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A Bicomplex Riemann Zeta Function

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Abstract. In this work we use a commutative generalization of complex numbers, called bicomplex numbers, to introduce a holomorphic Riemann zeta function of two complex variables satisfying the complexified Cauchy-Riemann equations. Furthermore, we establish a bicomplex Riemann hypothesis equivalent to the complex Riemann hypothesis of one variable and we obtain a bicomplex Euler Product.

1. Introduction

There exist several ways to generalize complex numbers into a real algebra of dimension four. However, it seems that perhaps only quaternions ([2], [4], [5], [12], [13]) and bicomplex numbers ([7], [9], [11]) enable us to well define analysis with such kind of generalizations. In fact, by the famous Frobenius theorem, we know that quaternions are the only possible four dimensional algebra without zero divisors (the same theorem says that there are no algebras without zero divisors in $\boldsymbol{R}^3$). However, quaternions are not commutative instead of bicomplex numbers which are commutative but with zero divisors.

Now, it is well known that the Riemann's Conjecture for the zeta function is presently one the most important conjecture in the whole mathematics. In that context, it is natural to look for a Riemann zeta function for such kind of hypercomplex numbers. In the case of quaternions, it is not obvious to generalize the Riemann zeta function because quaternions are not commutative. However, there exist a definition of a quaternionic Riemann zeta function using the Dirichlet series (see [16]).

In this article, we introduce a Riemann zeta function for bicomplex numbers. More precisely, we obtain a holomorphic Riemann zeta function of two complex variables satisfying the complexified Cauchy-Riemann equations. Furthermore, we establish a bicomplex Riemann hypothesis equivalent to the complex Riemann hypothesis of one variable and we obtain a bicomplex Euler Product. Finally, as corollary, we treat our results for the specific case of hyperbolic numbers ([3], [17]).

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2. Preliminaries

Here we introduce some of the basic results of the theory of bicomplex numbers. In 1892, in search for special algebras, Corrado Segre (1860–1924) published a paper [14] in which he treated an infinite family of algebras whose elements are called bicomplex numbers, tricomplex numbers,$\cdots$, n-complex numbers. We define bicomplex numbers (also called tetranumbers) as follows:

$$\boldsymbol{T} := \{a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} \mid \mathbf{i_1^2} = \mathbf{i_2^2} = -1, \mathbf{j}^2 = 1$$

and

$$\mathbf{i_2 j} = \mathbf{j i_2} = -\mathbf{i_1}, \mathbf{i_1 j} = \mathbf{j i_1} = -\mathbf{i_2}, \mathbf{i_2 i_1} = \mathbf{i_1 i_2} = \mathbf{j}\}$$

where $a, b, c, d \in \boldsymbol{R}$. The topology used on $\boldsymbol{T}$ is the topology of $\boldsymbol{R}^4$ induced by the Euclidean norm (also noted $|\ \ |$).

We remark that we can write a bicomplex number $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}$ as $(a + b\mathbf{i_1}) + (c + d\mathbf{i_1})\mathbf{i_2} = z_1 + z_2\mathbf{i_2}$ where $z_1, z_2 \in \boldsymbol{C}(\mathbf{i_1}) := \{x + y\mathbf{i_1} \mid \mathbf{i_1}^2 = -1\}$. Thus, $\boldsymbol{T}$ can be viewed as a kind of "duplication" of $\boldsymbol{C}(\mathbf{i_1})$. In particular, a bicomplex number can be seen as an element of $\boldsymbol{C}^2 \simeq \boldsymbol{C}^2(\mathbf{i_1})$. It is easy to see [7] that $\boldsymbol{T}$ is a commutative unitary ring with the following characterization for the non-invertible elements:

PROPOSITION 1. *Let $w = a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} \in \boldsymbol{T}$. Then w is non-invertible if and only if*

$$(a = -d \quad and \quad b = c) \quad or \quad (a = d \quad and \quad b = -c)\,.$$

It is also possible to define differentiability of a function at a point of $\boldsymbol{T}$ [7]:

DEFINITION 1. Let U be an open set of $\boldsymbol{T}$ and $w_0 \in U$. Then, $f : U \subseteq \boldsymbol{T} \to \boldsymbol{T}$ is said to be $\boldsymbol{T}$-differentiable at w_0 with derivative equal to $f'(w_0) \in \boldsymbol{T}$ if

$$\lim_{\substack{w \to w_0 \\ (w - w_0\ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0)\,.$$

We also say that the function f is $\boldsymbol{T}$-holomorphic on an open set U if and only if f is $\boldsymbol{T}$-differentiable at each point of U.

As we saw, a bicomplex number can be seen as an element of $\boldsymbol{C}^2$, so a function $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$ of $\boldsymbol{T}$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\boldsymbol{C}^2$. Here we have a characterization of such mappings:

THEOREM 1. *Let U be an open set and $f : U \subseteq \boldsymbol{T} \to \boldsymbol{T}$ such that $f \in C^1(U)$. Let also $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$. Then f is $\boldsymbol{T}$-holomorphic on U if and only if:*

$$f_1 \textit{ and } f_2 \textit{ are holomorphic in } z_1 \textit{ and } z_2$$

and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \textit{and} \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \textit{on } U.$$

Moreover, $f' = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\mathbf{i_2}$ *and* $f'(w)$ *is invertible if and only if* $det\mathcal{J}_f(w) \neq 0$.

This theorem can be obtained from results in [7] and [10]. Moreover, by the Hartogs theorem [15], it is possible to show that "$f \in C^1(U)$" can be dropped from the hypotheses. Hence, it is natural to define the corresponding class of mappings for $\boldsymbol{C}^2$:

DEFINITION 2. The class of $\boldsymbol{T}$-holomorphic mappings on a open set $U \subseteq \boldsymbol{C}^2$ is defined as follows:

$$TH(U) := \left\{ f : U \subseteq \boldsymbol{C}^2 \to \boldsymbol{C}^2 | f \in H(U) \quad \textit{and} \quad \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \textit{on } \ U \right\}.$$

It is the subclass of holomorphic mappings of $\boldsymbol{C}^2$ *satisfying the complexified Cauchy-Riemann equations.*

We remark that $f \in TH(U)$ in terms of $\boldsymbol{C}^2$ if and only if f is $\boldsymbol{T}$-differentiable on U. It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be non-invertible if and only if $z_1 - z_2\mathbf{i_1} = 0$ or $z_1 + z_2\mathbf{i_1} = 0$.

The notion of holomorphicity can also be seen with this kind of notation. For this we need to define the projections $P_1, P_2 : \boldsymbol{T} \to \boldsymbol{C}(\mathbf{i_1})$ as $P_1(z_1 + z_2\mathbf{i_2}) = z_1 - z_2\mathbf{i_1}$ and $P_2(z_1 + z_2\mathbf{i_2}) = z_1 + z_2\mathbf{i_1}$. Also, we need the following definition:

DEFINITION 3. We say that $X \subseteq \boldsymbol{T}$ is a $\boldsymbol{T}$-cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2 := \{z_1 + z_2\mathbf{i_2} \in \boldsymbol{T} : z_1 + z_2\mathbf{i_2} = w_1\mathbf{e_1} + w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}$.

In [7] it is shown that if X_1 and X_2 are domains of $\boldsymbol{C}(\mathbf{i_1})$ then $X_1 \times_e X_2$ is also a domain of $\boldsymbol{T}$. Now, it is possible to state the following striking theorems [7]:

THEOREM 2. *If* $f_{e1} : X_1 \to \boldsymbol{C}(\mathbf{i_1})$ *and* $f_{e2} : X_2 \to \boldsymbol{C}(\mathbf{i_1})$ *are holomorphic functions of* $\boldsymbol{C}(\mathbf{i_1})$ *on the domains* X_1 *and* X_2 *respectively, then the function* $f : X_1 \times_e X_2 \to \boldsymbol{T}$ *defined as*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}, \ \forall\, z_1 + z_2\mathbf{i_2} \in X_1 \times_e X_2$$

is $\boldsymbol{T}$*-holomorphic on the domain* $X_1 \times_e X_2$ *and*

$$f'(z_1 + z_2\mathbf{i_2}) = f'_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f'_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

$\forall\, z_1 + z_2\mathbf{i_2} \in X_1 \times_e X_2$.

THEOREM 3. *Let X be a domain in $\boldsymbol{T}$, and let $f : X \to \boldsymbol{T}$ be a $\boldsymbol{T}$-holomorphic function on X. Then there exist holomorphic functions $f_{e1} : X_1 \to \boldsymbol{C}(\mathbf{i_1})$ and $f_{e2} : X_2 \to \boldsymbol{C}(\mathbf{i_1})$ with $X_1 = P_1(X)$ and $X_2 = P_2(X)$, such that:*

$$f(z_1 + z_2\mathbf{i_2}) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \qquad \forall\, z_1 + z_2\mathbf{i_2} \in X\,.$$

We note here that X_1 and X_2 will also be domains of $\boldsymbol{C}(\mathbf{i_1})$.

3. Bicomplex Riemann zeta function

In this section we want to give a meaning of an expression of the form $\sum_{n=1}^{\infty} \frac{1}{n^w}$ where w is a bicomplex number. For this, we need to define what we mean by an integer to a bicomplex power.

DEFINITION 4. Let $n \in \boldsymbol{N}\backslash\{0\}$ and $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$. We define

$$n^w := e^{w \cdot \ln(n)}$$

where

$$e^{z_1+z_2\mathbf{i_2}} := e^{z_1} \cdot e^{z_2\mathbf{i_2}} \quad \text{and} \quad e^{z_2\mathbf{i_2}} := \cos(z_2) + \mathbf{i_2}\sin(z_2)\,.$$

Hence,

$$n^{z_1+z_2\mathbf{i_2}} = e^{z_1\cdot\ln(n)} \cdot [\cos(z_2 \cdot \ln(n)) + \mathbf{i_2}\sin(z_2 \cdot \ln(n))]$$

REMARK (see [7] and [10]).

- $e^{w_1+w_2} = e^{w_1} \cdot e^{w_2}\ \forall w_1, w_2 \in \boldsymbol{T}$
- e^{w} is invertible $\forall w \in \boldsymbol{T}$
- $e^{z_1+z_2\mathbf{i_2}} = (e^{z_1-z_2\mathbf{i_1}})\mathbf{e_1} + (e^{z_1+z_2\mathbf{i_1}})\mathbf{e_2}\ \forall z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$

We are now able to define a bicomplex Riemann zeta function.

DEFINITION 5. Let $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ with $\mathrm{Re}(z_1) > 1$ and $|\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1$. We define a bicomplex Riemann zeta function $\zeta(w)$ by the following convergent series:

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w}\,.$$

The last definition can be well justified by the following theorem.

THEOREM 4. *Let $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ with* $\mathrm{Re}(z_1 - z_2\mathbf{i_1}) > 1$ *and* $\mathrm{Re}(z_1 + z_2\mathbf{i_1}) > 1$. *Then $\sum_{n=1}^{\infty} \frac{1}{n^w}$ converges and*

$$\sum_{n=1}^{\infty}\frac{1}{n^w}=\left[\sum_{n=1}^{\infty}\frac{1}{n^{z_1-z_2\mathbf{i_1}}}\right]\mathbf{e_1}+\left[\sum_{n=1}^{\infty}\frac{1}{n^{z_1+z_2\mathbf{i_1}}}\right]\mathbf{e_2}\,.$$

Moreover,

$$\begin{aligned}&\{w\in \boldsymbol{T}\mid \mathrm{Re}(z_1-z_2\mathbf{i_1})>1 \quad \textit{and} \quad \mathrm{Re}(z_1+z_2\mathbf{i_1})>1\}\\ &=\{w\in \boldsymbol{T}\mid \mathrm{Re}(z_1)>1 \quad \textit{and} \quad |\mathrm{Im}(z_2)|<Re(z_1)-1\}\,.\end{aligned}$$

PROOF. From the last remarks, we obtain

$$\begin{aligned}n^{z_1+z_2\mathbf{i_2}}&=n^{(z_1-z_2\mathbf{i_1})\mathbf{e_1}+(z_1+z_2\mathbf{i_1})\mathbf{e_2}}\\ &=e^{((z_1-z_2\mathbf{i_1})\mathbf{e_1}+(z_1+z_2\mathbf{i_1})\mathbf{e_2})\ln(n)}\\ &=e^{(z_1-z_2\mathbf{i_1})\ln(n)\mathbf{e_1}+(z_1+z_2\mathbf{i_1})\ln(n)\mathbf{e_2}}\\ &=e^{(z_1-z_2\mathbf{i_1})\ln(n)}\mathbf{e_1}+e^{(z_1+z_2\mathbf{i_1})\ln(n)}\mathbf{e_2}\\ &=n^{z_1-z_2\mathbf{i_1}}\mathbf{e_1}+n^{z_1+z_2\mathbf{i_1}}\mathbf{e_2}\ \text{(invertible)}\,.\end{aligned}$$

Hence,

$$\frac{1}{n^{z_1+z_2\mathbf{i_2}}}=\frac{1}{n^{z_1-z_2\mathbf{i_1}}}\mathbf{e_1}+\frac{1}{n^{z_1+z_2\mathbf{i_1}}}\mathbf{e_2}\,.$$

Now, from the theory of the Riemann zeta function of one complex variable, it is well known that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^s}$$

converges in the half-plane $\mathrm{Re}(s)>1$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^{z_1-z_2\mathbf{i_1}}}$ and $\sum_{n=1}^{\infty}\frac{1}{n^{z_1+z_2\mathbf{i_1}}}$ converge, respectively, for $\mathrm{Re}(z_1-z_2\mathbf{i_1})>1$ and $\mathrm{Re}(z_1+z_2\mathbf{i_1})>1$. Hence,

$$\sum_{n=1}^{\infty}\frac{1}{n^w}=\left[\sum_{n=1}^{\infty}\frac{1}{n^{z_1-z_2\mathbf{i_1}}}\right]\mathbf{e_1}+\left[\sum_{n=1}^{\infty}\frac{1}{n^{z_1+z_2\mathbf{i_1}}}\right]\mathbf{e_2}$$

on $\{w\in \boldsymbol{T}\mid \mathrm{Re}(z_1-z_2\mathbf{i_1})>1 \text{ and } \mathrm{Re}(z_1+z_2\mathbf{i_1})>1\}$. Moreover, let $w=z_1+z_2\mathbf{i_2}=a+b\mathbf{i_1}+c\mathbf{i_2}+d\mathbf{j}$ (i.e. $z_1=a+b\mathbf{i_1}$ and $z_2=c+d\mathbf{i_1}$). Then, $\mathrm{Re}(z_1)=a$, $\mathrm{Im}(z_2)=d$, $\mathrm{Re}(z_1-z_2\mathbf{i_1})=a+d$ and $\mathrm{Re}(z_1+z_2\mathbf{i_1})=a-d$. Now, $\{w\in \boldsymbol{T}\mid \mathrm{Re}(z_1-z_2\mathbf{i_1})>1 \text{ and } \mathrm{Re}(z_1+z_2\mathbf{i_1})>1\}=\{w\in \boldsymbol{T}\mid \mathrm{Re}(z_1)>1 \text{ and } |\mathrm{Im}(z_2)|<\mathrm{Re}(z_1)-1\}$ since

$$a+d>1 \quad \text{and} \quad a-d>1 \Longleftrightarrow a>1 \quad \text{and} \quad |d|<a-1\,. \qquad \square$$

We will now determine the whole domain of existence of our bicomplex Riemann zeta function. In fact, if $\mathcal{O}_2$ denotes the set of non-invertible elements in $\boldsymbol{T}$, we extend $\zeta(w)$ as follows:

$$\zeta(w) := \zeta(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + \zeta(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$$

on the set $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$.

REMARK.
- $1 = 1\mathbf{e_1} + 1\mathbf{e_2}$
- $w \in 1 + \mathcal{O}_2 \Leftrightarrow z_1 - z_2\mathbf{i_1} = 1$ or $z_1 + z_2\mathbf{i_1} = 1$

The next theorems of this section will help us to better understand why we choose $\boldsymbol{T}\backslash\{1+\mathcal{O}_2\}$ to define our analytic continuation of $\zeta(w)$.

THEOREM 5. *The set $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$ is open and connected in $\boldsymbol{C}^2$.*

PROOF. It is easy to show that $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\} = (\boldsymbol{C}(\mathbf{i_1})\backslash\{1\}) \times_e (\boldsymbol{C}(\mathbf{i_1})\backslash\{1\})$. Hence, $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$ must be a domain in $\boldsymbol{C}^2$ because $\boldsymbol{C}(\mathbf{i_1})\backslash\{1\}$ is a domain in the complex plane. □

THEOREM 6. *The bicomplex Riemann zeta function $\zeta(w)$ is $\boldsymbol{T}$-holomorphic on $\boldsymbol{T}\backslash\{1+\mathcal{O}_2\}$.*

PROOF. Let $f_{e1}(z_1 - z_2\mathbf{i_1}) = \zeta(z_1 - z_2\mathbf{i_1})$ and $f_{e2}(z_1 + z_2\mathbf{i_1}) = \zeta(z_1 + z_2\mathbf{i_1})$ on $X_1 = X_2 = \boldsymbol{C}(\mathbf{i_1})\backslash\{1\}$. Now, by analytic continuation, the Riemann zeta function is holomorphic on $\boldsymbol{C}(\mathbf{i_1})\backslash\{1\}$ and, by Theorem 2, $\zeta(w) = f_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + f_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ is a $\boldsymbol{T}$-holomorphic mappings on the domain $X_1 \times_e X_2 = (\boldsymbol{C}(\mathbf{i_1})\backslash\{1\}) \times_e (\boldsymbol{C}(\mathbf{i_1})\backslash\{1\}) = \boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$. Therefore, $\zeta(w) \in TH(\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\})$. □

THEOREM 7. *The analytic continuation of $\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w}$ on $\boldsymbol{T}\backslash\{1+\mathcal{O}_2\}$ is unique.*

PROOF. From Theorems 5 and 6, $\zeta(w) := \zeta(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + \zeta(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ is, in particular, holomorphic on the open and connected set $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$. Hence, by the identity theorem of $\boldsymbol{C}^2$ (see [8]), the analytic continuation of $\sum_{n=1}^{\infty} \frac{1}{n^w}$ from the nonempty open set $\{w \in \boldsymbol{T} \mid \mathrm{Re}(z_1) > 1 \text{ and } |\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1\}$ to $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$ must be unique. In particular, $\zeta(w) := \zeta(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + \zeta(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ is the only one possible $\boldsymbol{T}$-holomorphic continuation. □

Finally, the following theorem will confirm that the domain $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$ is the best possible.

THEOREM 8. *Let $w_0 \in 1 + \mathcal{O}_2$ then*

$$\lim_{\substack{w \to w_0 \\ (w \notin 1+\mathcal{O}_2)}} |\zeta(w)| = \infty\,.$$

PROOF. Let $w = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ and $w_0 = (z_1^0 - z_2^0\mathbf{i_1})\mathbf{e_1} + (z_1^0 + z_2^0\mathbf{i_1})\mathbf{e_2}$. By hypothesis, $w_0 \in 1 + \mathcal{O}_2$. Hence, $z_1^0 - z_2^0\mathbf{i_1} = 1$ or $z_1^0 + z_2^0\mathbf{i_1} = 1$. Without loss of generality, let us suppose that $z_1^0 - z_2^0\mathbf{i_1} = 1$. Now, from this identity (see [7])

$$|z_1 + z_2\mathbf{i_2}| = \left(\frac{|z_1 - z_2\mathbf{i_1}|^2 + |z_1 + z_2\mathbf{i_1}|^2}{2}\right)^{1/2} \qquad \forall z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$$

it follows that $w \to w_0 \Rightarrow z_1 - z_2\mathbf{i_1} \to 1$ and $z_1 + z_2\mathbf{i_1} \to z_1^0 + z_2^0\mathbf{i_1}$.

Moreover, as shown by Riemann, $\zeta(s)$ extends to $\boldsymbol{C}$ as a meromorphic function with only a simple pole at $s = 1$. Therefore,

$$\lim_{z_1 - z_2\mathbf{i_1} \to 1} |\zeta(z_1 - z_2\mathbf{i_1})| = \infty\,.$$

Then,

$$\begin{aligned}
\lim_{\substack{w \to w_0 \\ (w \notin 1 + \mathcal{O}_2)}} |\zeta(w)| &= \lim_{\substack{w \to w_0 \\ (w \notin 1 + \mathcal{O}_2)}} |\zeta(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + \zeta(z_1 + z_2\mathbf{i_1})\mathbf{e_2}| \\
&= \lim_{\substack{w \to w_0 \\ (w \notin 1 + \mathcal{O}_2)}} \left(\frac{|\zeta(z_1 - z_2\mathbf{i_1})|^2 + |\zeta(z_1 + z_2\mathbf{i_1})|^2}{2}\right)^{1/2} \\
&= \infty\,.
\end{aligned}$$

□

4. Zeros of $\zeta(w)$

Let $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$. Then,

$$\zeta(w) = 0 \Longleftrightarrow \zeta(z_1 - z_2\mathbf{i_1}) = 0 \quad \text{and} \quad \zeta(z_1 + z_2\mathbf{i_1}) = 0\,.$$

Hence, from the trivial zeros of the complex Riemann zeta function we can obtain trivial zeros for $\zeta(w)$. More specifically:

THEOREM 9. *Let $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$. Then $z_1 - z_2\mathbf{i_1}$ and $z_1 + z_2\mathbf{i_1}$ are trivial zeros of the complex Riemann zeta function if and only if $z_1 + z_2\mathbf{i_2} = (-n_1 - n_2) + (-n_1 + n_2)\mathbf{j}$, where $n_1, n_2 \in \boldsymbol{N}\backslash\{0\}$.*

PROOF. The complex Riemann zeta function has zero at the negative even integers and one refers to them as the trivial zeros. Now, $z_1 - z_2\mathbf{i_1} = -2n_1$ and $z_1 + z_2\mathbf{i_1} = -2n_2$ where $n_1, n_2 \in \boldsymbol{N}\backslash\{0\}$ if and only if $z_1 = \frac{-2n_1 - 2n_2}{2} = -(n_1 + n_2)$ and $z_2 = \frac{(-2n_1 + 2n_2)\mathbf{i_1}}{2} = (-n_1 + n_2)\mathbf{i_1}$, that is $z_1 + z_2\mathbf{i_2} = (-n_1 - n_2) + (-n_1 + n_2)\mathbf{i_1}\mathbf{i_2} = (-n_1 - n_2) + (-n_1 + n_2)\mathbf{j}$. □

The definition of trivial zeros for the bicomplex Riemann zeta function follows from the last theorem.

DEFINITION 6. The set $z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ such that

$$z_1 + z_2\mathbf{i_2} = (-n_1 - n_2) + (-n_1 + n_2)\,\mathbf{j}\,,$$

where $n_1, n_2 \in \boldsymbol{N}\backslash\{0\}$, will be defined as the set of the trivial zeros for the bicomplex Riemann zeta function.

5. Bicomplex Riemann hypothesis

Let us recall the Riemann hypothesis.

Riemann hypothesis:
The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

In this section, we will establish a bicomplex Riemann hypothesis for $\zeta(w)$ equivalent to the Riemann hypothesis for $\zeta(s)$.

CONJECTURE 1. Let $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}\backslash\{1+\mathcal{O}_2\}$. If w is a nontrivial zeros of the bicomplex Riemann zeta function then:

$$(\mathrm{Re}(z_1), \mathrm{Im}(z_2)) = \left(\frac{1}{2}, 0\right)$$

or

$$(\mathrm{Re}(z_1), \mathrm{Im}(z_2)) = \left(\frac{1}{4} - n, \pm\left(\frac{1}{4} + n\right)\right) \qquad \text{where } n \in \boldsymbol{N}\backslash\{0\}.$$

THEOREM 10. *The Conjecture* 1 *is equivalent to the Riemann hypothesis.*

PROOF. If we supposed that all nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$ then the bicomplex Riemann zeta function has nontrivial zeros if and only if

$$z_1 - z_2\mathbf{i_1} = -2n_1\,, \quad n_1 \in \boldsymbol{N}\backslash\{0\} \quad \text{and} \quad z_1 + z_2\mathbf{i_1} = \frac{1}{2} + y_1\mathbf{i_1}\,, \quad y_1 \in \boldsymbol{R} \qquad (\mathbf{1})$$

or

$$z_1 - z_2\mathbf{i_1} = \frac{1}{2} + y_2\mathbf{i_1}\,, \quad y_2 \in \boldsymbol{R} \quad \text{and} \quad z_1 + z_2\mathbf{i_1} = -2n_2\,, \quad n_2 \in \boldsymbol{N}\backslash\{0\} \qquad (\mathbf{2})$$

or

$$z_1 - z_2\mathbf{i_1} = \frac{1}{2} + y_2\mathbf{i_1}\,, \quad y_2 \in \boldsymbol{R} \quad \text{and} \quad z_1 + z_2\mathbf{i_1} = \frac{1}{2} + y_1\mathbf{i_1}\,, \quad y_1 \in \boldsymbol{R} \qquad (\mathbf{3})$$

Now, from **(1)** we obtain that $z_1 = \frac{1}{2}(-2n_1 + (\frac{1}{2} + y_1\mathbf{i_1})) = -n_1 + \frac{1}{4} + \frac{y_1}{2}\mathbf{i_1}$ and $z_2 = \frac{1}{2}(-2n_1 - (\frac{1}{2} + y_1\mathbf{i_1}))\mathbf{i_1} = -n_1\mathbf{i_1} - \frac{\mathbf{i_1}}{4} + \frac{y_1}{2}$. Hence, $z_1 + z_2\mathbf{i_2} = (-n_1 + \frac{1}{4} + \frac{y_1}{2}\mathbf{i_1}) + (-n_1\mathbf{i_1} - \frac{\mathbf{i_1}}{4} + \frac{y_1}{2})\mathbf{i_2} = (\frac{1}{4} - n_1) + (\frac{y_1}{2})\mathbf{i_1} + (\frac{y_1}{2})\mathbf{i_2} - (\frac{1}{4} + n_1)\mathbf{j}$, i.e. $(\mathrm{Re}(z_1), \mathrm{Im}(z_2)) = (\frac{1}{4} - n_1, -(\frac{1}{4} + n_1))$. In the same way, from **(2)** we obtain that $z_1 + z_2\mathbf{i_2} = (\frac{1}{4} + \frac{y_2}{2}\mathbf{i_1} - n_2) + (\frac{\mathbf{i_1}}{4} - \frac{y_2}{2} + n_2\mathbf{i_1})\mathbf{i_2}$ $= (\frac{1}{4} - n_2) + (\frac{y_2}{2})\mathbf{i_1} - (\frac{y_2}{2})\mathbf{i_2} + (\frac{1}{4} + n_2)\mathbf{j}$, i.e. $(\mathrm{Re}(z_1), \mathrm{Im}(z_2)) = (\frac{1}{4} - n_2, \frac{1}{4} + n_2)$. Finally, from **(3)** we obtain that $z_1 + z_2\mathbf{i_2} = \frac{1}{2} + \frac{(y_1+y_2)}{2}\mathbf{i_1} + \frac{(y_1-y_2)}{2}\mathbf{i_2} + 0\mathbf{j}$ i.e. $(\mathrm{Re}(z_1), \mathrm{Im}(z_2)) = (\frac{1}{2}, 0)$.

Conversely, we want to prove that if Conjecture 1 is true then the Riemann hypothesis must be true. For that, we will suppose that there exist a nontrival zero for $\zeta(s)$ with real part different from $\frac{1}{2}$ and we will find a contadiction with Conjecture 1. Let s^* be a nontrivial zero for $\zeta(s)$ with $\mathrm{Re}(s^*) = a \neq \frac{1}{2}$. Hence, $w^* = z_1^* + z_2^*\mathbf{i_2} := s^*\mathbf{e_1} + s^*\mathbf{e_2}$ must be a nontrival zero for $\zeta(w)$. However, $w^* = s^* = \mathrm{Re}(s^*) + \mathrm{Im}(s^*)\mathbf{i_1} + 0\mathbf{i_2} + 0\mathbf{j} = (a + \mathrm{Im}(s^*)\mathbf{i_1}) + (0)\mathbf{i_2}$. Then, $\mathrm{Re}(z_1^*) = a \neq \frac{1}{2}$ and $\mathrm{Im}(z_2^*) = 0 \neq \pm(\frac{1}{4} + n)$ $\forall n \in \boldsymbol{N}\backslash\{0\}$. Therefore,

$$(\mathrm{Re}(z_1^*), \mathrm{Im}(z_2^*)) \neq \left(\frac{1}{2}, 0\right)$$

and

$$(\mathrm{Re}(z_1^*), \mathrm{Im}(z_2^*)) \neq \left(\frac{1}{4} - n, \pm\left(\frac{1}{4} + n\right)\right) \quad \forall n \in \boldsymbol{N}\backslash\{0\}\,. \qquad \square$$

6. Bicomplex Euler product

In the complex plane, an infinite product is said to converge if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero. In the bicomplex case we have to pay attention to the divisors of zero.

DEFINITION 7. A bicomplex infinite product is said to converge if and only if at most a finite number of the factors are non-invertible, and if the partial products formed by the invertible factors tend to a finite limit which is invertible.

The following lemma establishes a connection between the bicomplex infinite product and the complex infinite product for sequences.

LEMMA 1. *Let $w_n = z_{1,n} + z_{2,n}\mathbf{i_2} \in \boldsymbol{T}\backslash\mathcal{O}_2$ be a sequence of invertible bicomplex numbers. Then, $\prod_{n=1}^{\infty} w_n$ converges if and only if*

$$\prod_{n=1}^{\infty}(z_{1,n} - z_{2,n}\mathbf{i_1}) \quad \textit{and} \quad \prod_{n=1}^{\infty}(z_{1,n} + z_{2,n}\mathbf{i_1}) \quad \textit{converge.}$$

Moreover, in case of convergence, we obtain:

$$\prod_{n=1}^{\infty} w_n = \prod_{n=1}^{\infty}(z_{1,n} - z_{2,n}\mathbf{i_1})\mathbf{e_1} + \prod_{n=1}^{\infty}(z_{1,n} + z_{2,n}\mathbf{i_1})\mathbf{e_2}\,.$$

PROOF. By definition, $\prod_{n=1}^{\infty} w_n = \lim_{n\to\infty} \prod_{k=1}^{n} w_k = \lim_{n\to\infty} \prod_{k=1}^{n}(z_{1,k} + z_{2,k}\mathbf{i_2}) = \lim_{n\to\infty} \prod_{k=1}^{n}[(z_{1,k} - z_{2,k}\mathbf{i_1})\mathbf{e_1} + (z_{1,k} + z_{2,k}\mathbf{i_1})\mathbf{e_2}]$ where $z_{1,k} - z_{2,k}\mathbf{i_1} \neq 0$ and $z_{1,k} + z_{2,k}\mathbf{i_1} \neq 0$ $\forall k \geq 1$. Moreover, the idempotent representation implies that $\prod_{k=1}^{n}[(z_{1,k} - z_{2,k}\mathbf{i_1})\mathbf{e_1} + (z_{1,k} + z_{2,k}\mathbf{i_1})\mathbf{e_2}] = [\prod_{k=1}^{n}(z_{1,k} - z_{2,k}\mathbf{i_1})]\mathbf{e_1} + [\prod_{k=1}^{n}(z_{1,k} + z_{2,k}\mathbf{i_1})]\mathbf{e_2}$ $\forall n \geq 1$. This complete the proof because a sequence of bicomplex numbers $\{s_n\} = \{s_{1,n}\mathbf{e_1} + s_{2,n}\mathbf{e_2}\}$ converges to a point $s = s_1\mathbf{e_1} + s_2\mathbf{e_2}$ whenever $n \to \infty$ if and only if $\{s_{1,n}\}$ and $\{s_{2,n}\}$ converge respectively to s_1 and s_2 in the complex plane (see [7]). $\square$

Using this last result, we are able to establish a bicomplex Euler product:

THEOREM 11. *Let* $w = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ *with* $\mathrm{Re}(z_1) > 1$ *and* $|\mathrm{Im}(z_2)| < \mathrm{Re}(z_1) - 1$. *Then*

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^w}} .$$

Where $p_1, p_2, \cdots, p_n, \cdots$ *is the ascending sequence of prime numbers.*

PROOF. From Theorem 4, we know that the bicomplex Riemann zeta function will converges at w and

$$\sum_{n=1}^{\infty} \frac{1}{n^w} = \left[\sum_{n=1}^{\infty} \frac{1}{n^{z_1 - z_2\mathbf{i_1}}}\right]\mathbf{e_1} + \left[\sum_{n=1}^{\infty} \frac{1}{n^{z_1 + z_2\mathbf{i_1}}}\right]\mathbf{e_2} .$$

Moreover, it is well known (see [1]) that Riemann has extended Euler's formula to a complex variable. In fact, in the complex plane we have:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^s}} ,$$

for every complex number s with $\mathrm{Re}(s) > 1$. Therefore,

$$\zeta(w) = \left[\prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^{z_1 - z_2\mathbf{i_1}}}}\right]\mathbf{e_1} + \left[\prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^{z_1 + z_2\mathbf{i_1}}}}\right]\mathbf{e_2} .$$

Hence, by Lemma 1,

$$\zeta(w) = \prod_{n=1}^{\infty} \left[\left[\frac{1}{1 - \frac{1}{p_n^{z_1 - z_2\mathbf{i_1}}}}\right]\mathbf{e_1} + \left[\frac{1}{1 - \frac{1}{p_n^{z_1 + z_2\mathbf{i_1}}}}\right]\mathbf{e_2}\right] = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^w}} . \qquad \square$$

7. Hyperbolic Riemann zeta function

It has been proven (see [3]) that there exist essentially three possible ways to "naturally" generalize real numbers into real algebras of dimension two. In fact, each possible system can be reduced to one of the following:

1. numbers $a + b\mathbf{i}$ with $\mathbf{i}^2 = -1$ (the complex numbers);
2. numbers $a + b\mathbf{j}$ with $\mathbf{j}^2 = 1$ (the hyperbolic numbers);
3. numbers $a + b\mathbf{k}$ with $\mathbf{k}^2 = 0$ (the dual numbers).

Now, from the definition of bicomplex numbers, we remark that the complex numbers and the hyperbolic numbers (also called duplex numbers) are included in $\boldsymbol{T}$ as subrings.

Hence, it is also possible to define a Riemann zeta function for the specific sub-case of hyperbolic numbers. In fact, most of the properties will come directly from the properties of the bicomplex Riemann zeta function.

Let $\boldsymbol{D} := \{c + d\mathbf{j} : c, d \in \boldsymbol{R}, \mathbf{j}^2 = 1\}$. Using Definition 4, we obtain

$$n^h = e^{h\cdot\ln(n)} \ \forall h \in \boldsymbol{D},$$

where

$$\begin{aligned} e^{c+d\mathbf{j}} &= e^c \cdot e^{d\mathbf{j}} \\ &= e^c \cdot e^{(d\mathbf{i_1})\mathbf{i_2}} \\ &= e^c \cdot [\cos(d\mathbf{i_1}) + \mathbf{i_2}\sin(d\mathbf{i_1})] \\ &= e^c \cdot [\cosh(d) + \mathbf{i_2}(\mathbf{i_1}\sinh(d))] \\ &= e^c \cdot \left[\cosh(d) + \mathbf{j}\sinh(d)\right]. \end{aligned}$$

Therefore, by Theorems 4 and 11, if $c > 1$ and $|d| < c - 1$ then

$$\zeta(c + d\mathbf{j}) = \sum_{n=1}^{\infty} \frac{1}{n^{c+d\mathbf{j}}} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^{c+d\mathbf{j}}}} \quad \text{converge},$$

where $p_1, p_2, \cdots, p_n, \cdots$ is the ascending sequence of prime numbers. It is also possible to define differentiability of a function at a point of $\boldsymbol{D}$ as follows:

DEFINITION 8. Let U be an open set of $\boldsymbol{D}$ and $h_0 \in U$. Then, $f : U \subseteq \boldsymbol{D} \to \boldsymbol{D}$ is said to be $\boldsymbol{D}$-differentiable at h_0 with derivative equal to $f'(h_0) \in \boldsymbol{D}$ if

$$\lim_{\substack{h\to h_0 \\ (h-h_0 \ inv.)}} \frac{f(h) - f(h_0)}{h - h_0} = f'(h_0).$$

We will also say that the function f is $\boldsymbol{D}$-holomorphic on an open set U if and only if f is $\boldsymbol{D}$-differentiable at each point of U.

In particular, from Theorems 6 and 8, $\zeta(c + d\mathbf{j})$ can be $\boldsymbol{D}$-holomorphically "extended" on $\boldsymbol{D}\backslash\{\boldsymbol{D} \cap \{1 + \mathcal{O}_2\}\} = \boldsymbol{D}\backslash\{(1 + c) + d\mathbf{j} : |c| = |d|\} = \boldsymbol{D}\backslash\{1 + \mathcal{O}_1\}$ as follows:

$$\zeta(c + d\mathbf{j}) := \zeta(c + d)\mathbf{e_1} + \zeta(c - d)\mathbf{e_2}$$

with

$$\lim_{\substack{h\to h_0 \\ (h\notin 1+\mathcal{O}_1)}} |\zeta(h)| = \infty \quad \text{whenever} \quad h_0 \in 1 + \mathcal{O}_1$$

where $\mathcal{O}_1 := \boldsymbol{D} \cap \mathcal{O}_2$.

However, such kind of hyperbolic extension is not unique. For example, let us consider:

$$\Upsilon(x) := \begin{cases} \zeta(x), & \text{if } x > 1, x \in \boldsymbol{R} \\ -\frac{1}{1-x} & \text{if } x < 1, x \in \boldsymbol{R}. \end{cases}$$

Let, $\Upsilon(c+d\mathbf{j}) := \Upsilon(c+d)\mathbf{e_1} + \Upsilon(c-d)\mathbf{e_2}$. Hence, if $c > 1$ and $|d| < c-1$, then $\Upsilon(c+d\mathbf{j}) = \sum_{n=1}^{\infty} \frac{1}{n^{c+d\mathbf{j}}}$. Moreover, $\Upsilon(c+d\mathbf{j})$ is $\boldsymbol{D}$-holomorphic on $\boldsymbol{D}\backslash\{1+\mathcal{O}_1\}$. In fact,

$$\lim_{\substack{c+d\mathbf{j}\to c_0+d_0\mathbf{j} \\ [(c+d\mathbf{j})-(c_0+d_0\mathbf{j})\ inv.]}} \frac{\Upsilon(c+d\mathbf{j}) - \Upsilon(c_0+d_0\mathbf{j})}{(c+d\mathbf{j})-(c_0+d_0\mathbf{j})}$$

$$\begin{aligned}
&= \lim_{\substack{c+d\mathbf{j}\to c_0+d_0\mathbf{j} \\ [(c+d\mathbf{j})-(c_0+d_0\mathbf{j})\ inv.]}} \left[\frac{\Upsilon(c+d)-\Upsilon(c_0+d_0)}{(c-c_0)+(d-d_0)}\mathbf{e_1} + \frac{\Upsilon(c-d)-\Upsilon(c_0-d_0)}{(c-c_0)-(d-d_0)}\mathbf{e_2}\right] \\
&= \lim_{\substack{c+d\mathbf{j}\to c_0+d_0\mathbf{j} \\ [(c+d\mathbf{j})-(c_0+d_0\mathbf{j})\ inv.]}} \left[\frac{\Upsilon(c+d)-\Upsilon(c_0+d_0)}{(c+d)-(c_0+d_0)}\mathbf{e_1} + \frac{\Upsilon(c-d)-\Upsilon(c_0-d_0)}{(c-d)-(c_0-d_0)}\mathbf{e_2}\right] \\
&= \lim_{\substack{c+d\mathbf{j}\to c_0+d_0\mathbf{j} \\ [(c+d\mathbf{j})-(c_0+d_0\mathbf{j})\ inv.]}} \left[\frac{\Upsilon(c+d)-\Upsilon(c_0+d_0)}{(c+d)-(c_0+d_0)}\right]\mathbf{e_1} \\
&\quad + \lim_{\substack{c+d\mathbf{j}\to c_0+d_0\mathbf{j} \\ [(c-d\mathbf{j})-(c_0-d_0\mathbf{j})\ inv.]}} \left[\frac{\Upsilon(c-d)-\Upsilon(c_0-d_0)}{(c-d)-(c_0-d_0)}\right]\mathbf{e_2} \\
&= \Upsilon'(c_0+d_0)\mathbf{e_1} + \Upsilon'(c_0-d_0)\mathbf{e_2}
\end{aligned}$$

because $c+d\mathbf{j} \to c_0+d_0\mathbf{j} \Leftrightarrow c+d \to c_0+d_0$ and $c-d \to c_0-d_0$ ($(c+d\mathbf{j})-(c_0+d_0\mathbf{j})$ is invertible if and only if $c+d \neq c_0+d_0$ and $c-d \neq c_0-d_0$), and $c_0+d_0\mathbf{j} \in \boldsymbol{D}\backslash\{1+\mathcal{O}_1\} \Leftrightarrow c_0+d_0 \neq 1$ and $c_0-d_0 \neq 1$. Finally, we can see that

$$\lim_{\substack{c+d\mathbf{j}\to c_0+d_0\mathbf{j} \\ (c+d\mathbf{j}\notin 1+\mathcal{O}_1)}} |\Upsilon(c+d\mathbf{j})| = \infty \quad \text{whenever} \quad c_0+d_0\mathbf{j} \in 1+\mathcal{O}_1$$

since $|\zeta(x)| \to \infty$ and $|-\frac{1}{1-x}| \to \infty$ whenever $x \to 1$.

Hence, the bicomplex Riemann zeta function enable us to give a "natural" definition of the Riemann zeta function for the hyperbolic case. Moreover, the trivial zeros for our hyperbolic Riemann zeta function are exactly the same than for the bicomplex Riemann zeta function, i.e.

$$\{(-n_1-n_2)+(-n_1+n_2)\mathbf{j} : n_1, n_2 \in \boldsymbol{N}\backslash\{0\}\} \in \boldsymbol{D}\,.$$

However, in this case, it is not possible to obtain a Riemann hypothesis:

THEOREM 12. *Every zeros of the hyperbolic Riemann zeta function are trivial.*

PROOF. By definition $\zeta(c+d\mathbf{j}) := \zeta(c+d)\mathbf{e_1} + \zeta(c-d)\mathbf{e_2}$ $\forall c+d\mathbf{j} \in \boldsymbol{D}\backslash\{1+\mathcal{O}_1\}$. We note that $c+d$ and $c-d$ are real. Moreover, on the real line, $\zeta(\sigma) = 0 \Leftrightarrow \sigma = -2n$ with $n \in \boldsymbol{N}\backslash\{0\}$ (see [6]). Therefore, $\zeta(c+d\mathbf{j}) = 0 \Leftrightarrow c+d\mathbf{j} \in \{(-n_1-n_2)+(-n_1+n_2)\mathbf{j} : n_1, n_2 \in \boldsymbol{N}\backslash\{0\}\}$. □

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