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On Integral Geometry in the Four Dimensional Complex Projective Space

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1. Introduction and result

The name "Integral Geometry" was introduced by W. Blaschke in his book "Vorlesungen über Integralgeometrie" and later elaborated by him and his colleagues in their subsequent papers in the 30's. Integral geometry treats integrations of geometric invariants of geometric objects such as points, lines, submanifolds and elements of transformation groups.

Let G be a Lie group and K a closed subgroup of G. If M and N are submanifolds of the Riemannian homogeneous space G/K. Then one of main topics in the present work will compute the following integral

$$\int_G \mathrm{vol}(M \cap gN) d\mu(g) .$$

The Poincaré formula means equalities which represent the above integral by some geometric invariants of submanifolds M and N of G/K. For example, in the case that G is the group of isometries of Euclidean space $\mathbf{R}^n$, and M and N are submanifolds of $\mathbf{R}^n$; then the results of above integral lead to remarkable integral formulas by Poincaré, Crofton and other integral geometers. When G is the unitary group $U(n+1)$ acting on complex projective space $\mathbf{C}P^n$, M and N are complex submanifolds of $\mathbf{C}P^n$; then the evaluation of above integral leads to the results obtained by L. A. Santaló [8] and R. Howard [4]. In the same case, if M is a totally real submanifold and N a complex one, and M, N are totally real submanifolds, then the evaluation of above integral gives the results of R. Howard [4]. The present author and H. Tasaki [5], [6] gave the Poincaré formulas of real surfaces and complex hypersurfaces of $\mathbf{C}P^n$, and of two real surfaces of $\mathbf{C}P^2$ using the Kähler angle.

Recently, H. Tasaki [10] generalized the notion of the Kähler angle. Using this generalized Kähler angle, he obtained the Poincaré formula (see Section 3) of general submanifolds, which are neither complex nor totally real submanifolds of $\mathbf{C}P^n$. Although this formula holds under the general situation, it is difficult to give an explicit description through the concrete

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computation using the generalized Kähler angle of submanifolds, which is said to be multiple Kähler angle. In the present paper, we attempt to explicitly describe this formula.

Most of the fundamental definitions, theorems and properties, which will be necessary later, are listed in Sections 2 and 3. We shall review the multiple Kähler angle of a real vector subspace of a complex vector space and its properties in Section 2, and the Poincaré formulas obtained by R. Howard and H. Tasaki in Section 3.

In section 4 we give the Poincaré formula for any real 4-dimensional real submanifold and any complex 2-dimensional complex submanifold of $\mathbf{C}P^4$. More specifically,

THEOREM 1.1. *Let M be a real 4-dimensional submanifold and N a complex 2-dimensional submanifold of $\mathbf{C}P^4$. Then we have*

$$\int_{U(5)} \sharp(M \cap gN) d\mu(g) = \frac{\mathrm{vol}(U(5))\mathrm{vol}(N)}{\mathrm{vol}(\mathbf{C}P^2)^2} \cdot \int_M \left(\frac{1}{4}(1+\cos^2(\theta_1)_x)(1+\cos^2(\theta_2)_x) + \frac{1}{8}\sin^2(\theta_1)_x \sin^2(\theta_2)_x \right) d\mu(x)\,,$$

where $(\theta_1)_x, (\theta_2)_x$ is the multiple Kähler angle of M at x.

On the other hand, Lê Hông Vân proved the following:

THEOREM 1.2 ([7]). *For a real 4-dimensional submanifold M of $\mathbf{C}P^4$, we have*

$$\int_{U(5)} \sharp(M \cap g\mathbf{C}P^2) d\mu(g) \leq \frac{\mathrm{vol}(U(5))}{\mathrm{vol}(\mathbf{C}P^2)} \mathrm{vol}(M)\,.$$

Moreover, the inequality becomes an equality if and only if M is a complex submanifold.

This inequality immediately follows from our theorem.

2. The multiple Kähler angle

In this section, we shall study the multiple Kähler angle generalized by H. Tasaki [10] and its properties.

Let $\mathbf{C}^n$ be an n-dimensional complex vector space with the standard real inner product $\langle \cdot, \cdot \rangle$ and almost complex structure J. The natural action of the unitary group $U(n)$ on $\mathbf{C}^n$ induces its action on the Grassmann manifold $G^{\mathbf{R}}_{2k}(\mathbf{C}^n)$ that consists of real $2k$-dimensional subspaces in $\mathbf{C}^n$. We defined the multiple Kähler angle using the standard Kähler form ω on $\mathbf{C}^n$ defined by $\omega(u, v) = \langle Ju, v \rangle$ for u, v in $\mathbf{C}^n$.

Let V be a real $2k$-dimensional vector subspace in $\mathbf{C}^n$ with $2k \leq n$. Then we take a canonical form of $\omega|_V$ as an alternating 2-form, that is, we can take an orthonormal basis $\alpha^1, \cdots, \alpha^{2k}$ of the dual space V^* which satisfies

$$\omega|_V = \sum_{i=1}^{k} \cos\theta_i \alpha^{2i-1} \wedge \alpha^{2i}\,, \quad 0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \pi/2\,.$$

We set

$$\theta_V = (\theta_1 \cdots, \theta_k) .$$

We call θ_V the **multiple Kähler angle** of V. We here remark that the action of $U(n)$ preserves the multiple Kähler angle of V. If $k = 1$ then the multiple Kähler angle is nothing but the Kähler angle (See [5] for definition). In the case where $\theta_1 = \cdots = \theta_k = \text{constant} = \theta$, the multiple Kähler angle was introduced as the **s**lant angle by B.-Y. Chen and Y. Tazawa [1], [2]. In particular, V is a complex k-dimensional vector subspace if and only if $\theta_V = (0, \cdots, 0)$, that is, the slant angle $\theta = 0$. And V is $2k$-dimensional totally real vector subspace if and only if the slant angle $\theta = \pi/2$.

If $n < 2k < 2n - 1$ then we shall define the multiple Kähler angle of real $2k$-dimensional vector subspace in $\mathbf{C}^n$ as that of its orthogonal complement $V^\perp$. Namely $\theta_V = \theta_{V^\perp}$.

The linear isotropy action of the complex projective space

$$\mathbf{C}P^n = U(n+1)/(U(1) \times U(n))$$

on the tangent space at the origin is equivalent to the action of $U(1) \times U(n)$ on $\mathbf{C}^n$ defined by

$$(z, A)v = zvA^*$$

for $(z, A) \in U(1) \times U(n)$ and $v \in \mathbf{C}^n$.

LEMMA 2.1 ([10]). *Let $G_{2k,\theta}$ be the set of real $2k$-dimensional vector subspaces with multiple Kähler angle $\theta = (\theta_1, \cdots, \theta_k)$ of $\mathbf{C}^n$. Then, $U(n)$ acts transitively on $G_{2k,\theta}$. Moreover, we put*

$$V_\theta^{2k} = \sum_{i=1}^{k} \operatorname{span}_{\mathbf{R}}\{\mathbf{e}_{2i-1}, \cos\theta_i \sqrt{-1}\mathbf{e}_{2i-1} + \sin\theta_i \mathbf{e}_{2i}\} ,$$

where $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is the standard unitary basis of $\mathbf{C}^n$. Then, we have $G_{2k,\theta} = U(n) \cdot V_\theta^{2k}$.

The assumption, even dimensional, of the definition of multiple Kähler angle in this section is not necessary. However, in the general case the definition becomes cluttered with factors involving the symbol [ ], where $[x]$ means the greatest integer $[x]$ not greater than x. In fact, H. Tasaki [10] defined it without assuming the dimension.

3. The Poincaré formula

In this section, we shall review the Poincaré formula on Riemannian homogeneous spaces given by R. Howard [4], and on complex projective spaces given by H. Tasaki [10].

Let E be a finite dimensional real vector space with an inner product. For two vector subspaces V and W of dimension p and q in E, take orthonormal bases $v_1, \cdots, v_p$ and $w_1, \cdots, w_q$ of V and W, respectively, and define

$$\sigma(V, W) = |v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q| .$$

This definition is independent of the choice of orthonormal bases. Furthermore, if $p + q = \dim E$, then

$$\sigma(V, W) = \sigma(V^{\perp}, W^{\perp}) .$$

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of elements of K. This metric induces a G-invariant Riemannian metric on G/K. We denote by o the origin of G/K. For x and y in G/K and vector subspaces V and W in $T_x(G/K)$ and $T_y(G/K)$, we define $\sigma_K(V, W)$ by

$$\sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1} V, dk_o^{-1}(dg_y)_o^{-1} W) d\mu(k)$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$. With these facts, the Poincaré formula for homogeneous spaces can be stated.

THEOREM 3.1 ([4]). *Let M and N be submanifolds of G/K. Assume that* $\dim M + \dim N = \dim(G/K)$ *and that G is unimodular. Then*

$$\int_G \sharp(M \cap gN) d\mu(g) = \int_{M \times N} \sigma_K(T_x^{\perp} M, T_y^{\perp} N) d\mu(x, y) ,$$

where $\sharp(X)$ denotes the number of points in X.

In general the actions of K on the Grassmann manifolds are not transitive. The function $\sigma_K(\cdot, \cdot)$ is defined on the product of the Grassmann manifolds consisting of subspaces in the tangent space at the origin. By the invariance of $\sigma_K(\cdot, \cdot)$ under the action of K, we can consider $\sigma_K(\cdot, \cdot)$ as a function defined on the product of the orbit spaces of the actions of K on the Grassmann manifolds. We can apply this and the argument on the multiple Kähler angle to the complex projective spaces. The following theorem is a special case of the Poincaré formula of Theorem 8 in [10].

THEOREM 3.2 ([10]). *For two even numbers p, q with $p + q = 2n$, we define*

$$\sigma_{p,q}^n(\theta, \tau) = \int_{U(1)\times U(n)} \sigma(V_\theta^p, k^{-1} \cdot V_\tau^q) d\mu(k) ,$$

where $\theta = (\theta_1, \cdots, \theta_{p/2})$ and $\tau = (\tau_1, \cdots, \tau_{q/2})$. Let M and N be any real p-dimensional and real q-dimensional submanifolds of $\mathbf{C}P^n$. Then we have

$$\int_{U(n+1)} \sharp(M \cap gN) d\mu(g) = \int_{M \times N} \sigma_{p,q}^n(\theta_{T_x M}, \tau_{T_y N}) d\mu(x, y) .$$

Theorems 3.1 and 3.2 hold in a general situation. However, σ_K in Theorem 3.1 and $\sigma_{p,q}^n$ in Theorem 3.2 are not in concrete enough forms to be easily used. Moreover, there are few results of the concrete calculation for σ_K or $\sigma_{p,q}^n$. We now list the examples on $\mathbf{C}P^4$. We may rewrite them in the sense of $\sigma_{p,q}^n$.

THEOREM 3.3 ([4], [8]). *Let M and N be complex 2-dimensional submanifolds of $\mathbf{C}P^4$. Then we have*

$$\sigma^4_{4,4}(0,0,0,0) = \frac{\mathrm{vol}(U(5))}{\mathrm{vol}(\mathbf{C}P^2)^2}\,.$$

THEOREM 3.4 ([4]). *Let M and N be complex 2-dimensional and totally real 4-dimensional submanifolds of $\mathbf{C}P^4$. Then we have*

$$\sigma^4_{4,4}\left(0,0,\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{\mathrm{vol}(U(5))}{\mathrm{vol}(\mathbf{C}P^2)\mathrm{vol}(\mathbf{R}P^4)}\,,$$

where $\mathbf{R}P^4$ is the 4-dimensional real projective space.

For $\mathbf{C}P^2$, we have the following:

THEOREM 3.5 ([6]). *For any real surfaces M and N of $\mathbf{C}P^2$, we have*

$$\sigma^2_{2,2}(\theta,\tau) = \frac{\mathrm{vol}(U(3))}{\mathrm{vol}(\mathbf{R}P^2)^2}(2 + 2\cos^2\theta_x\cos^2\tau_y + \sin^2\theta_x\sin^2\tau_y)\,.$$

Up to this point, we unrestrainedly used the notation vol (M), the volume of the manifold M. These values are, for example,

$$\mathrm{vol}(\mathbf{R}P^n) = \frac{1}{2}\mathrm{vol}(S^n),$$
$$\mathrm{vol}(\mathbf{C}P^n) = \frac{1}{2\pi}\mathrm{vol}(S^{2n+1}),$$
$$\mathrm{vol}(U(n+1)) = \frac{1}{\sqrt{2}}\mathrm{vol}(U(n))\cdot\mathrm{vol}(S^{2n+1})\,.$$

4. Proof of the main theorem

Let $\mathbf{C}P^n$ be an n-dimensional complex projective space with almost complex structure J, and let M a real 4-dimensional submanifold of $\mathbf{C}P^n$. For x in M, let $\theta_x = ((\theta_1)_x, (\theta_2)_x)$ be the multiple Kähler angle of T_xM in $T_x\mathbf{C}P^n$. We call $\theta_x = ((\theta_1)_x, (\theta_2)_x)$ the multiple Kähler angle of M at x.

Take a complex submanifold N of complex dimension 2. By Theorem 3.2, we have

$$\int_{U(5)} \sharp(M\cap gN)d\mu(g) = \int_{M\times N}\sigma^4_{4,4}(\theta_{T_xM},\tau_{T_yN})\,d\mu(x,y)\,.$$

We can simply write

$$\sigma((\theta_1)_x,(\theta_2)_x) = \sigma^4_{4,4}(\theta_{T_xM},\tau_{T_yN})$$

by Corollary 2.1. We shall identify the tangent space of $\mathbf{C}P^4$ with $\mathbf{C}^4$ and that of N with $\mathbf{C}^2$. By the action of $U(1)\times U(4)$, we can identify T_xM with V_θ^4 which is spanned by

$$\begin{aligned}\mathbf{e}_1 &= (1,0,0,0)\,, \quad e_1(\theta_1) = (\sqrt{-1}\cos\theta_1, \sin\theta_1, 0, 0)\,,\\ \mathbf{e}_3 &= (0,0,1,0)\,, \quad e_3(\theta_2) = (0,0,\sqrt{-1}\cos\theta_2, \sin\theta_2)\,.\end{aligned}$$

Then, we have

$$\begin{aligned}\sigma((\theta_1)_x, (\theta_2)_x) &= \int_{U(1)\times U(4)} \sigma(V_\theta^4, k^{-1}\cdot \mathbf{C}^2)d\mu(k)\\ &= \mathrm{vol}(U(1))\int_{U(4)} \sigma(V_\theta^4, k^{-1}\cdot\mathbf{C}^2)d\mu(k)\,.\end{aligned}$$

Let $\mathbf{e}_1,\cdots,\mathbf{e}_4$ be the standard unitary basis of $\mathbf{C}^4$. Then, we have

$$\begin{aligned}\sigma(V_\theta^4, k^{-1}\cdot\mathbf{C}^2) &= |(\mathbf{e}_1\wedge e_1(\theta_1)\wedge\mathbf{e}_3\wedge e_3(\theta_2))\wedge k^{-1}\cdot(\mathbf{e}_1\wedge\sqrt{-1}\mathbf{e}_1\wedge\mathbf{e}_2\wedge\sqrt{-1}\mathbf{e}_2)|\\ &= |(E_1(\theta_1)\wedge E_3(\theta_2))\wedge k^{-1}\cdot E_{12}|\,,\end{aligned}$$

where, to simplify notation, we have set

$$\begin{aligned}E_1(\theta_1) &:= \mathbf{e}_1\wedge e_1(\theta_1)\,,\\ E_3(\theta_2) &:= \mathbf{e}_3\wedge e_3(\theta_2)\,,\\ E_{12} &:= \mathbf{e}_1\wedge\sqrt{-1}\mathbf{e}_1\wedge\mathbf{e}_2\wedge\sqrt{-1}\mathbf{e}_2\,.\end{aligned}$$

In order to integrate this over $U(4)$, we shall use the compact symmetric pair $(U(4), U(2)\times U(2))$. See Section 3 in [11] for the statement of this compact symmetric pair.

$$\mathfrak{u}(4) = (\mathfrak{u}(2)+\mathfrak{u}(2))+\mathfrak{m}\,,\quad \mathfrak{m} = \left\{\begin{bmatrix}0 & X\\ -X^* & 0\end{bmatrix}\,\middle|\, X\in M_2(\mathbf{C})\right\}$$

is the canonical orthogonal direct sum decomposition of $\mathfrak{u}(4)$ associated with the compact symmetric pair $(U(4), U(2)\times U(2))$. We here define the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ by

$$\mathfrak{a} = \left\{\begin{bmatrix}0 & 0 & \phi_1 & 0\\ 0 & 0 & 0 & \phi_2\\ -\phi_1 & 0 & 0 & 0\\ 0 & -\phi_2 & 0 & 0\end{bmatrix}\,\middle|\, \phi_1,\phi_2\in\mathbf{R}\right\}.$$

Then we have the following set of positive restricted roots with respect to $\mathfrak{a}$

$$\{\phi_1-\phi_2, \phi_1+\phi_2, 2\phi_1, 2\phi_2\}$$

for a suitable ordering. Multiplicities of these roots are as follows:

$$\phi_1-\phi_2 : 2\,,\quad \phi_1+\phi_2 : 2\,,\quad 2\phi_1 : 1\,,\quad 2\phi_2 : 1\,.$$

Hence the fundamental cell in $\mathfrak{a}$ is defined by $\pi/2 \geq \phi_1 \geq \phi_2 \geq 0$. Let C be the image of the fundamental cell by exponential mapping, and put $B = U(2) \times U(2)$. We define a mapping $\rho : B \times C \times B \to U(4)$ by

$$\rho(s, a, t) = sat .$$

To apply the coarea formula to this mapping ρ, we give the following:

LEMMA 4.1 ([11]). *Under the above situation, we have*

$$\mathrm{vol}(\rho^{-1}(sat)) = 4\mathrm{vol}(U(1))^2 ,$$

$$J d\rho_{(s,a,t)} = 2 \sin^2(\phi_1 - \phi_2) \sin^2(\phi_1 + \phi_2) \sin 2\phi_1 \sin 2\phi_2 .$$

By Lemma 4.1 and the coarea formula we obtain

$$\int_{B\times C\times B} |E_1(\theta_1) \wedge E_3(\theta_2) \wedge E_{12}sat| \, J d\rho d\mu(s, a, t)$$
$$= 4\mathrm{vol}(U(1))^2 \int_{U(4)} \sigma(V^4_\theta, k^{-1} \cdot \mathbf{C}^2) d\mu(k) .$$

We first integrate on $B \times B$; then we get

$$\int_{B\times B} |E_1(\theta_1) \wedge E_3(\theta_2) \wedge (E_{12})sat| d\mu(s, t)$$
$$= \mathrm{vol}(U(2))^2 \int_B |(E_1(\theta_1) \wedge E_3(\theta_2))t^{-1} \wedge (E_{12})a| d\mu(t)$$

since $(E_{12})s = E_{12}$ for all $s \in U(2) \times U(2)$. From $t = (t_1, t_2) \in U(2) \times U(2)$, it follows

$$|E_1(\theta_1)t^{-1} \wedge E_3(\theta_2)t^{-1} \wedge (E_{12})a| = |\langle E_1(\theta_1)t_1^{-1}, *(E_3(\theta_2)t_2^{-1} \wedge (E_{12})a)| ,$$

where $*$ is Hodge star operator from $\wedge^6_{\mathbf{R}}(\mathbf{C}^4)$ to $\wedge^2_{\mathbf{R}}(\mathbf{C}^4)$. We here set

$$\eta := *(E_3(\theta_2)(t_2)^{-1} \wedge (E_{12})a) .$$

Since the action of B is $\mathbf{C}^2$ invariant,

$$\int_B |(E_1(\theta_1) \wedge E_3(\theta_2))t^{-1} \wedge (E_{12})a| d\mu(t)$$
$$= \int_{U(2)\times U(2)} |\langle E_1(\theta_1)t_1^{-1}, P(\eta)\rangle| d\mu(t_1, t_2) ,$$

where $P : \mathbf{C}^4 \to \mathbf{C}^2$ is orthogonal projection. Let ψ be the Kähler angle of $P(\eta)$. Needless to say, ψ is a function with respect to θ_2, t_2 and a. By Theorem 3.5 we obtain

$$\mathrm{vol}(U(1)) \int_{U(2)\times U(2)} |\langle E_1(\theta_1)t_1^{-1}, P(\eta)\rangle| d\mu(t_1, t_2)$$
$$= \frac{\mathrm{vol}(U(3))}{\mathrm{vol}(\mathbf{R}P^2)^2} \int_{U(2)} |P(\eta)|(2 + 2\cos^2\theta_1 \cos^2\psi + \sin^2\theta_1 \sin^2\psi) d\mu(t_2) .$$

In the sequel, we have to compute the following:

$$\int_{U(2)} |P(\eta)|(2 + 2\cos^2\theta_1 \cos^2\psi + \sin^2\theta_1 \sin^2\psi)\, d\mu(t_2)$$

Now in order to evaluate this we shall examine $U(2)\cdot(\mathbf{e}_1 \wedge e_1(\tau))$ in some detail. We take an orientation on $\mathbf{C}^2$ such that $\mathbf{e}_1, \sqrt{-1}\mathbf{e}_1, \mathbf{e}_2, \sqrt{-1}\mathbf{e}_2$ is a positive basis of $\mathbf{C}^2$ and the inner product on $\wedge^2_{\mathbf{R}}(\mathbf{C}^2)$ induced by that on $\mathbf{C}^2$. Let $*$ be the Hodge star operator on $\wedge^2_{\mathbf{R}}(\mathbf{C}^2)$. Put

$$\wedge^2_+ = \{\xi \in \wedge^2_{\mathbf{R}}(\mathbf{C}^2) \mid *\xi = \xi\}\,, \quad \wedge^2_- = \{\xi \in \wedge^2_{\mathbf{R}}(\mathbf{C}^2) \mid *\xi = -\xi\}\,.$$

Then we have an orthogonal direct sum decomposition

$$\wedge^2_{\mathbf{R}}(\mathbf{C}^2) = \wedge^2_+ \oplus \wedge^2_-\,.$$

We define orthonormal bases A_i and B_i of $\wedge^2_+$ and $\wedge^2_-$ by

$$A_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \sqrt{-1}\mathbf{e}_1 + \mathbf{e}_2 \wedge \sqrt{-1}\mathbf{e}_2)\,,$$
$$A_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 - \sqrt{-1}\mathbf{e}_1 \wedge \sqrt{-1}\mathbf{e}_2)\,,$$
$$A_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \sqrt{-1}\mathbf{e}_2 + \sqrt{-1}\mathbf{e}_1 \wedge \mathbf{e}_2)\,,$$
$$B_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \sqrt{-1}\mathbf{e}_1 - \mathbf{e}_2 \wedge \sqrt{-1}\mathbf{e}_2)\,,$$
$$B_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 + \sqrt{-1}\mathbf{e}_1 \wedge \sqrt{-1}\mathbf{e}_2)\,,$$
$$B_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \sqrt{-1}\mathbf{e}_2 - \sqrt{-1}\mathbf{e}_1 \wedge \mathbf{e}_2)\,.$$

Then we obtain

$$\wedge^2_+ = \mathrm{Span}_{\mathbf{R}}\{A_1, A_2, A_3\}\,, \quad \wedge^2_- = \mathrm{Span}_{\mathbf{R}}\{B_1, B_2, B_3\}\,.$$

By a simple calculation we have

$$U(2)\cdot(\mathbf{e}_1 \wedge e_1(\tau)) = \left(\frac{\cos\tau}{\sqrt{2}}A_1 + S^1\left(\frac{\sin\tau}{\sqrt{2}}\right)\right) \times S^2\left(\frac{1}{\sqrt{2}}\right)\,,$$

where $S^1(\sin\tau/\sqrt{2})$ is the circle of radius $\sin\tau/\sqrt{2}$ in $\mathrm{Span}_{\mathbf{R}}\{A_2, A_3\}$ and $S^2(1/\sqrt{2})$ is the 2-dimensional sphere of radius $1/\sqrt{2}$ in $\wedge^2_-$.

Now we define a mapping $p : U(2) \to (\mathbf{e}_3 \wedge e_3(\theta_2))U(2)$ by

$$p(k) = (\mathbf{e}_3 \wedge e_3(\theta_2))k\,.$$

As shown in [6], we have $Jdp = 2\sqrt{2}\sin\theta_2$. By the coarea formula we have

$$\frac{2\sqrt{2}\sin\theta_2}{\mathrm{vol}(SO(2))}\int_{U(2)}|P(\eta)|(2+2\cos^2\theta_1\cos^2\psi+\sin^2\theta_1\sin^2\psi)d\mu(t_2)$$
$$=\int_{(E_3(\theta_2))U(2)}|P(*(\xi\wedge E_{12}a))|\,(2+2\cos^2\theta_1\cos^2\psi+\sin^2\theta_1\sin^2\psi)d\mu(\xi)\,.$$

The variable $\xi \in (E_3(\theta_2))U(2)$ of above integral is represented by

$$\xi = \frac{\cos\theta_2}{\sqrt{2}}A_1' + x_2A_2' + x_3A_3' + y_1B_1' + y_2B_2' + y_3B_3'$$
$$\left((x_2)^2+(x_3)^2 = \frac{\sin^2\theta_2}{2},\ (y_1)^2+(y_2)^2+(y_3)^2 = \frac{1}{2}\right).$$

Here A_i' and B_i' take $\mathbf{e}_{j+2}, \sqrt{-1}\mathbf{e}_{j+2}$ instead of $\mathbf{e}_j, \sqrt{-1}\mathbf{e}_j$ in above notation A_i and B_i. We set

$$a := \begin{bmatrix} \cos\phi_1 & 0 & \sin\phi_1 & 0 \\ 0 & \cos\phi_2 & 0 & \sin\phi_2 \\ -\sin\phi_1 & 0 & \cos\phi_1 & 0 \\ 0 & -\sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix} \in C\,.$$

Then we obtain

$$\begin{aligned}
P(*(A_1'\wedge(E_{12})a)) &= \cos^2\phi_1\sin^2\phi_2\cdot\frac{A_1-B_1}{2}+\sin^2\phi_1\cos^2\phi_2\cdot\frac{A_1+B_1}{2}\\
&= \frac{1}{2}(\cos^2\phi_1\sin^2\phi_2+\sin^2\phi_1\cos^2\phi_2)A_1+\frac{1}{2}(\cos^2\phi_2-\cos^2\phi_1)B_1\\
P(*(B_1'\wedge(E_{12})a)) &= \cos^2\phi_1\sin^2\phi_2\cdot\frac{A_1-B_1}{2}-\sin^2\phi_1\cos^2\phi_2\cdot\frac{A_1+B_1}{2}\\
&= \frac{1}{2}(\cos^2\phi_1-\cos^2\phi_2)A_1-\frac{1}{2}(\cos^2\phi_1\sin^2\phi_2+\sin^2\phi_1\cos^2\phi_2)B_1
\end{aligned}$$

Similarly we get

$$\begin{aligned}
P(*(A_2'\wedge(E_{12})a)) &= -\cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2A_2\,,\\
P(*(A_3'\wedge(E_{12})a)) &= \cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2A_3\,,\\
P(*(B_2'\wedge(E_{12})a)) &= \cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2B_2\,,\\
P(*(B_3'\wedge(E_{12})a)) &= -\cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2B_3\,.
\end{aligned}$$

Hence we have

$$2P(*(\xi\wedge(E_{12})a))$$
$$=\left(\frac{\cos\theta_2}{\sqrt{2}}(\cos^2\phi_1\sin^2\phi_2+\sin^2\phi_1\cos^2\phi_2)+(\cos^2\phi_1-\cos^2\phi_2)y_1\right)A_1$$

$$
\begin{aligned}
&- 2\cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2 x_2 A_2 + 2\cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2 x_3 A_3\\
&+ \left(\frac{\cos\theta_2}{\sqrt{2}}(\cos^2\phi_2 - \cos^2\phi_1) - (\cos^2\phi_1\sin^2\phi_2 + \sin^2\phi_1\cos^2\phi_2)y_1\right)B_1\\
&+ 2\cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2 y_2 B_2 - 2\cos\phi_1\cos\phi_2\sin\phi_1\sin\phi_2 y_3 B_3 .
\end{aligned}
$$

To simplify notation, we put

$$
f(y_1) := \frac{\cos\theta_2}{2}(\cos^2\phi_1\sin^2\phi_2 + \sin^2\phi_1\cos^2\phi_2) + \frac{1}{\sqrt{2}}(\cos^2\phi_1 - \cos^2\phi_2)y_1
$$

then

$$
|P(*(\xi \wedge (E_{12})a))|^2 = f(y_1)^2 + \cos^2\phi_1\cos^2\phi_2\sin^2\phi_1\sin^2\phi_2\sin^2\theta_2 .
$$

Furthermore, from the expression of $P(*(\xi \wedge (E_{12})a))$, we have

$$
\cos\psi = \frac{f(y_1)}{|P(*(\xi \wedge (E_{12})a))|} .
$$

Hence the integrand of the above integral over $(\mathbf{e}_3 \wedge e_3(\theta_2))U(2)$ is

$$
\begin{aligned}
&|P(*(\xi \wedge (E_{12})a))|(2 + 2\cos^2\theta_1\cos^2\psi + \sin^2\theta_1\sin^2\psi)\\
&\quad = 2(1+\cos^2\theta_1)|P(*(\xi \wedge (E_{12})a))|\\
&\quad\quad - (3\cos^2\theta_1 - 1)\frac{\cos^2\phi_1\cos^2\phi_2\sin^2\phi_1\sin^2\phi_2\sin^2\theta_2}{|P(*(\xi \wedge (E_{12})a))|} .
\end{aligned}
$$

Therefore it is sufficient to compute the following:

$$
\begin{aligned}
\int_{S^1(\sin\theta_2/\sqrt{2})\times S^2(1/\sqrt{2})} \alpha(f(y_1), a)d\mu(x,y) &= 2\pi \cdot \frac{\sin\theta_2}{\sqrt{2}} \int_{S^2(1/\sqrt{2})} \alpha(f(y_1), a)d\mu(y)\\
&= 2\pi^2 \sin\theta_2 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \alpha(f(t), a)dt ,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(f(y_1), a) &= 2(1+\cos^2\theta_1)\,|P(*(\xi \wedge (E_{12})a))|\\
&\quad - (3\cos^2\theta_1 - 1)\frac{\cos^2\phi_1\cos^2\phi_2\sin^2\phi_1\sin^2\phi_2\sin^2\theta_2}{|P(*(\xi \wedge (E_{12})a))|} .
\end{aligned}
$$

Then, long but simple calculation with some elementary integrals yields

$$
\begin{aligned}
&\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \alpha(f(t), a)dt\\
&= \frac{\sqrt{2}}{2} \cdot (1+\cos^2\theta_1)(1+\cos^2\theta_2)\frac{\cos^4\phi_1\sin^4\phi_2 - \sin^4\phi_1\cos^4\phi_2}{\cos^2\phi_1 - \cos^2\phi_2}
\end{aligned}
$$

$$+ 2\sqrt{2}\sin^2\theta_1 \sin^2\theta_2 \frac{\sin^2\phi_1 \sin^2\phi_2 \cos^2\phi_1 \cos^2\phi_2}{\cos^2\phi_1 - \cos^2\phi_2} \cdot \log \frac{\cos^2\phi_1 \sin^2\phi_2}{\sin^2\phi_1 \cos^2\phi_2} .$$

It is known that

$$\begin{aligned} J d\rho &= 2\sin^2(\phi_1 - \phi_2)\sin^2(\phi_1 + \phi_2)\sin 2\phi_1 \sin 2\phi_2 \\ &= 2(\cos^2\phi_1 - \cos^2\phi_2)^2 \sin 2\phi_1 \sin 2\phi_2 . \end{aligned}$$

Hence by routine computations we have

$$\int_C \frac{\cos^4\phi_1 \sin^4\phi_2 - \sin^4\phi_1 \cos^4\phi_2}{\cos^2\phi_1 - \cos^2\phi_2} J d\rho d\phi_1 d\phi_2 = \frac{1}{9}$$

and

$$\int_C \frac{\sin^2\phi_1 \sin^2\phi_2 \cos^2\phi_1 \cos^2\phi_2}{\cos^2\phi_1 - \cos^2\phi_2} \cdot \log \frac{\cos^2\phi_1 \sin^2\phi_2}{\sin^2\phi_1 \cos^2\phi_2} J d\rho d\phi_1 d\phi_2 = \frac{1}{72} .$$

Summarizing, we obtain

$$\begin{aligned} \sigma((\theta_1), (\theta_2)) = &\frac{\pi^2 \mathrm{vol}(U(2))^2 \mathrm{vol}(U(3)) \mathrm{vol}(SO(2))}{4\sqrt{2}\mathrm{vol}(U(1))^2 \mathrm{vol}(\mathbf{R}P^2)^2} \\ &\times \left[\frac{(1+\cos^2\theta_1)(1+\cos^2\theta_2)}{2} \cdot \frac{1}{9} + 2\sin^2\theta_1 \sin^2\theta_2 \cdot \frac{1}{72} \right]. \end{aligned}$$

Here

$$\frac{\pi^2 \mathrm{vol}(U(2))^2 \mathrm{vol}(U(3)) \mathrm{vol}(SO(2))}{4\mathrm{vol}(U(1))^2 \mathrm{vol}(\mathbf{R}P^2)^2} = \frac{9}{2} \cdot \frac{\mathrm{vol}(U(5))}{\mathrm{vol}(\mathbf{C}P^2)^2} ,$$

so we have

$$\sigma((\theta_1), (\theta_2)) = \frac{\mathrm{vol}(U(5))}{\mathrm{vol}(\mathbf{C}P^2)^2} \left[\frac{1}{4}(1+\cos^2\theta_1)(1+\cos^2\theta_2) + \frac{1}{8}\sin^2\theta_1 \sin^2\theta_2 \right].$$

This completes the proof.

COROLLARY 4.2. *Under the hypothesis of our Theorem, if M is a slant submanifold of $\mathbf{C}P^4$ then*

$$\begin{aligned} &\int_{U(5)} \sharp(M \cap gN) d\mu(g) \\ &\quad = \frac{\mathrm{vol}(U(5))}{\mathrm{vol}(\mathbf{C}P^2)^2} \left(\frac{1}{4}(1+\cos^2\theta)^2 + \frac{1}{8}\sin^4\theta \right) \mathrm{vol}(M) \mathrm{vol}(N) , \end{aligned}$$

where θ is the slant angle of M.

REMARK 4.3. Complex and totally real submanifolds have constant multiple Kähler angles 0 and $\pi/2$, respectively. Thus the formulas in Theorems 3.3 and 3.4 are special cases of our theorem.

REMARK 4.4. By the transfer principle in integral geometry (see [4] paragraph 3.5 on pages 14–15), it is clear that our theorem holds for all complex space forms with isotropy subgroup $U(1) \times U(4)$.

References

[1] B.-Y. CHEN, Slant immersions, Bull. Austral. Math. Soc. **41** (1990), 135–147.

[2] B.-Y. CHEN and Y. TAZAWA, Slant submanifolds in complex Euclidean spaces, Tokyo. J. Math. **14** (1991), 101–120.

[3] S. HELGASON, *Differential geometry*, *Lie groups*, *and Symmetric spaces*, Academic Press (1978).

[4] R. HOWARD, The kinematic formula in Riemannian homogeneous spaces, Mem. Amer. Math. Soc. **509** (1993).

[5] H. J. KANG and H. TASAKI, Integral geometry of real surfaces in complex projective spaces, Tsukuba J. Math. **25** (2001), 155–164.

[6] H. J. KANG and H. TASAKI, Integral geometry of real surfaces in the complex projective plane, Geom. Dedicata **90** (2002), 99–106.

[7] LÊ HÔNG VÂN, Application of integral geometry to minimal surfaces, Int. J. Math. **4** (1993), 89–111.

[8] L. A. SANTALÓ, Integral geometry in Hermitian spaces, Amer. J. Math. **74** (1952), 423–434.

[9] M. TAKEUCHI, On conjugate loci and cut locus of compact symmetric spaces I, Tsukuba J. Math. **2** (1978), 35–68.

[10] H. TASAKI, Generalization of Kähler angle and integral geometry in complex projective spaces, *Steps in Differential Geometry*, Proceedings of Colloquium on Differential Geometry (Debrecen, 2000) (2001), Inst. Math. Inform. 349–361.

[11] H. TASAKI, Integral geometry of submanifolds of real dimension and codimension two in complex projective spaces, *Differential geometry and integrable systems*, Contemp. Math. **308** (2002), Amer. Math. Soc. 315–327.

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