Boundary Theta Curves in $S^3$

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Abstract. It is introduced a split extension of groups $1 \to P_2 \to C_{1,2} \to \Theta \to 1$, where $P_2$ is the group of pure braids in 2 strings, $C_{1,2}$ is the group of cobordism classes of (pure) 2-string links and $\Theta$ is the group of cobordism classes of theta curves. The concept of boundary theta curve is introduced and it is proved that the group of boundary cobordism classes of boundary theta curves is isomorphic to the group of boundary cobordism classes of boundary string links in 2 strings.

1. Introduction

One way of trying to detect how intertwined are the different components of a link is the concept of boundary link (that is, a link whose components bound disjoint Seifert surfaces). This paper introduces a similar concept for theta curves.

It is divided in two parts. In the first part we relate the group of cobordism of theta curves with the group of cobordism classes of 2-string links. String links were introduced in [2] and there it was observed that since the group of cobordism classes of (pure) $n$-string links contains the group of (pure) $n$-braids, it is non abelian if $n \geq 3$, on the other side, if $n = 1$, it coincides with the group of cobordism classes of knots and therefore is abelian. The case $n = 2$ was open, but it follows from Proposition 2 below that it is non abelian.

In the second part we introduce the group of boundary cobordism classes of boundary theta curves and prove that it is isomorphic to the group BSL(2) of boundary cobordism classes of boundary 2-string links, introduced in [1].

2. Theta curves in $S^3$

The group of cobordism of theta curves was introduced in [5]. We recall it. We work in the piecewise linear category.

A labelled theta curve, or simply theta curve, is a graph $\theta$ with two vertices $v_1$ and $v_2$ and three edges $e_1$, $e_2$ and $e_3$ each of which joins $v_1$ and $v_2$. We give an orientation from $v_1$ to $v_2$ to each edge.
Let $f : \theta \to S^3$ be an embedding of $\theta$ into the three-sphere $S^3$. Then $f$ is called a spatial embedding and the image $f(\theta)$ is called a spatial theta curve. Then $f(e_2 \cup e_3)$, $f(e_3 \cup e_1)$ and $f(e_1 \cup e_2)$ together with the orientation of $e_2$, $e_3$ and $e_1$ respectively are oriented knots in $S^3$ which are called constituent knots of $f$ and denoted by $k_1(f)$, $k_2(f)$ and $k_3(f)$ respectively.

Two embedding $f, g : \theta \to S^3$ are said to be cobordant if there is a 'locally flat' embedding $\Phi : \theta \times [0, 1] \to S^3 \times [0, 1]$ such that $\Phi|_{\theta \times \{0\}} : \theta \times \{0\} \to S^3 \times \{0\}$ equals to $f$ and $\Phi|_{\theta \times \{1\}} : \theta \times \{1\} \to S^3 \times \{1\}$ equals to $g$, where 'locally flat' means that the image of $\Phi$ in $S^3 \times [0, 1]$ is locally homeomorphic to the standard disk pair $(D^4, D^2)$ or $(D^3, Y) \times D^1$ where $(D^3, Y)$ is shown in Figure 1.

The cobordism class of $f$ is denoted by $[f]$. For two spatial embeddings $f : \theta \to S^3_1$, $g : \theta \to S^3_2$, remove small balls centered at $f(v_2)$ and $g(v_1)$ from $S^3_1$ and $S^3_2$ respectively, then identify the boundaries such that the images of the $i$-th edge are joined for each $i$. Then we obtain a new embedding of $\theta$ into $S^3 = S^3_1 \# S^3_2$ which is called the vertex connected sum of $f$ and $g$ and denoted by $f \# g$. The vertex connected sum is well defined up to ambient isotopy [6].

**Proposition 1** (Taniyama). The cobordism classes of embeddings of $\theta$ into $S^3$ form a group under the vertex connected sum.

The inverse of $[f]$ is $[f!]$ where $f!$ is the reflected inverse of $f$. See Figure 2.

We denote this group by $\Theta$.

For an embedding $f : \theta \to S^3$, choose a regular neighbourhood $N$ of $f(e_i)$ in $S^3$, $i \in \{1, 2, 3\}$. Then the pair $(N, N \cap f(\theta))$ is homeomorphic to $(D^3, A)$ of Figure 3.

Let $h_i : (D^3, A) \to (N, N \cap f(\theta))$ be a homeomorphism such that $(f(\theta) \cap N) \cup h_i(B)$ is a 2-component link with linking number zero, where $B$ is a pair of strings in $D^3$ as illustrated in Figure 4.

$h_i$ is called a 0-framing of $f(e_i)$ and the link is called the $i$-th parallel link of $f$ and denoted by $\ell_i(f)$.

The following result was suggested to us by J. Levine.

**Proposition 2.** There is a split extension of groups

$$1 \to P_2 \to C_{1,2} \xrightarrow{\alpha} \Theta \to 1$$
where $P_2$ is the group of pure braids in 2 strings and $C_{1,2}$ is the group of cobordism classes of 2-string links. (see [1] for definitions and notation)

**Proof.** Considering $D \times I \subseteq \mathbb{R}^3 \subseteq S^3$ and three intervals $I_0$, $I_1$, $I_2$ embedded in $S^3$, $I_0$ connecting $(a_1, 0)$ to $(a_2, 0)$, $I_1$ connecting $(a_1, 1)$ to $(a_2, 1)$ and $I_2$ connecting the midpoints of $I_0$ and $I_1$, we can associate to each 2-string link a theta curve (see Figure 5).

A cobordism between 2-string links can clearly be extended to a cobordism between the respective theta curves. Besides this association provides a homomorphism $\alpha : C_{1,2} \to \Theta$. $\alpha$ is surjective since given a theta curve one can, without changing its cobordism class, push one of its vertices along one of its edges until obtain one unknotted edge that does not undercross nor overcross the others.

According to [2], the inclusion map $j_k : P_k \to C_{1,k}$ is a monomorphism, that is, two pure braids in $k$ strings are ambient isotopic if and only if they are cobordant as $k$-string links.

Let $b(f)$ be the unique pure braid in 2 strings whose linking number between the strings is the same as the linking number between the 2 strings of a given 2-string link $f$.

By [5] (Theorem 5), the class of cobordism of $f$ is in the kernel of $\alpha$ if and only if the closure of $b(f^{-1}) \cdot f$ is slice but, by [3], this happens if and only if $b(f^{-1}) \cdot f$ is slice (i.e., cobordant to the trivial string link). Since $P_2$ is in the center of $C_{1,2}$, the map $\beta : C_{1,2} \to C_{1,2}$ that sends the cobordism class of $f$ to the cobordism class of $b(f^{-1}) \cdot f$ is also a homomorphism. The cobordism class of $f$ belongs to $\ker \beta$ if and only if $f$ is cobordant to $b(f^{-1})^{-1} = b(f)$, that is, if and only if $f \in \im j_2$. Therefore $\ker \alpha = \ker \beta = \im j_2 \cong P_2$ and we have an extension of groups

$$1 \to P_2 \to C_{1,2} \xrightarrow{\alpha} \Theta \to 1.$$
It is clear that $\alpha \circ \beta = \alpha$. This equation together with the fact that $\ker \alpha = \ker \beta$ implies that $\alpha$ maps the image of $\beta$ isomorphically onto $\Theta$. Thus the extension splits as stated. \hfill \square

Le Dimet has observed that $C_{1,1}$ is commutative (the group of cobordism of knots) while $C_{1,k}$ for $k \geq 3$ is not (since there is a monomorphism $j_k : P_k \rightarrow C_{1,k}$). From Proposition 2 it follows that $C_{1,2}$ is not commutative since, by [4], the theta curve cobordism group $\Theta$ is not commutative.

3. Boundary theta curves

DEFINITION 1. A Seifert surface for a $\theta$-curve $f$ is a pair $(S_1, S_2)$ of oriented connected surfaces in $S^3$ such that $\partial S_1$ and $\partial S_2$ are two different constituent knots of $f$, $S_i \cap f(\theta) = \partial S_i$ ($i = 1, 2$) and $(S_1 \setminus \partial S_1) \cap (S_2 \setminus \partial S_2) = \emptyset$. A $\theta$-curve that has a Seifert surface is a boundary $\theta$-curve.$^1$

If $f$ is a boundary $\theta$-curve, given any two constituent knots of $f$ there are Seifert surfaces as in the definition. This can be seen geometrically but we shall follow a different approach.

Let us represent a general $\theta$-curve $f$ as in Figure 6.

Let us call top meridians of $f$ the following elements $x_1, x_2$ and $x_3$ of $\pi_1(f)$ (where $\pi_1(f)$ stands for the fundamental group of the complement of $f(\theta)$.) They are obtained by taking the homotopy class of the following loops: beginning at the eye of the reader the loop goes straight to the beginning of one arrow, follows that arrow until its end and goes back to the eye of the reader.

Similarly one defines bottom meridians $y_1, y_2$ and $y_3$ (see Figure 8).

DEFINITION 2. Let $F(2)$ be the free group in 2 generators $\alpha_1$ and $\alpha_2$. An epimorphism $\eta : \pi_1(f) \rightarrow F(2)$ that sends two top meridians $x_i \neq x_j$, $i, j \in \{1, 2, 3\}$, to $\alpha_1$ and $\alpha_2$ respectively and send the corresponding bottom meridians $y_i$ to $\alpha_1$ and $y_j$ to $\alpha_2$ is called a splitting for the $\theta$-curve $f$.

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$^1$At the time of revision it was brought to our attention that R. Shinjo and R. Nikkuni are also working with this concept, in a different direction.
It follows from Thom-Pontryagin construction that a \( \theta \)-curve has a splitting if and only if it is a boundary \( \theta \)-curve.

By composing \( \eta \) with automorphisms of \( F(2) \) we see that the existence of the splitting does not depend upon which meridians have index \( i \) and \( j \).

From this it follows easily that the vertex connected sum of boundary \( \theta \)-curves is a boundary \( \theta \)-curve.

If \( F \) is a cobordism between \( \theta \)-curves \( f_1 \) and \( f_2 \), there are homomorphism \( i_j : \pi_1(f_j) \to \pi_1(F) \) induced by the inclusion maps. If \( f_1 \) and \( f_2 \) are boundary \( \theta \)-curves with splittings \( \eta_1 \) and \( \eta_2 \) respectively, an epimorphism \( \eta : \pi_1(F) \to F(2) \) such that \( \eta \circ i_j = \eta_j \), \( j = 1, 2 \), is called a splitting for the cobordism \( F \). A cobordism \( F \) that has a splitting \( \eta \) is called a boundary cobordism between \( f_1 \) and \( f_2 \) and in this case, \( f_1 \) and \( f_2 \) are said to be boundary cobordant (boundary) \( \theta \)-curves.

Of course the reflected inverse of a boundary \( \theta \)-curve is a boundary \( \theta \)-curve and we have a group \( B\Theta \) of boundary cobordism classes of boundary \( \theta \)-curves.

If \( f \) is a boundary string-link in two strings, its associated \( \theta \)-curve is a boundary \( \theta \)-curve.

Clearly \( f \) is boundary slice (that is, represents the unit element of \( B\Theta \)) if and only if \( \ell_1(f) \) is a boundary slice link, but \( \ell_1(f) = \hat{f} \) and, by [1], Theorem 14, \( \hat{f} \) is boundary slice if and only if \( f \) is. Therefore we have

**Proposition 3.** \( \beta : BSL(2) \to B\Theta \) is an isomorphism.

A question that still remains is if \( B\Theta \) is abelian.

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