TOKYO J. MATH.
VOL. 28, NO. 2, 2005

A Characterization of Certain Einstein Kähler Hypersurfaces in a Complex Grassmann manifold of 2-planes

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(Communicated by Y. Ohnita)

1. Introduction

Denote by $G_r(\mathbf{C}^n)$ the complex Grassmann manifold of r-planes in $\mathbf{C}^n$, equipped with the Kähler metric of maximal holomorphic sectional curvature c.

One of the simplest typical examples of submanifolds of $G_r(\mathbf{C}^n)$ is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [4, 5] determined maximal totally geodesic submanifolds of $G_2(\mathbf{C}^n)$. I. Satake and S. Ihara in [11, 6] determined all (equivariant) holomorphic, totally geodesic imbeddings of a symmetric domain into another symmetric domain. When an ambient symmetric domain is of type $(\mathrm{I})_{p,q}$, taking a compact dual symmetric space, we obtain the complete list of maximal totally geodesic Kähler submanifolds of $G_r(\mathbf{C}^n)$.

Let M be a maximal totally geodesic Kähler submanifold of $G_r(\mathbf{C}^n)$ given by a Kähler immersion $\varphi : M \to G_r(\mathbf{C}^n)$. Since M is a symmetric space, denote by (G, K) the compact symmetric pair of M. Then there exists a certain unitary representation $\rho : G \to \tilde{G} = SU(n)$, such that $\varphi(M)$ is given by the orbit of $\rho(G)$ through the origin in $G_r(\mathbf{C}^n)$.

Denote by $\mathbf{C}P^n$ and Q^n, an n-dimensional complex projective space and an n-dimensional complex quadric respectively.

EXAMPLE 1 ([4, 5, 11, 6]). Let $M = G/K$ be a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbf{C}^n)$, ρ a corresponding unitary representation of G to $SU(n)$. Then, M and ρ are one of the following (up to isomorphism).

(1) $M_1 = G_r(\mathbf{C}^{n-1}) \hookrightarrow G_r(\mathbf{C}^n), \quad 1 \leqq r \leqq n-2$

(2) $M_2 = G_{r-1}(\mathbf{C}^{n-1}) \hookrightarrow G_r(\mathbf{C}^n), \quad 2 \leqq r \leqq n-1$

(3) $M_3 = G_{r_1}(\mathbf{C}^{n_1}) \times G_{r_2}(\mathbf{C}^{n_2}) \hookrightarrow G_{r_1+r_2}(\mathbf{C}^{n_1+n_2}), \quad 1 \leqq r_i \leqq n_i - 1,\ i = 1, 2$

(4) $M_4 = M_{4,p} = Sp(p)/U(p) \hookrightarrow G_p(\mathbf{C}^{2p}), \quad p \geqq 2$

(5) $M_5 = M_{5,p} = SO(2p)/U(p) \hookrightarrow G_p(\mathbf{C}^{2p}), \quad p \geqq 4$

(6) $M_{6,m} = \mathbf{C}P^p \hookrightarrow G_r(\mathbf{C}^n),\ r = \binom{p}{m-1},\ n = \binom{p+1}{m},\ 2 \leqq m \leqq p-1,$
$\rho_{6,m} : SU(p+1) \to SU(n)$: the exterior representation of degree m

Received March 2, 2004

(7) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbf{C}^4)$,
$\rho_7 : Spin(5) \to SU(4)$: spin representation
(8) $M_8 = M_{8,2l} = Q^{2l} \hookrightarrow G_r(\mathbf{C}^{2r})$, $l \geqq 3$,
$\rho_8^{\pm} : Spin(2l+2) \to SU(2^l)$: (two) spin representations

Notice that $\rho_1, \cdots, \rho_5$ are the identical representations, and notice that $M_{4,2} = M_7$ and $M_{5,4} = M_{8,6}$.

A submanifold M of $G_r(\mathbf{C}^n)$ is parallel if the second fundamental form of M is parallel. H. Nakagawa and R. Takagi in [10] classified parallel Kähler submanifolds of a complex projective space $\mathbf{C}P^{n-1} = G_1(\mathbf{C}^n)$. K. Tsukada in [14] showed that, in parallel Kähler submanifolds of $G_r(\mathbf{C}^n)$, the above classification is essential. Moreover, if $r \neq 1, n-1$, then a parallel Kähler submanifold M of $G_r(\mathbf{C}^n)$ is a parallel Kähler submanifolds of some totally geodesic Kähler submanifold of $G_r(\mathbf{C}^n)$, i.e, M is a parallel Kähler submanifold of one of $\{M_i, i = 1, \cdots, 8\}$. Notice that a Hermitian symmetric submanifolds of $G_r(\mathbf{C}^n)$ is parallel.

Another one of the simplest typical examples of submanifolds of $G_r(\mathbf{C}^n)$ is a homogeneous Kähler hypersurface. K. Konno in [8] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

EXAMPLE 2 ([8]). Let M be a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbf{C}^n)$. Then, M are one of the following (up to isomorphism).

(1) $M_9 = \mathbf{C}P^{n-2} \hookrightarrow \mathbf{C}P^{n-1} = G_1(\mathbf{C}^n)$
(2) $M_{10} = Q^{n-2} \hookrightarrow \mathbf{C}P^{n-1} = G_1(\mathbf{C}^n)$
(3) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbf{C}^4)$
(4) $M_{11} = M_{11,l} = Sp(l)/U(2) \cdot Sp(l-2) \hookrightarrow G_2(\mathbf{C}^{2l})$: Kähler C-space of type (C_l, α_2), $l \geqq 2$

M_9 and M_7 are totally geodesic. M_9, M_{10} and M_7 are symmetric spaces. M_{10} is not totally geodesic but parallel. If $l = 2$, then M_{11} is congruent to M_7. If $l > 2$, M_{11} is neither symmetric nor parallel.

Notice that all manifolds in Examples 1 and 2 are Einstein manifolds.

The purpose of this paper is, without the assumption of homogeneity, to characterize a Kähler hypersurface M_{11}.

M_{11} satisfies another interesting, extrinsic property as follows. It is known that $G_2(\mathbf{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^{\perp}M$ of a Kähler hypersurface M in $G_2(\mathbf{C}^n)$, $\mathfrak{J}T^{\perp}M$ is a vector bundle of real rank 6 over M. We consider a Kähler hypersurface M of $G_2(\mathbf{C}^n)$ satisfying the property that $\mathfrak{J}T^{\perp}M$ is a subbundle of the tangent bundle TM of M, i.e, $\mathfrak{J}T^{\perp}M \subset TM$. The Kähler hypersurface $M_{11,l}$ satisfies this condition. In [9], the author showed that if M is compact, then the first eigenvalue λ_1 of the Laplacian satisfies $\lambda_1 \leqq c(n - \frac{n-1}{2n-5})$. The equality holds if and only if $n = 4$ and M is congruent to $M_{11,2} = Q^3$.

One of the simplest questions is as follows: *What is M satisfying $\mathfrak{J}T^{\perp}M \subset TM$?* Without the assumption of homogeneity, we shall show the following result.

THEOREM 1.1. *If an Einstein Kähler hypersurface M of $G_2(\mathbf{C}^n)$ satisfies the condition $\mathfrak{J}T^{\perp}M \subset TM$, then n is even and M is locally congruent to $M_{11,\,n/2}$.*

The author wishes to thank Professors K. Ogiue and Y. Ohnita for many valuable comments and suggestions.

NOTATIONS. $M_{r,s}(\mathbf{C})$ denotes the set of all $r \times s$ matrices with entries in $\mathbf{C}$, and $M_r(\mathbf{C})$ stands for $M_{r,r}(\mathbf{C})$. I_r and O_r denote the identity r-matrix and the zero r-matrix.

2. Preliminaries

In this section, we review well-known geometries of complex Grassmann manifolds of 2-planes. For details, see [7] and [2].

Let $M_2(\mathbf{C}^n)$ be the complex Stiefel manifold which is the set of all unitary 2-systems of $\mathbf{C}^n$, i.e.,

$$M_2(\mathbf{C}^n) = \{Z \in M_{n,2}(\mathbf{C}) \mid Z^*Z = I_2\}.$$

The complex 2-plane Grassmann manifold $G_2(\mathbf{C}^n)$ is defined by

$$G_2(\mathbf{C}^n) = M_2(\mathbf{C}^n)/U(2).$$

The origin $\tilde{o}$ of $G_2(\mathbf{C}^n)$ is defined by $\pi(Z_0)$, where $Z_0 = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$ is an element of $M_2(\mathbf{C}^n)$, and $\pi : M_2(\mathbf{C}^n) \to G_2(\mathbf{C}^n)$ is the natural projection.

The left action of the unitary group $\tilde{G} = SU(n)$ on $G_2(\mathbf{C}^n)$ is transitive, and the isotropy subgroup at the origin $\tilde{o}$ is

$$\begin{aligned} \tilde{K} &= S(\,U(2)\cdot U(n-2)\,) \\ &= \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \,\middle|\, U_1 \in U(2),\ U_2 \in U(n-2),\ \det U_1 \det U_2 = 1 \right\}, \end{aligned}$$

so that $G_2(\mathbf{C}^n)$ is identified with a homogeneous space $\tilde{M} = \tilde{G}/\tilde{K}$.

Set $\tilde{\mathfrak{g}} = \mathfrak{su}(n)$ and

$$\begin{aligned} \tilde{\mathfrak{k}} &= \mathbf{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(n-2) \\ &= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + a \begin{pmatrix} -\frac{1}{2}\sqrt{-1}I_2 & 0 \\ 0 & \frac{1}{n-2}\sqrt{-1}I_{n-2} \end{pmatrix} \,\middle|\, a \in \mathbf{R},\ \begin{array}{c} u_1 \in \mathfrak{su}(2) \\ u_2 \in \mathfrak{su}(n-2) \end{array} \right\}. \end{aligned}$$

Then $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ are the Lie algebras of $\tilde{G}$ and $\tilde{K}$, respectively. Define a linear subspace $\tilde{\mathfrak{m}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \,\middle|\, \xi \in M_{n-2,2}(\mathbf{C}) \right\}.$$

Then $\tilde{\mathfrak{m}}$ is identified with the tangent space $T_{\tilde{o}}(G_2(\mathbf{C}^n))$. The $\tilde{G}$-invariant complex structure J of $G_2(\mathbf{C}^n)$ and the $\tilde{G}$-invariant Kähler metric $\tilde{g}_c$ of $G_2(\mathbf{C}^n)$ of the maximal holomorphic sectional curvature c are given by

$$J\begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix},$$

$$\tilde{g}_c(X,\ Y) = -\frac{2}{c} trXY\,, \quad X, Y \in \tilde{\mathfrak{m}}\,. \tag{2.1}$$

Notice that $\tilde{g}_c$ satisfies

$$\tilde{g}_c = -\frac{2}{c}\frac{1}{2n}B_{\tilde{\mathfrak{g}}} = -\frac{2}{c}\frac{L(\tilde{\mathfrak{g}})}{2}B_{\tilde{\mathfrak{g}}} \tag{2.2}$$

on $\tilde{\mathfrak{m}}$, where $B_{\tilde{\mathfrak{g}}}$ is the Killing form of $\tilde{\mathfrak{g}}$, and $L(\tilde{\mathfrak{g}})$ is the squared length of the longest root of $\tilde{\mathfrak{g}}$ relative to the Killing form.

We denote by X^* an vector field on $\tilde{M}$ generated by $X \in \tilde{\mathfrak{g}}$, i.e.,

$$(X^*)_p = \left[\frac{d}{dt}\exp tX \cdot p\right]_{t=0}, \quad p = g\tilde{o} \in \tilde{M}\,, \quad g \in \tilde{G}\,.$$

The Riemannian connection $\tilde{\nabla}$ is described in terms of the Lie derivative as follows:

$$(L_{X^*} - \tilde{\nabla}_{X^*})_{\tilde{o}}\tilde{Y} = \begin{cases} -ad(X)\tilde{Y}_{\tilde{o}}\,, & \text{if } X \in \tilde{\mathfrak{k}}\,, \\ 0\,, & \text{if } X \in \tilde{\mathfrak{m}}\,, \end{cases} \tag{2.3}$$

where $\tilde{Y}$ is a vector field on $\tilde{M}$.

The complex 2-plane Grassmann manifold $G_2(\mathbf{C}^n)$ admits another geometric structure named the quaternionic Kähler structure $\mathfrak{J}$. $\mathfrak{J}$ is a $\tilde{G}$-invariant subbundle of $End(T(G_2(\mathbf{C}^n)))$ of rank 3, where $End(T(G_2(\mathbf{C}^n)))$ is the $\tilde{G}$-invariant vector bundle of all linear endmorphisms of the tangent bundle $T(G_2(\mathbf{C}^n))$. Under the identification with $T_{\tilde{o}}(G_2(\mathbf{C}^n))$ and $\tilde{\mathfrak{m}}$, the fiber $\mathfrak{J}_{\tilde{o}}$ at the origin $\tilde{o}$ is given by

$$\mathfrak{J}_{\tilde{o}} = \{J_{\tilde{\varepsilon}} = ad(\tilde{\varepsilon}) \mid \tilde{\varepsilon} \in \tilde{\mathfrak{k}}_q\}\,,$$

where $\tilde{\mathfrak{k}}_q$ is an ideal of $\tilde{\mathfrak{k}}$ defined by

$$\tilde{\mathfrak{k}}_q = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, u_1 \in \mathfrak{su}(2) \right\} \cong \mathfrak{su}(2)\,.$$

Define a basis $\{\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\}$ of $\mathfrak{su}(2)$ by

$$\varepsilon_1=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix},\quad \varepsilon_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \varepsilon_3=\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}.$$

Then ε_1, ε_2 and ε_3 satisfy

$$[\varepsilon_1,\ \varepsilon_2]=2\varepsilon_3\,,\quad [\varepsilon_2,\ \varepsilon_3]=2\varepsilon_1\,,\quad [\varepsilon_3,\ \varepsilon_1]=2\varepsilon_2\,.$$

Set $\tilde{\varepsilon}_i=\begin{pmatrix}\varepsilon_i & 0\\ 0 & 0\end{pmatrix}$ and $J_i=J_{\tilde{\varepsilon}_i}$ for $i=1,2,3$. Then the basis $\{J_1,\ J_2,\ J_3\}$ is a canonical basis of $\mathfrak{J}_{\tilde{o}}$ satisfying

$$J_i^2=-id_{\tilde{\mathfrak{m}}}\quad\text{for } i=1,2,3\,,$$
$$J_1J_2=-J_2J_1=J_3\,,\quad J_2J_3=-J_3J_2=J_1\,,\quad J_3J_1=-J_1J_3=J_2\,,$$
$$\tilde{g}_c(J_iX,\ J_iY)=\tilde{g}_c(X,\ Y)\,,\quad \text{for } X,Y\in\tilde{\mathfrak{m}} \text{ and } i=1,2,3\,.$$

Since J is given by

$$J=ad(\tilde{\varepsilon}_{\mathbf{C}})\,,\quad \tilde{\varepsilon}_{\mathbf{C}}=\frac{2(n-2)}{n}\begin{pmatrix}-\frac{1}{2}\sqrt{-1}I_2 & 0\\ 0 & \frac{1}{n-2}\sqrt{-1}I_{n-2}\end{pmatrix}$$

on $\mathfrak{m}$, and since $\tilde{\varepsilon}_{\mathbf{C}}$ is an element of the center of $\tilde{\mathfrak{k}}$, J is commutable with $\mathfrak{J}$. Moreover, the property

$$trJJ'=0 \tag{2.4}$$

holds for any $J'\in\mathfrak{J}$.

In [2], J. Berndt showed that the curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$\begin{aligned}\tilde{R}(X,\ Y)Z=\frac{c}{8}\Big[&\tilde{g}_c(Y,Z)X-\tilde{g}_c(X,Z)Y\\ &+\tilde{g}_c(JY,Z)JX-\tilde{g}_c(JX,Z)JY+2\tilde{g}_c(X,JY)JZ\\ &+\sum_{k=1}^{3}\{\tilde{g}_c(J_kY,Z)J_kX-\tilde{g}_c(J_kX,Z)J_kY+2\tilde{g}_c(X,J_kY)J_kZ\}\\ &+\sum_{k=1}^{3}\{\tilde{g}_c(JJ_kY,Z)JJ_kX-\tilde{g}_c(JJ_kX,Z)JJ_kY\}\Big]\end{aligned}\tag{2.5}$$

for any vector fields X, Y and Z of $\tilde{M}$.

Let $(M,\ g)$ be a Riemannian submanifold of $\tilde{M}$. Denote by ∇ the Riemannian connection of M, and by σ, A and $\nabla^{\perp}$ the second fundamental form, the Weingarten map and the normal connection of M in $G_2(\mathbf{C}^{2l})$ respectively. We have the Gauss' formula and the Weingarten's formula are:

$$\tilde{\nabla}_XY=\nabla_XY+\sigma(X,Y)\,,\quad \tilde{\nabla}_X\xi=-A_\xi X+\nabla_X^{\perp}\xi\,,\tag{2.6}$$

where X, Y and Z are tangent vector fields and ξ is a normal vector field. Moreover, we see

$$g(A_\xi X,\ Y) = \tilde{g}_c(\sigma(X, Y),\ \xi)\,.$$

If M is a Kähler submanifold of $\tilde{M}$, then the following hold.

$$\sigma(X,\ JY) = \sigma(JX,\ Y) = J\sigma(X,\ Y)\,, \tag{2.7}$$

$$A_\xi J = -JA_\xi = -A_{J\xi}\,. \tag{2.8}$$

M is called a *quaternionic submanifold*, if the tangent space T_pM is invariant under the action of $\mathfrak{J}$ for each p in M. M is called a *totally real submanifold*, if JT_pM is a subspace of the normal space $T_p^\perp M$ for each p in M.

3. The second fundamental form of $Sp(l)/U(2)\cdot Sp(l-2)$ in $G_2(\mathbf{C}^{2l})$

In this section, we will consider a Kähler C-space $M_{11,\,l} = Sp(l)/U(2)\cdot Sp(l-2)$ as a Kähler submanifold of $G_2(\mathbf{C}^{2l})$ (cf. [3], [12]).

First, we study an intrinsic geometry of $M_{11,\,l}$. Let us set

$$\begin{aligned} G &= Sp(l)\\ &= \left\{ g\in SU(2l) \;\middle|\; {}^tg\begin{pmatrix} 0 & -I_l\\ I_l & 0\end{pmatrix} g = \begin{pmatrix} 0 & -I_l\\ I_l & 0\end{pmatrix}\right\}\\ &= \left\{ \begin{pmatrix} A & -\overline{C}\\ C & \overline{A}\end{pmatrix}\in SU(2l) \;\middle|\; A, C\in M_l(\mathbf{C})\right\}\end{aligned}$$

and

$$\begin{aligned} K &= U(2)\cdot Sp(l-2)\\ &= \left\{ \begin{pmatrix} A & 0 & 0 & 0\\ 0 & A' & 0 & -\overline{C'}\\ 0 & 0 & \overline{A} & 0\\ 0 & C' & 0 & \overline{A'}\end{pmatrix} \;\middle|\; \begin{matrix} A\in U(2)\,, \quad A', C'\in M_{l-2}(\mathbf{C})\,,\\ \begin{pmatrix} A' & -\overline{C'}\\ C' & \overline{A'}\end{pmatrix}\in Sp(l-2)\end{matrix}\right\}.\end{aligned}$$

Then K is a closed subgroup of G. The Lie algebra $\mathfrak{g}$, the complexification $\mathfrak{g}^{\mathbf{C}}$ and the Lie algebra $\mathfrak{k}$ are given by

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sp}(l)\\ &= \left\{ \begin{pmatrix} A & -\overline{C}\\ C & \overline{A}\end{pmatrix} \;\middle|\; \begin{matrix} A, C\in M_l(\mathbf{C}),\\ A^* = -A,\ {}^tC = C\end{matrix}\right\},\\ \mathfrak{g}^{\mathbf{C}} &= \mathfrak{sp}(l, \mathbf{C})\\ &= \left\{ \begin{pmatrix} A & B\\ C & -{}^tA\end{pmatrix} \;\middle|\; \begin{matrix} A, B, C\in M_l(\mathbf{C}),\\ {}^tB = B,\ {}^tC = C\end{matrix}\right\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{k} &= \mathfrak{u}(2) + \mathfrak{sp}(l-2) \\ &= \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -\overline{C'} \\ 0 & 0 & \overline{A} & 0 \\ 0 & C' & 0 & \overline{A'} \end{pmatrix} \;\middle|\; \begin{matrix} A \in M_2(\mathbf{C}), \\ A', C' \in M_{l-2}(\mathbf{C}), \\ A^* = -A,\ A'^* = -A',\ {}^tC' = C' \end{matrix} \right\} .\end{aligned}$$

$\mathfrak{g}$ is a compact semisimple Lie algebra of type C_l.

For $x, y \in M_{l-2,2}(\mathbf{C})$ and $z \in M_2(\mathbf{C})$ with ${}^tz = z$, define

$$\eta(x, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & {}^ty & 0 & -{}^tx \\ y & 0 & 0 & 0 \end{pmatrix}$$

and

$$X(x, y, z) = \eta(x, y, z) - \eta(x, y, z)^* .$$

Define a subspace $\mathfrak{m}$ of $\mathfrak{g}$ by

$$\mathfrak{m} = \{X(x, y, z)\} ,$$

then $\mathfrak{m}$ is an $ad(\mathfrak{k})$-invariant subspace and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} .$$

$\mathfrak{m}$ is identified with the tangent space $T_o(M_{11,l})$. Set

$$\mathfrak{m}^+ = \{\eta(x, y, z)\} , \quad \mathfrak{m}^- = \{{}^t\eta(x, y, z)\} ,$$

then $\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$ and $\mathfrak{m}^\pm$ are $\pm\sqrt{-1}$-eigenspaces of the complex structure J of $M_{11,l}$.

For $X = X(x, y, z),\ X' = X(x', y', z') \in \mathfrak{m}$, define a Hermitian inner product g_o on $\mathfrak{m}$ by

$$g_o(X, X') = \frac{4}{c}\mathrm{Re}\, tr(x'^* x + y'^* y + \overline{z'} z) ,$$

then g_o is $ad(\mathfrak{k})$-invariant, so that g_o induces a G-invariant Kähler metric g on $M_{11,l}$. $(M_{11,l},\ J,\ g)$ is an Einstein Kähler manifold.

The natural inclusion $G \to \tilde{G}$ defines a G-equivariant Kähler immersion φ of $M_{11,l}$ into $\tilde{M} = G_2(\mathbf{C}^{2l})$, by $\varphi(g \cdot K) = g \cdot \tilde{K},\ g \in G$. The complex codimension of φ is 1, so that $M_{11,l}$ is a complex hypersurface of $G_2(\mathbf{C}^{2l})$.

For $X = X(x, y, z) \in \mathfrak{m}$, let's set

$$X_{\tilde{\mathfrak{k}}}(x, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{y} & 0 \\ 0 & {}^t y & 0 & -{}^t x \\ 0 & 0 & \bar{x} & 0 \end{pmatrix}, \quad X_{\tilde{\mathfrak{m}}}(x, y, z) = \begin{pmatrix} 0 & -x^* & -z^* & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix}.$$

Denote by φ_*, the differential of φ. Then, the image of the tangent space $T_o(M_{11,l})$ is given by

$$\varphi_{*o} T_o(M_{11,l}) = \varphi_{*o}\mathfrak{m} = \{X_{\tilde{\mathfrak{m}}}(x, y, z)\} \subset \tilde{\mathfrak{m}} = T_{\tilde{o}}(G_2(\mathbf{C}^n)). \tag{3.1}$$

For $z \in M_2(\mathbf{C})$ with ${}^t z = -z$, set

$$\xi(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we can identify the normal space $T_o^\perp(M_{11,l})$ with the subspace

$$\mathfrak{m}^\perp = \{\xi(z)\} \tag{3.2}$$

of $\tilde{\mathfrak{m}}$. Since φ is G-equivariant, the normal space at $g \cdot o$ is given by

$$T_{g\cdot o}^\perp(M_{11,l}) = \left\{ \left[\frac{d}{dt} g\exp(t\xi) \cdot \tilde{o}\right]_{t=0} \,\middle|\, \xi \in \mathfrak{m}^\perp \right\}.$$

For $X = X(x, y, z) \in T_o(M_{11,l})$, the curve $c(t) = \exp(tX) \cdot \tilde{o}$ is a curve in $M_{11,l}$, so that the vector field X^* generated by X is tangent to $M_{11,l}$. Define a unit normal vector field along $c(t)$ by

$$\xi(t) = (\exp tX)_{*\tilde{o}}\xi_0, \quad \xi_0 = \xi(z_0), \quad z_0 = \sqrt{\frac{c}{8}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(2.3) implies

$$(L_{X^*}\xi(t) - \tilde{\nabla}_{X^*}\xi(t))_{\tilde{o}} = -[X_{\tilde{\mathfrak{k}}}(x, y, z), \ \xi_0].$$

By the definition of the Lie derivative,

$$(L_{X^*}\xi(t))_{\tilde{o}} = [X^*, \ \xi(t)]_{\tilde{o}} = \left[\frac{d}{dt}\exp(-tX)_{*c(t)}\xi(t)\right]_{t=0} = \left[\frac{d}{dt}\xi_0\right]_{t=0} = 0,$$

so that we obtain

$$\tilde{\nabla}_{\varphi_{*o}X}\xi(t) = [X_{\tilde{\mathfrak{k}}}(x, y, z), \ \xi_0] = \begin{pmatrix} 0 & -z_0{}^t y & 0 & z_0{}^t x \\ -\bar{y}z_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{x}z_0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{m}}.$$

From (3.1) and (3.2), we obtain the following.

PROPOSITION 3.1. *$\tilde{\nabla}_{\varphi_{*o}X}\xi(t)$ is tangent to $M_{11,l}$. Moreover, the unit normal vector field $\xi(t)$ is parallel at o, and the Weingarten map satisfies*

$$A_{\xi_0}X(x, y, z) = X(\bar{y}z_0, -\bar{x}z_0, 0) \tag{3.3}$$

for any $X(x, y, z) \in \mathfrak{m}$.

Define three subspaces of $T_o(M_{11,l})$ by

$$V_0(o, \xi_0) = \{X(0, 0, z) \mid {}^t z = z,\ z \in M_2(\mathbf{C})\},$$

$$V_+(o, \xi_0) = \{X(x, y, 0) \mid x = (x_1, x_2),\ y = (-\overline{x_2}, \overline{x_1}),\ x_i \in M_{l-2,1}(\mathbf{C})\}$$

and

$$V_-(o, \xi_0) = \{X(x, y, 0) \mid x = (x_1, x_2),\ y = (\overline{x_2}, -\overline{x_1}),\ x_i \in M_{l-2,1}(\mathbf{C})\}.$$

We have the eigenspace decomposition of the tangent space $T_p(M_{11,l})$ as follows.

PROPOSITION 3.2. *For any point $p \in M_{11,l}$ and any unit normal vector $\xi \in T_p^{\perp}(M_{11,l})$, there exist three subspaces V_0, V_+ and V_- of $T_p(M_{11,l})$, such that the following properties hold.*

(1) *V_0 is a J-invariant 0-eigenspace of A_ξ satisfying*

$$V_0 = \mathfrak{J}_p T_p^{\perp}(M_{11,l}).$$

(2) *$V_\pm$ are $\mathfrak{J}$-invariant $\pm\sqrt{\frac{c}{8}}$-eigenspaces of A_ξ satisfying*

$$JV_+ = V_-.$$

(3) *The eigenspace decomposition*

$$T_p(M_{11,l}) = V_0 \oplus V_+ \oplus V_-$$

holds.

PROOF. In the case that $p = o$ and $\xi = \xi_0$, put $V_0 = V_0(o, \xi_0)$ and $V_\pm = V_\pm(o, \xi_0)$. By simple calculation of matrices, we can easily see that V_0, V_+ and V_- satisfy the properties of this proposition.

In the case that $p = o$ and ξ is arbitrary, (2.8) implies this proposition.

Since the structures J and $\mathfrak{J}$ are $\tilde{G}$-invariant, and since the immersion φ is G-equivariant, this proposition holds for arbitrary p and ξ. □

4. A second fundamental form of an Einstein Kähler hypersurface

In this section, we study an Einstein Kähler hypersurface of $G_2(\mathbf{C}^n)$, and under some assumption, determine its second fundamental form.

Let M be a Kähler hypersurface of $\tilde{M} = G_2(\mathbf{C}^n)$. The complex dimension m of M is equal to $2n-5$. Let p be any fixed point of M, and ξ be a local unit normal vector field around p, and set $\xi_1 = \xi,\ \xi_2 = J\xi$, so that $\{\xi_1, \xi_2\}$ is a local orthonormal frame field of the normal bundle $T^{\perp}M$.

Denote by R the curvature tensor field of M. Then we have the Gauss equation

$$(4.1)\qquad g(R(X,\ Y)Z,\ W) = \sum_{\alpha=1}^{2} \left\{ g(A_{\xi_\alpha}X,\ W)g(A_{\xi_\alpha}Y,\ Z) - g(A_{\xi_\alpha}X,\ Z)g(A_{\xi_\alpha}Y,\ W) \right\} + \tilde{g}_c(\tilde{R}(X,\ Y)Z,\ W)$$

for any tangent vector fields X, Y, Z and W of M.

For any vector field X along M, denote by X^T and $X^{\perp}$, the tangential part of X and the normal part of X, respectively. Then, we obtain the following.

LEMMA 4.1. *The Ricci curvature tensor Ric satisfies*

$$(4.2)\qquad \begin{aligned} Ric(Y,\ Z) =& -2g(A_\xi^2 Y,\ Z) \\ &+ \frac{c}{8}\Bigg\{(2m+2)g(Y,\ Z) + 3\sum_{k=1}^{3} g((J_kY)^T,\ (J_kZ)^T) \\ &- \sum_{k=1}^{3} g((JJ_kY)^T,\ (JJ_kZ)^T) + 2\sum_{k=1}^{3} \tilde{g}_c(J\xi,\ J_k\xi)\,\tilde{g}_c(JJ_kY,\ Z)\Bigg\} \end{aligned}$$

for any tangent vector fields Y and Z.

PROOF. Let $\{e_1, \cdots, e_{2m}\}$ be a local orthonormal basis of TM. Note that A_{ξ_α} is symmetric. Moreover, from (2.8), $trA_{\xi_\alpha} = 0$ and $A_{\xi_1}^2 = A_{\xi_2}^2 = A_\xi^2$. So we get, from (4.1),

$$(4.3)\qquad \begin{aligned} Ric(Y,\ Z) &= \sum_{i=1}^{2m} g(R(e_i,\ Y)Z,\ e_i) \\ &= \sum_{i=1}^{2m}\sum_{\alpha=1}^{2} \left\{ g(A_{\xi_\alpha}e_i,\ e_i)g(A_{\xi_\alpha}Y,\ Z) - g(A_{\xi_\alpha}e_i,\ Z)g(A_{\xi_\alpha}Y,\ e_i)\right\} \\ &\quad + \sum_{i=1}^{2m} \tilde{g}_c(\tilde{R}(e_i,\ Y)Z,\ e_i) \\ &= \sum_{\alpha=1}^{2} \left\{ (trA_{\xi_\alpha})\ g(A_{\xi_\alpha}Y,\ Z) - g(A_{\xi_\alpha}Y,\ A_{\xi_\alpha}Z)\right\} + \sum_{i=1}^{2m} \tilde{g}_c(\tilde{R}(e_i,\ Y)Z,\ e_i) \end{aligned}$$

$$= -2g(A_\xi^2 Y,\ Z) + \sum_{i=1}^{2m} \tilde{g}_c(\tilde{R}(e_i,\ Y)Z,\ e_i)\,.$$

From (2.5), we can see that

(4.4)

$$\begin{aligned}
&\sum_{i=1}^{2m}\tilde{g}_c(\tilde{R}(e_i,\ Y)Z,\ e_i)\\
&\quad= \frac{c}{8}\sum_{i=1}^{2m}\Big[\tilde{g}_c(e_i,\ e_i)\tilde{g}_c(Y,\ Z) - \tilde{g}_c(e_i,\ Z)\tilde{g}_c(Y,\ e_i)\\
&\qquad + \tilde{g}_c(Je_i,\ e_i)\tilde{g}_c(JY, Z) - \tilde{g}_c(Je_i, Z)\tilde{g}_c(JY,\ e_i) + 2\tilde{g}_c(e_i,\ JY)\tilde{g}_c(JZ,\ e_i)\\
&\qquad + \sum_{k=1}^{3}\big\{\tilde{g}_c(J_k e_i,\ e_i)\tilde{g}_c(J_k Y, Z) - \tilde{g}_c(J_k e_i, Z)\tilde{g}_c(J_k Y,\ e_i) + 2\tilde{g}_c(e_i,\ J_k Y)\tilde{g}_c(J_k Z,\ e_i)\big\}\\
&\qquad + \sum_{k=1}^{3}\big\{\tilde{g}_c(JJ_k e_i,\ e_i)\tilde{g}_c(JJ_k Y,\ Z) - \tilde{g}_c(JJ_k e_i,\ Z)\tilde{g}_c(JJ_k Y,\ e_i)\big\}\Big]\\
&\quad= \frac{c}{8}\Big[(2m+2)g(Y,\ Z) + 3\sum_{k=1}^{3}\tilde{g}_c\Big(\sum_{i=1}^{2m}\tilde{g}_c(J_k Z,\ e_i)e_i,\ J_k Y\Big)\\
&\qquad + \sum_{k=1}^{3}\sum_{i=1}^{2m}\tilde{g}_c(JJ_k e_i,\ e_i)\tilde{g}_c(JJ_k Y,\ Z) - \sum_{k=1}^{3}\tilde{g}_c\Big(\sum_{i=1}^{2m}\tilde{g}_c(JJ_k Z,\ e_i)e_i,\ JJ_k Y\Big)\Big]\\
&\quad= \frac{c}{8}\Big[(2m+2)g(Y,\ Z) + 3\sum_{k=1}^{3}\tilde{g}_c((J_k Z)^T,\ J_k Y)\\
&\qquad + \sum_{k=1}^{3}\sum_{i=1}^{2m}\tilde{g}_c(JJ_k e_i,\ e_i)\tilde{g}_c(JJ_k Y,\ Z) - \sum_{k=1}^{3}\tilde{g}_c((JJ_k Z)^T,\ JJ_k Y)\Big].
\end{aligned}$$

Since $\{e_1, \cdots, e_{2m}\ \xi,\ J\xi\}$ is a local orthonormal frame of $T\tilde{M}$, (2.4) implies

$$\sum_{i=1}^{2m}\tilde{g}_c(JJ_k e_i,\ e_i) = -\tilde{g}_c(JJ_k\xi,\ \xi) - \tilde{g}_c(JJ_k(J\xi),\ J\xi) = 2\tilde{g}_c(J\xi,\ J_k\xi)\,. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we see that (4.2) holds. □

From now on, we assume that $\mathfrak{J}T^\perp M$ is a vector subbundle of the tangent bundle TM, i.e,

$$\mathfrak{J}T^\perp M \subset TM\,. \tag{4.6}$$

This condition is equivalent to the condition that $J_p\nu \perp \mathfrak{J}_p\nu$, where p is any point of M and ν is any normal vector at p.

Set $V_0 = \mathfrak{J}T^{\perp}M$. For any unit normal vector ξ, $\{J_1\xi, J_2\xi, J_3\xi, JJ_1\xi, JJ_2\xi, JJ_3\xi\}$ is an orthonormal basis of V_0, i.e.,

$$V_0 = Span_{\mathbf{R}}\{J_1\xi, J_2\xi, J_3\xi, JJ_1\xi, JJ_2\xi, JJ_3\xi\}, \tag{4.7}$$

so that V_0 is J-invariant. Let's define V be the orthogonal complement of V_0 in TM. Then we have an orthogonal decomposition

$$TM = V_0 \oplus V .$$

It is easy to see that V is J-invariant and $\mathfrak{J}$-invariant.

For a fiber bundle $\mathfrak{F}$, denote by $\Gamma(\mathfrak{F})$ the linear space of all smooth sections of $\mathfrak{F}$.

LEMMA 4.2.

(1) *V_0 is a subspace of 0-eigenspace of A_ξ, i.e., $A_\xi Y = 0$ for any $Y \in \Gamma(V_0)$.*

(2) *For any $X \in \Gamma(TM)$, $Y \in \Gamma(V)$ and $J' \in \Gamma(\mathfrak{J})$,*

$$g(\nabla_X Y, J'\xi) = -g(A_\xi X, J'Y). \tag{4.8}$$

PROOF. For any $X \in \Gamma(TM)$ and $J' \in \Gamma(\mathfrak{J})$, since $J'\xi$ is a section of V_0, (2.6) implies

$$\begin{aligned} \nabla_X(J'\xi) + \sigma(X, J'\xi) &= \tilde{\nabla}_X(J'\xi) = (\tilde{\nabla}_X J')\xi + J'(\tilde{\nabla}_X\xi) \\ &= (\tilde{\nabla}_X J')\xi - J'A_\xi X + J'\nabla^{\perp}_X\xi . \end{aligned} \tag{4.9}$$

Since $\mathfrak{J}$ is parallel, $\tilde{\nabla}_X J' \in \mathfrak{J}$. Thus, under our assumption (4.6), we see that $(\tilde{\nabla}_X J')\xi$ and $J'\nabla^{\perp}_X\xi$ are tangent to M. Therefore, the normal component of (4.9) is given by

$$\begin{aligned} \sigma(X, J'\xi) &= -\tilde{g}_c\left(J'A_\xi X, \xi\right)\xi - \tilde{g}_c\left(J'A_\xi X, J\xi\right)J\xi \\ &= g\left(A_\xi X, J'\xi\right)\xi + g\left(A_\xi X, J'J\xi\right)J\xi \\ &= \tilde{g}_c(\sigma(X, J'\xi), \xi)\xi + \tilde{g}_c(\sigma(X, J'J\xi), \xi)J\xi , \end{aligned}$$

which, from (2.7), is equivalent to

$$\tilde{g}_c(\sigma(X, J'\xi), \xi)\xi - \tilde{g}_c(\sigma(X, J'\xi), J\xi)J\xi ,$$

so that we have

$$\tilde{g}_c(\sigma(X, J'\xi), J\xi) = 0 . \tag{4.10}$$

Exchanging X for $JX \in \Gamma(TM)$, we get $\tilde{g}_c(\sigma(JX, J'\xi), J\xi) = 0$, so that

$$\tilde{g}_c(\sigma(X, J'\xi), \xi) = 0 . \tag{4.11}$$

From (4.10) and (4.11), we get $\sigma(X, J'\xi) = 0$. Therefore, (2.7) and (4.7) imply $\sigma(X, Y) = 0$ for any $Y \in \Gamma(V_0)$, namely, $A_\xi Y = 0$.

Next, we consider the V-component of (4.9). The assumption (4.6) implies that $(\tilde{\nabla}_X J')\xi$ and $J'\nabla^{\perp}_X\xi$ are sections of V_0, so that, for any $Y \in \Gamma(V)$, we get

$$g\left(\nabla_X(J'\xi),\ Y\right) = -\tilde{g}_c\left(J'A_\xi X,\ Y\right)\,.$$

Since $J'\xi \perp Y$, this implies (4.8) immediately. □

For any tangent vector field X of M, denote by X_0 and X_V, the V_0-component of X and V-component of X, respectively. Then, we obtain the following.

LEMMA 4.3. *Under the assumption* (4.6), *the Ricci curvature tensor Ric satisfies*

$$Ric(Y,\ Z) = -2g(A_\xi^2 Y_V,\ Z_V) + \frac{c}{8}\left\{(4n-4)g(Y_0,\ Z_0) + (4n-2)g(Y_V,\ Z_V)\right\} \tag{4.12}$$

for any tangent vector fields Y and Z.

PROOF. Lemma 4.2 (1) implies that

$$g(A_\xi^2 Y,\ Z) = g(A_\xi^2 Y_V,\ Z) = g(A_\xi^2 Y_V,\ Z_V)\,. \tag{4.13}$$

Since V is $\mathfrak{J}$-invariant, $J_k Y_V$ is a section of V, so that

$$\begin{aligned}(J_kY)^{\perp} &= (J_kY_0)^{\perp}\\ &= \tilde{g}_c(J_kY_0,\ \xi)\,\xi + \tilde{g}_c(J_kY_0,\ J\xi)\,J\xi\\ &= -g(Y_0,\ J_k\xi)\,\xi - g(Y_0,\ J_kJ\xi)\,J\xi\,.\end{aligned}$$

Then, we get

$$\begin{aligned}g\big((J_kY)^T, (J_kZ)^T\big) &= \tilde{g}_c(J_kY,\ J_kZ) - \tilde{g}_c\big((J_kY)^{\perp},\ (J_kZ)^{\perp}\big)\\ &= g(Y,\ Z) - g(Y_0,\ J_k\xi)\,g(Z_0,\ J_k\xi) - g(Y_0,\ J_kJ\xi)\,g(Z_0,\ J_kJ\xi)\,,\end{aligned}$$

so that, from (4.7), we have

$$\begin{aligned}\sum_{k=1}^{3} g\big((J_kY)^T,\ (J_kZ)^T\big) &= 3g(Y,\ Z) - g(Y_0,\ Z_0)\\ &= 2g(Y_0,\ Z_0) + 3g(Y_V,\ Z_V)\,.\end{aligned} \tag{4.14}$$

Exchanging Y and Z for JY and JZ respectively, we get

$$\sum_{k=1}^{3} g\big((JJ_kY)^T,\ (JJ_kZ)^T\big) = 2g(Y_0,\ Z_0) + 3g(Y_V,\ Z_V)\,. \tag{4.15}$$

Since $J\xi \perp J_k\xi$, combining (4.2), (4.13), (4.14) and (4.15), we see that (4.12) holds. □

In the next stage, we consider the Codazzi's equation

$$g((\nabla_X A)_\xi Y - (\nabla_Y A)_\xi X,\ Z) = \tilde{g}_c(\tilde{R}(X,\ Y)Z,\ \xi) \tag{4.16}$$

for any tangent vector fields X, Y and Z of M.

Let μ be a non-zero eigenvalue of A_ξ, and Y be an eigenvector corresponding to μ. We can assume that μ is a local smooth function on M, and Y is a local smooth section of TM. Then, for any $X \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_X A)_\xi Y &= \nabla_X(A_\xi Y) - A_{\nabla^\perp_X \xi} Y - A_\xi(\nabla_X Y)\\ &= d\mu(X)Y + \mu\nabla_X Y - A_{\nabla^\perp_X \xi} Y - A_\xi(\nabla_X Y)\,,\end{aligned}$$

so that, from Lemma 4.2 (1), since Y is a local section of V, we see

$$\begin{aligned}g\big((\nabla_X A)_\xi Y,\ J'\xi\big) &= \mu g\big(\nabla_X Y,\ J'\xi\big) - g\big(A_{\nabla^\perp_X \xi} Y,\ J'\xi\big) - g(A_\xi(\nabla_X Y),\ J'\xi)\\ &= \mu g(\nabla_X Y,\ J'\xi) - g\big(Y,\ A_{\nabla^\perp_X \xi} J'\xi\big) - g(\nabla_X Y,\ A_\xi J'\xi)\\ &= \mu g(\nabla_X Y,\ J'\xi)\end{aligned}$$

for any $J' \in \Gamma(\mathfrak{J})$. By Lemma 4.2 (2), we see

$$g\big((\nabla_X A)_\xi Y,\ J'\xi\big) = -\mu g(A_\xi X,\ J'Y)\,.$$

If X is also an eigenvector of A_ξ corresponding to a non-zero eigenvalue λ, we get

$$g\big((\nabla_X A)_\xi Y,\ J'\xi\big) = -\lambda\mu g(X,\ J'Y) = \lambda\mu g(J'X,\ Y) \tag{4.17}$$

and

$$g\big((\nabla_Y A)_\xi X,\ J'\xi\big) = \lambda\mu g(J'Y,\ X) = -\lambda\mu g(J'X,\ Y)\,. \tag{4.18}$$

On the other hand, from (2.5), we can see that, for above X and Y,

$$\begin{aligned}&\tilde{g}_c(\tilde{R}(X,\ Y)J'\xi,\ \xi)\\ &= \frac{c}{8}\Big[\tilde{g}_c(X,\ \xi)\tilde{g}_c(Y,\ J'\xi) - \tilde{g}_c(X,\ J'\xi)\tilde{g}_c(Y,\ \xi)\\ &\quad + \tilde{g}_c(JX,\ \xi)\tilde{g}_c(JY,\ J'\xi) - \tilde{g}_c(JX,\ J'\xi)\tilde{g}_c(JY,\ \xi) + 2\tilde{g}_c(X,\ JY)\tilde{g}_c(JJ'\xi,\ \xi)\\ &\quad + \sum_{k=1}^{3}\big\{\tilde{g}_c(J_kX,\ \xi)\tilde{g}_c(J_kY,\ J'\xi) - \tilde{g}_c(J_kX,\ J'\xi)\tilde{g}_c(J_kY,\ \xi)\\ &\qquad\qquad + 2\tilde{g}_c(X,\ J_kY)\tilde{g}_c(J_kJ'\xi,\ \xi)\big\}\\ &\quad + \sum_{k=1}^{3}\big\{\tilde{g}_c(JJ_kX,\ \xi)\tilde{g}_c(JJ_kY,\ J'\xi) - \tilde{g}_c(JJ_kX,\ J'\xi)\tilde{g}_c(JJ_kY,\ \xi)\big\}\Big]\\ &= \frac{c}{4}\sum_{k=1}^{3}\tilde{g}_c(X,\ J_kY)\tilde{g}_c(J_kJ'\xi,\ \xi)\,.\end{aligned}$$

Since $\{J_1, J_2, J_3\}$ is a basis of $\mathfrak{J}$, there exist real numbers a^l, $l = 1, 2, 3$, such that $J' = \sum_{l=1}^{3} a^l J_l$, so that we see $\tilde{g}_c(J_k J'\xi, \xi) = \sum_{l=1}^{3} a^l \tilde{g}_c(J_k J_l \xi, \xi) = -a^k$ and

$$\begin{aligned}\tilde{g}_c(\tilde{R}(X, Y)J'\xi, \xi) &= -\frac{c}{4}\sum_{k=1}^{3} \tilde{g}_c(X, a^k J_k Y) \\ &= -\frac{c}{4}\tilde{g}_c(X, J'Y) = \frac{c}{4} g(J'X, Y)\,. \end{aligned}\tag{4.19}$$

From (4.16), (4.17), (4.18) and (4.19), we obtain the following.

LEMMA 4.4. *Under the assumption* (4.6), *the equality*

$$\left(\lambda\mu - \frac{c}{8}\right) g(J'X, Y) = 0 \tag{4.20}$$

holds, where X and Y are eigenvectors of A_ξ corresponding to non-zero eigenvalues λ and μ respectively, and J' is any section of $\mathfrak{J}$.

The following proposition is a goal of this section.

PROPOSITION 4.5. *If an Einstein Kähler hypersurface M of $G_2(\mathbf{C}^n)$ satisfies the condition $\mathfrak{J}T^\perp M \subset TM$, then, for any point $p \in M$ and any unit normal vector $\xi \in T_p^\perp M$, there exist three subspaces V_0, V_+ and V_- of T_pM such that the following properties hold*:

(1) *V_0 is a J-invariant* 0*-eigenspace of A_ξ satisfying*

$$V_0 = \mathfrak{J}_p T_p^\perp M\,.$$

(2) *$V_\pm$ are $\mathfrak{J}_p$-invariant $\pm\sqrt{\frac{c}{8}}$-eigenspaces of A_ξ satisfying*

$$JV_+ = V_-\,.$$

(3) *The eigenspace decomposition*

$$T_pM = V_0 \oplus V_+ \oplus V_-$$

holds.

Moreover, n must be even.

PROOF. Let $A_\xi|_V$ be the restriction of A_ξ to V. Denote by ρ, the scalar curvature of M. Since the Ricci curvature Ric satisfies the Einstein condition $Ric = \frac{\rho}{2m} g$, Lemma 4.3 implies

$$g(A_\xi^2 Y_V, Z_V) = \frac{c}{16}\left\{\left(4n - 4 - \frac{4\rho}{cm}\right) g(Y_0, Z_0) + \left(4n - 2 - \frac{4\rho}{cm}\right) g(Y_V, Z_V)\right\} \tag{4.21}$$

for any tangent vector fields Y and Z. Choosing Y and Z as $Y = Z \in V_0$, we get $\rho = cm(n-1) = c(n-1)(2n-5)$. Therefore, (4.21) implies

$$g(A_\xi^2 Y_V, Z_V) = \frac{c}{8}\, g(Y_V, Z_V)\,,$$

equivalently, all eigenvalues of $A_\xi|_V$ are $\pm\sqrt{\frac{c}{8}}$. In particular, 0 is not an eigenvalue of $A_\xi|_V$, which, together with Lemma 4.2 (1), implies that V_0 is a 0-eigenspace of A_ξ. Denote by $V_\pm$, eigenspaces corresponding to $\pm\sqrt{\frac{c}{8}}$ respectively. Then V is a diagonal sum of subspaces $V_\pm : V = V_+ \oplus V_-$. From (2.8), we easily see $JV_+ = V_-$.

For any $X \in V_+$, $Y \in V_-$ and $J' \in \mathfrak{J}_p$, Lemma 4.4 implies $g(J'X, Y) = 0$. Since $J'X \in V$, we get $J'X \in V_+$, so that V_+ is $\mathfrak{J}_p$-invariant. Similarly, we can see that V_- is also $\mathfrak{J}_p$-invariant.

Since the real dimension of V_0 is 6, we have $\dim_{\mathbf{R}} V = 2m-6 = 4n-16$ and $\dim_{\mathbf{R}} V_\pm = \frac{1}{2}\dim_{\mathbf{R}} V = 2n-8$. Since $V_\pm$ are $\mathfrak{J}_p$-invariant, $2n-8$ is a multiple of 4, so that n is even. □

5. A focal variety

Let M be an Einstein Kähler hypersurface M of $\tilde{M} = G_2(\mathbf{C}^n)$ satisfies the condition $\mathfrak{J}T^\perp M \subset TM$. By Proposition 4.5, n must be even, so that we put $n = 2l$. In this section, we study the first focal set of M, and prove our main theorem.

We will use the same notations as those in the section 4. Moreover, for any point $p \in M$ and any unit normal vector ξ, define subspaces of $T_p\tilde{M}$ by

$$\begin{aligned} V_{0,+} &= \mathfrak{J}\xi = Span_{\mathbf{R}}\{J_1\xi, J_2\xi, J_3\xi\}, \\ V_{0,-} &= J\mathfrak{J}\xi = Span_{\mathbf{R}}\{JJ_1\xi, JJ_2\xi, JJ_3\xi\}, \\ \perp_+ &= Span_{\mathbf{R}}\{\xi\}, \\ \perp_- &= Span_{\mathbf{R}}\{J\xi\}. \end{aligned}$$

By direct computation, (2.5) implies the following. Also see [2, Theorem 3].

LEMMA 5.1. *Let $\tilde{R}_\xi$ be the curvature operator with respect to ξ, i.e, $\tilde{R}_\xi$ is defined by $\tilde{R}_\xi(X) = \tilde{R}(X, \xi)\xi$ for any $X \in T_p\tilde{M}$. Let κ be an eigenvalue of $\tilde{R}_\xi$, and T_κ be an eigenspace corresponding to κ. Then, we have the following complete table.*

κ	T_κ
0	$\perp_+ \oplus V_{0,-}$
$\frac{c}{8}$	$V_+ \oplus V_-$
$\frac{c}{2}$	$\perp_- \oplus V_{0,+}$

Let $U^\perp M$ be the unit normal bundle of M with a natural projection π, i.e., $U^\perp M$ is the subbundle of all unit normal vectors of M. For $\xi \in U^\perp M$, let $\gamma_\xi(t)$ be the geodesic of $G_2(\mathbf{C}^n)$, such that $\gamma_\xi(0) = \pi(\xi)$ and $\gamma'_\xi(0) = \xi$. For $r > 0$, define a smooth map F_r from $U^\perp M$ into $G_2(\mathbf{C}^n)$ by $F_r(\xi) = \gamma_\xi(r)$. If r is sufficiently small, the image $N_r = F_r(U^\perp M)$ is a tube around M with radius r, which is a real hypersurface of $G_2(\mathbf{C}^n)$. If $\operatorname{rank}(F_{r*})_\xi < \dim_{\mathbf{R}} \tilde{M} - 1$ for some r and ξ, a point $F_r(\xi)$ is called a "focal point". $F_r(\xi)$ is called the first focal point if $F_t(\xi)$ is not a focal point for any t with $0 < t < r$.

Let $\xi(s)$ be a curve in $U^\perp M$ with $\xi(0) = \xi$ and $\xi'(0) = \hat{X} \in T_\xi(U^\perp M)$. Define a smooth map ψ by $\psi(t, s) = F_t(\xi(s))$, and define a vector field $Z(t)$ along γ_ξ by

$$Z(t) = \left(F_{t*}\right)_\xi \hat{X} = \left[\frac{d}{ds} F_t(\xi(s))\right]_{s=0} = \left[\frac{\partial}{\partial s}\,\psi\right]_{s=0}.$$

Since ψ is a variation of a geodesic γ_ξ, $Z(t)$ is a Jacobi field along γ_ξ, i.e, $Z(t)$ satisfies the Jacobi equation

$$\tilde{\nabla}_t^2 Z(t) + \tilde{R}(Z(t),\ \gamma_\xi'(t))\gamma_\xi'(t) = 0\,.$$

$Z(t)$ must satisfy the initial condition

$$Z(0) = \pi_{*\xi}\hat{X}\,, \quad Z'(0) = \left[\tilde{\nabla}_s \xi(s)\right]_{s=0}.$$

We remark that the image $\left(F_{t*}\right)_\xi \left(T_\xi(U^\perp M)\right)$ are spanned by above Jacobi fields.

To get a basic Jacobi field, set $Z(t) = f(t)P(t)$, where P is a parallel vector field along γ_ξ, and f is a smooth function. Since $\gamma_\xi'(t)$ and the curvature tensor $\tilde{R}$ are also parallel, the function f satisfies $f''(t)P(t) + f(t)\,\tau_t(\tilde{R}(P(0),\ \xi)\xi) = 0$, where τ_t is a parallel displacement along $\gamma_\xi(t)$. In particular, if $P(0) \in T_\kappa$ and $P(0) \neq 0$, then f satisfies $f''(t) + \kappa f(t) = 0$.

LEMMA 5.2. *For each of the cases below, there exists a curve $\xi(s)$ in $U^\perp M$, such that f satisfies*

(5.1) $$f''(t) + \kappa f(t) = 0\,, \quad f(0)P(0) = \pi_{*\xi}\xi'(0)\,, \quad f'(0)P(0) = \left[\tilde{\nabla}_s \xi(s)\right]_{s=0}.$$

(1) $P(0) \in \perp_-$ *and* $f(t) = \sqrt{\tfrac{2}{c}}\,\sin\sqrt{\tfrac{c}{2}}t$.

(2) $P(0) \in V_{0,+}$ *and* $f(t) = \cos\sqrt{\tfrac{c}{2}}t$.

(3) $P(0) \in V_{0,-}$ *and* $f(t) \equiv 1$.

(4) $P(0) \in V_+$ *and* $f(t) = \sqrt{2}\cos\left(\sqrt{\tfrac{c}{8}}t + \tfrac{\pi}{4}\right)$.

(5) $P(0) \in V_-$ *and* $f(t) = \sqrt{2}\cos\left(\sqrt{\tfrac{c}{8}}t - \tfrac{\pi}{4}\right)$.

PROOF. In the case (1), there exists $a \in \mathbf{R}$, such that $P(0) = aJ\xi$. Set $\xi(s) = \cos as \cdot \xi + \sin as \cdot J\xi$. Then, we see $\pi_{*\xi}\xi'(0) = 0$ and $\left[\tilde{\nabla}_s \xi(s)\right]_{s=0} = aJ\xi$. From Lemma 5.1, we have $\kappa = \frac{c}{2}$. Therefore, the equation (5.1) is equivalent to $f'' + \frac{c}{2} f = 0$, $f(0) = 0$, $f'(0) = 1$, which has a unique solution $f(t) = \sqrt{\frac{2}{c}}\,\sin\sqrt{\frac{c}{2}}t$.

In other cases, $X = P(0)$ is tangent to M. Let $c(s)$ be a curve in M with $c'(0) = X$, and $\xi(s)$ be a parallel normal vector field along $c(s)$, satisfying $\xi(0) = \xi$. Then, we see $\pi_{*\xi}\xi'(0) = X$ and $\big[\tilde{\nabla}_s \xi(s)\big]_{s=0} = -A_\xi X$.

Let's assume $X \in V_+$. Lemma 5.1 implies $\kappa = \frac{c}{8}$, and Proposition 4.5 implies $\big[\tilde{\nabla}_s \xi(s)\big]_{s=0} = -\sqrt{\frac{c}{8}}X$. Therefore, the equation (5.1) is equivalent to $f'' + \frac{c}{8}f = 0$, $f(0) = 1$, $f'(0) = -\sqrt{\frac{c}{8}}$, which has a unique solution $f(t) = \sqrt{2}\cos\big(\sqrt{\frac{c}{8}}t + \frac{\pi}{4}\big)$, so that the case (4) is proved.

The remaining cases are similarly proved. □

Let's set $r_1 = \sqrt{\frac{2}{c}}\frac{\pi}{2}$. Then, any point of N_{r_1} is the first focal point, the image of $(F_{r_1 *})_\xi$ is a vector space $\tau_{r_1}(\perp_- \oplus V_{0,-} \oplus V_-)$, and $\operatorname{rank}(F_{r_1 *})_\xi = \frac{1}{2}\dim_{\mathbf{R}} \tilde{M}$, so that the first focal set N_{r_1} is a submanifold of $\tilde{M}$. The tangent space of N_{r_1} at $q = F_{r_1}(\xi)$ is given by

$$T_q N_{r_1} = \tau_{r_1}(\perp_- \oplus V_{0,-} \oplus V_-)\,,$$

which is $\mathfrak{J}$-invariant. It is easy to see that the real dimension of N_{r_1} is equal to $\frac{1}{2}\dim_{\mathbf{R}} \tilde{M}$. Moreover, the normal space of N_{r_1} at q is given by

$$T_q^{\perp} N_{r_1} = \tau_{r_1}(\perp_+ \oplus V_{0,+} \oplus V_+)\,,$$

so that we see

$$J T_q N_{r_1} = T_q^{\perp} N_{r_1}\,.$$

Therefore, we obtain the following.

PROPOSITION 5.3. *The first focal set N_{r_1} of M is a quaternionic Kähler, totally real submanifold of $G_2(\mathbf{C}^{2l})$. The real dimension of N_{r_1} is one half of $dim_{\mathbf{R}}\, G_2(\mathbf{C}^{2l})$.*

In [13], H. Tasaki showed that any complete, quaternionic Kähler, totally real submanifold of $G_2(\mathbf{C}^{2l})$ is congruent to a quaternionic projective space. Then, for some fixed $q \in N_{r_1}$, there exists a quaternionic projective space $\mathbf{H}P^{l-1}$, such that $q \in \mathbf{H}P^{l-1}$ and $T_q N_{r_1} = T_q \mathbf{H}P^{l-1}$. In [1], Alekseevskii proved that a quaternionic submanifold in a quaternionic Kähler manifold is totally geodesic. Therefore, N_{r_1} is a open portion of $\mathbf{H}P^{l-1}$.

By Proposition 3.2, $M_{11,l}$ satisfies the same assumption as M. Then, the first focal set of $M_{11,l}$ is congruent to $\mathbf{H}P^{l-1}$ up to the automorphism of $G_2(\mathbf{C}^{2l})$, so that M and $M_{11,l}$ are locally congruent. Therefore, we complete the proof of Theorem 1.1.

References

[1] D. V. ALEKSEEVSKII, Compact quaternion spaces, Functional Anal. Appl. **2** (1968), 109–114.

[2] J. BERNDT, Riemannian geometry of complex two-plane Grassmannians, Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), 19–83.

[3] A. BOREL and F. HIRZEBRUCH, Characteristic classes and homogeneous spaces I, Amer. J. Math. **80** (1958), 458–538.

[4] B. Y. CHEN and T. NAGANO, Totally geodesic submanifolds of symmetric spaces, I, Duke Math. J. **44** (1977), 745–755.

[5] B. Y. CHEN and T. NAGANO, Totally geodesic submanifolds of symmetric spaces, II, Duke Math. J. **45** (1978), 405–425.

[6] S. IHARA, Holomorphic imbeddings of symmetric domains, J. Math. Soc. Japan **19** (1967), 261–302.

[7] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry I, II*, John Wiley and Sons, Interscience, New York, 1969.

[8] K. KONNO, Homogeneous hypersurfaces in Kähler C-spaces with $b_2 = 1$, J. Math. Soc. Japan **40** (1988), 687–703.

[9] Y. MIYATA, Spectral geometry of Kähler hypersurfaces in a complex Grassmann manifold, Tokyo J. Math. **28** (2005).

[10] H. NAKAGAWA and R. TAKAGI, On locally symmetric Kaehler submanifolds in a complex projective space, J. Math. Soc. Japan **28** (1976), 638–667.

[11] I. SATAKE, Holomorphic imbeddings of symmetric domains into a Siegel space, Amer. J. Math. **87** (1965), 425–461.

[12] M. TAKEUCHI, Homogeneous Kähler submanifolds in complex projective spaces, Japan. J. Math. **4** (1978), 171–219.

[13] H. TASAKI, Quaternionic submanifolds in quaternionic symmetric spaces, Tohoku Math. J. **38** (1986), 513–538.

[14] K. TSUKADA, Parallel Kaehler submanifolds of Hermitian symmetric spaces, Math. Z. **190** (1985), 129–150.

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