}\phi''\left(-\frac{b}{a}\right)A + \frac{a}{|a|}m^3 - \Delta_N a^3 = 0\,, \tag{16}$$

where $\Delta_N = N_{K/\mathbf{Q}}\big(A\theta + \frac{m}{|a|} + \frac{bA}{a}\big)$. This also holds for $A = 0$. If $\eta_{i,m,n_i} + l$ is a unit, then $A\theta + \frac{m}{|a|} + \frac{bA}{a}$ is also a unit because $a\theta + b$ is a unit. So we set $\Delta_N = \pm 1$ and regard the left-hand side of (16) as a polynomial in A, and denote it by $\psi(A)$. To prove Lemma 8, we may show that there exist no integral roots of $\psi(A)$ with $1 \le m \le |a|$ for which $(-1)^i(\eta_{i,m,n_i} + l) \in C_i$ other than $A = 0$ with $m = |a|$ and $\Delta_N = 1$. For that, we are going to see

$$\psi(1)\psi(-1) > 0\,, \quad |\psi(\pm 1)| > |\psi(0)| \tag{17}$$

and

$$\psi\left(-m|a|a\phi'\left(-\frac{b}{a}\right) + 1\right)\psi\left(-m|a|a\phi'\left(-\frac{b}{a}\right) - 1\right) < 0\,. \tag{18}$$

If (17) holds, then $\psi(A) = 0$ has only one root out of $(-1, 1)$. Moreover, if (18) holds, then the root is in

$$\left(-m|a|a\phi'\left(-\frac{b}{a}\right) - 1, -m|a|a\phi'\left(-\frac{b}{a}\right) + 1\right).$$

If $\phi''\left(-\frac{b}{a}\right) \neq 0$, then we have $\left|\phi''\left(-\frac{b}{a}\right)\right| = \frac{2}{|a|}|ae - 3b| \geq \frac{2}{|a|}$, and hence

$$\begin{aligned}\left|\psi\left(-m|a|a\phi'\left(-\frac{b}{a}\right)\right)\right| &= \left|\frac{1}{2}m^3|a|^3\phi'\left(-\frac{b}{a}\right)\phi''\left(-\frac{b}{a}\right) + \frac{a}{|a|}m^3 - \Delta_N a^3\right| \\ &\geq \left|a^2\phi'\left(-\frac{b}{a}\right)\right| - 2|a|^3 \\ &> 0\end{aligned}$$

by (2). Therefore the root is not an integer because $-m|a|a\phi'\left(-\frac{b}{a}\right)$ is a unique integer in the above interval. If $\phi''\left(-\frac{b}{a}\right) = 0$ and $\psi\left(-m|a|a\phi'\left(-\frac{b}{a}\right)\right) = 0$, then we have $e = 3\frac{b}{a}$, $m = |a|$ and $\Delta_N = 1$. Hence by (1) we have

$$-m|a|a\phi'\left(-\frac{b}{a}\right) = -a^3\left(-\frac{3b^2}{a^2} + f\right) = a\left(b^2 - \frac{ga^3 + 1}{b}\right).$$

On the other hand by the definition of A, we have

$$A = a\left(-\frac{ga^3 + 1}{b} - \left\lfloor \frac{ga^2}{\theta^{(i)}} \right\rfloor + l\right).$$

Hence l must be equal to $b^2 + \left\lfloor \frac{ga^2}{\theta^{(i)}} \right\rfloor$. Then we have $\eta_{i,|a|,n_i} + l = (a\theta + b)^2$. By Lemma 2, we have $|a\theta^{(i)} + b| < 1$, and hence $(-1)^i(\eta_{i,|a|,n_i} + l) \notin C_i$. Hence there exist no integral roots of $\psi(A)$ with $1 \leq m \leq |a|$ for which $(-1)^i(\eta_{i,m,n_i} + l) \in C_i$ other than $A = 0$. Therefore Lemma 8 holds. Now we shall show (17) and (18). Let δ be either 0 or 1. We have

$$\begin{aligned}&\psi\left(-m|a|a\phi'\left(-\frac{b}{a}\right)\delta \pm 1\right) \\ &= \left(-m|a|a\phi'\left(-\frac{b}{a}\right)\delta \pm 1\right)\left\{\pm(-1)^\delta m|a|a\phi'\left(-\frac{b}{a}\right) - \frac{m^2a}{2}\phi''\left(-\frac{b}{a}\right) + 1\right\} + \psi(0).\end{aligned}$$

Put

$$\beta = \pm(-1)^\delta m|a|a\phi'\left(-\frac{b}{a}\right) \quad \text{and} \quad \gamma = -\frac{m^2a}{2}\phi''\left(-\frac{b}{a}\right) + 1 + \psi(0).$$

By (2) and $1 \leq m \leq |a|$, we have

$$\begin{aligned}|\beta| - |\gamma| - |\psi(0)| &> ma^2\left(\left|\phi'\left(-\frac{b}{a}\right)\right| - \frac{m}{2|a|}\left|\phi''\left(-\frac{b}{a}\right)\right| - \frac{1}{ma^2} - \frac{2m^2}{a^2} - \frac{2|a|}{m}\right) \\ &> 0.\end{aligned}$$

Hence by $\left|-m|a|a\phi'\left(-\frac{b}{a}\right)\delta \pm 1\right| \geq 1$ and Lemma 1, we have $|\psi(\pm 1)| > |\psi(0)|$ and

$$\begin{aligned}
&\operatorname{sgn}\left(\psi\left(-m|a|a\phi'\left(-\frac{b}{a}\right)\delta \pm 1\right)\right)\\
&\quad = \operatorname{sgn}\left\{\left(-m|a|a\phi'\left(-\frac{b}{a}\right)\delta \pm 1\right)\left(\pm(-1)^{\delta} m|a|a\phi'\left(-\frac{b}{a}\right)\right)\right\}\\
&\quad = \begin{cases} \pm 1 & \text{if } \delta = 1\,,\\ \operatorname{sgn}\left(a\phi'\left(-\frac{b}{a}\right)\right) & \text{if } \delta = 0\,. \end{cases}
\end{aligned}$$

These mean that (17) and (18) hold. This completes the proof of Lemma 8. □

LEMMA 9. *If the assumptions in Theorem 1 or 2 hold, then we have* $\delta_i < \frac{1}{2}$.

PROOF. By Lemma 4 or 5 and $|a| \geq 2$, we have

$$\begin{aligned}
\delta_i &= \frac{1}{|\theta^{(i')} - \theta^{(i'')}|}\left(\frac{1}{|\theta^{(i)} - \theta^{(i')}|} + \frac{1}{|\theta^{(i)} - \theta^{(i'')}|}\right)\\
&= \frac{1}{|\theta_2 - \theta_3|}\left(\frac{1}{|\theta_1 - \theta_2|} + \frac{1}{|\theta_1 - \theta_3|}\right)\\
&< \begin{cases} \dfrac{1}{4\max\{|a|, |c|\} - \frac{1}{2}}\left(\dfrac{3}{2}\min\{|a|, |c|\} + \dfrac{1}{4\max\{|a|, |c|\} - \frac{1}{2}}\right) & \text{if } 2 \leq |c| < |d|\\ \dfrac{1}{4|a| - \frac{1}{2}}\left(1 + \dfrac{24}{53}\right) & \text{if } |c| = 1, d = 0 \end{cases}\\
&< \frac{1}{2}\,.
\end{aligned}$$

□

Hence by (14), Lemmas 7,8,9 and Theorem G, $(-1)^i(a\theta + b)^{-1}$ is the fundamental C_i unit. On the other hand, if $2 \leq |c| < |d|$, then c and d satisfy the same conditions with respect to a and b; therefore $(-1)^{i'}(c\theta + d)^{-1}$ is also the fundamental $C_{i'}$ unit. Hence, by Theorem B, we have $E_\theta^+ = \langle a\theta + b, c\theta + d\rangle$ and this completes the proof of Theorem 1. Finally, we shall show that $(-1)^{i'}(c\theta)^{-1} = -(-1)^{i'} g\theta^{-1}$ is the fundamental $C_{i'}$ unit if $d = 0$. If a, b, c, d and $\phi(x)$ satisfy the assumptions in Theorem 2, then so do $-a, b, -c, d$ and $-\phi(-x)$, and the last polynomial has three real roots $-\theta^{(0)} < -\theta^{(1)} < -\theta^{(2)}$. Hence we may assume that $|\theta^{(2)}| < \theta^{(0)}$ and $1 < \theta^{(0)}$ without loss of generality. Now we use Theorem T to determine the fundamental $C_{i'}$ unit. If $1 < \theta^{(1)}$, then $i' = 2$, i.e. $\theta^{(2)} = \theta^{(i')} = \theta_2$. Hence, by Lemma 5, we have

$$\begin{aligned}
(\theta^{(0)} - \theta^{(1)})(1 + g\theta^{(2)}) &= |\theta_3 - \theta_1|(1 + g\theta_2)\\
&> \frac{53}{24}\left(1 - \frac{1}{4|a|}\right)\\
&> 2\,.
\end{aligned}$$

Hence $-g\theta^{-1}$ is the fundamental C_2 unit. Next suppose $\theta^{(1)} \leq 1$. By Lemmas 2 and 5, we have $|\theta_1| > 1$ and

$$\begin{aligned} |\theta_3| &= |-e-\theta_1-\theta_2| \\ &> \left|e-\frac{b}{a}\right| - \frac{1}{3a^2} - \frac{1}{4|a|} \\ &> 1\,. \end{aligned}$$

Therefore the absolute values of two of three roots : $\theta^{(2)} < \theta^{(1)} < \theta^{(0)}$ are greater than 1. Hence we have $\theta^{(2)} < -1$ and $i' = 1$, i.e., $\theta^{(1)} = \theta^{(i')} = \theta_2$. By Theorem B and Theorem T, we obtain $E_\theta^+ = \langle a\theta + b, c\theta\rangle$. This completes the proof of Theorem 2.

In the end we shall prove Theorem 3. By Remark 2, we can construct infinitely many polynomials which satisfy (1)–(5) or (6)–(10) using a polynomial $\phi(x)$ which satisfies (1). Let $\Phi(x)$ be a cubic monic polynomial in x. Then the following two statements are equivalent:

1. $\Phi(x)$ satisfies (1),
2. $\Phi\big(-\frac{b}{a}\big) = \big(-\frac{1}{a}\big)^3, \Phi\big(-\frac{d}{c}\big) = \big(-\frac{1}{c}\big)^3$.

Now for a rational integer n, put $A = a, B = an + b, C = c, D = cn + d$ and $\Phi(x) = \phi(x+n)$. Then $\Phi(x)$ is a cubic monic polynomial in x and satisfies the second condition of the above for A, B, C, D. And let θ be a root of $\phi(x)$ and put $\Theta = \theta - n$. Then Θ is a root of $\Phi(x)$ and $A\Theta + B = a\theta + b, C\Theta + D = c\theta + d$. Hence we may assume $|a| < |b|$ and $|c| < |d|$ without loss of generality. This completes the proof of Theorem 3 for $|a| \geq 2, |c| \geq 2$. Next suppose $|c| = 1$ and put $n = -cd$. Then $D = 0$ and $|B| = |ad - bc|$. Hence if $|c| = 1$, we may assume $d = 0$ without loss of generality. Suppose $d = 0$ and put $A = bc, B = ac, C = c, D = 0$ and $\Phi(x) = -c\phi\big(\frac{1}{x}\big)x^3$. Then $\Phi(x)$ satisfies the second condition of the above for A, B, C, D and $\Theta = \frac{1}{\theta}$ is a root of $\Phi(x)$. Furthermore we have

$$\begin{aligned} \langle A\Theta + B, C\Theta + D\rangle &= \left\langle bc\frac{1}{\theta} + ac, c\frac{1}{\theta}\right\rangle \\ &= \langle a\theta + b, c\theta\rangle\ , \end{aligned}$$

$$|A| \lesseqgtr |B| \Leftrightarrow |a| \gtreqless |b|\,.$$

Hence if $|c| = 1$, we may consider $a, -acd + b$ (we again note that its absolute value is equal to $|ad - bc|$),c, 0 instead of a, b, c, d and assume $|a| < |ad - bc|$ without loss of generality. This completes the proof of Theorem 3 for $|a| \geq 2$ and $|c| = 1$.

References

[1] W. E. H. BERWICK, Algebraic number fields with two independent units, Proc. London Math. Soc. **34** (1932), 360–378.

[2] H. COHEN, *A Course in Computational Algebraic Number Theory*, *Second Corrected Printing*, GTM **138** (1995), Springer.

[3] H. G. GRUNDMAN, Systems of fundamental units in cubic orders, J. Number Theory **50** (1995), 119–127.
[4] G. H. HARDY and E. M. WRIGHT, *An introduction to the theory of numbers*, *Fifth edition*, Oxford Science Publications (1979).
[5] K. MINEMURA, On totally real cubic fields whose unit groups are of type $\{\theta + r, \theta + s\}$, Proc. Japan Acad. **74**A (1998), 156–159.
[6] H. J. STENDER, Einheiten für eine allgemeine Klasse total reeller algebraischer Zahlkörper, J. reine angew. Math. **257** (1972), 151–178.
[7] E. THOMAS, Fundamental units for orders in certain cubic number fields, J. reine angew. Math. **310** (1979), 33–55.
[8] M. WATABE, On certain cubic fields I, III, VI, Proc. Japan Acad. **59**A (1983), 66–69, 260–262; **60**A (1984), 331–332.

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