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Isometric Immersions with Geodesic Normal Sections in Semi-Riemannian Geometry

Kouhei MIURA

Tokyo University of Science

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Abstract. We study an isometric immersion $f : M \to \bar{M}$ with geodesic normal sections, where $\bar{M}$ is a semi-Riemannian space form. In Riemannian geometry, it is known that f is helical, in particular, all geodesics of M have the same proper order in $\bar{M}$. However this does not hold in general, when $\bar{M}$ is indefinite semi-Riemannian. We give sufficient conditions for an isometric immersion with geodesic normal sections to be helical.

1. Introduction

Let M be an n-dimensional submanifold in a Euclidean space $\mathbf{R}^{n+q}$. For any point $p \in M$ and unit tangent vector x in the unit tangent sphere U_pM at p, let $E(p,x)$ be the affine $(q+1)$-dimensional subspace of $\mathbf{R}^{n+q}$ through p spanned by x and the normal space $T_p^{\perp}M$. The intersection of M and $E(p,x)$ gives rise to a unit speed curve β_x with $\beta_x(0) = p$ and $\beta_x'(0) = x$ defined on an open interval containing 0. This curve β_x is called the *normal section* at (p,x). In Chen and Verheyen [3], M is said to have geodesic normal sections if, for any $p \in M$ and $x \in U_pM$, the normal section β_x at (p,x) is geodesic in a neighborhood of 0. Another important concept used in this paper, called helical immersions, originated from Besse [1]. Let $f : M \to \bar{M}$ be an isometric immersion between Riemannian manifolds. If, for each unit speed geodesic γ of M, the curve $f \circ \gamma$ in $\bar{M}$ is a helix of order d with curvatures $\lambda_1, \dots, \lambda_{d-1}$ which are independent of γ, then f is called a helical geodesic immersion of order d. Chen and Verheyen [3] proved that a helical submanifold in a Euclidean space has geodesic normal sections using a result for helical geodesic immersions in Sakamoto [10]. Verheyen proved its converse in [11]. Their results were proved in the case where the ambient space is a Riemannian space form by Hong and Houh [5].

Also in semi-Riemannian geometry, the notions of submanifolds with geodesic normal sections and helical geodesic immersions can be introduced. Kim [6] classified semi-Riemannian surfaces in $\mathbf{R}^5$ with vanishing mean curvature and geodesic normal sections. His

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classification shows that there exist surfaces with geodesic normal sections which are not helical. The present author [7] studied helical geodesic immersions between semi-Riemannian manifolds and showed that a helical immersed submanifold in a semi-Riemannian space form has geodesic normal sections.

In this paper, we study semi-Riemannian isometric immersions with geodesic normal sections Let $f : M \to \bar{M}$ be a semi-Riemannian isometric immersion into a semi-Riemannian space form. In contrast to the Riemannian case, there exist space-like, time-like and null geodesics and normal sections on M. Thus we say that f has *space-like* (resp. *time-like*) *geodesic normal sections* if any space-like (resp. time-like) unit speed normal section β_x is locally geodesic. We show that f has space-like geodesic normal sections if and only if f has time-like geodesic normal sections. Furthermore, we prove that if f has space-like geodesic normal sections, then there exist some $d \in \mathbf{N}$, positive constants $\lambda_1, \dots, \lambda_{d-1} \in \mathbf{R}$ such that, for any space-like unit speed geodesic γ of M, $f \circ \gamma$ has Frenet curvatures $\lambda_1, \dots, \lambda_{d-1}$ in $\bar{M}$. In semi-Riemannian geometry, note that $f \circ \gamma$ is not necessarily of proper order d. As a corollary of this result, we can give some sufficient conditions for an isometric immersion with geodesic normal sections to be helical.

In Section 2 we prepare basic notations and equations that we use later. The definitions of isometric immersions with geodesic normal sections and helical geodesic immersions in semi-Riemannian geometry are also given. Section 3 is devoted to the study of such isometric immersions.

2. Preliminaries

Let $f : M \to \bar{M}$ be an isometric immersion of a connected n-dimensional semi-Riemannian manifold M into an $(n+q)$-dimensional semi-Riemannian manifold $\bar{M}$ of constant sectional curvature. For all local formulas and computations we may regard f as an embedding and thus we shall often identify $p \in M$ with $f(p) \in \bar{M}$ and the tangent space T_pM the subspace $f_*(T_pM)$ of $T_p\bar{M}$. We denote the normal space of f at p by $T_p^{\perp}M$. Let $\bar{\nabla}$ (resp. ∇) be the Levi-Civita connection of $\bar{M}$ (resp. M), B the second fundamental form, A the shape tensor, and $\nabla^{\perp}$ the normal connection. Clearly A is related to B as $\langle A_\xi X, Y\rangle = \langle B(X,Y), \xi\rangle$, where $\langle\ ,\ \rangle$ is the semi-Riemannian metric of $\bar{M}$, and X and Y are vector fields tangent to M, and ξ is a vector field normal to M.

The k-th ($k \geq 1$) covariant derivative D^kB of B with respect to ∇ and $\nabla^{\perp}$ is defined by

$$\begin{aligned}(D^kB)(X_1, \dots, X_{k+2}) := {} & \nabla^{\perp}_{X_1}((D^{k-1}B)(X_2, \dots, X_{k+2})) \\ & - \sum_{i=2}^{k+2}(D^{k-1}B)(X_2, \dots, \nabla_{X_1}X_i, \dots, X_{k+2}),\end{aligned}$$

where $D^0B = B$. Then the Ricci identity for $(D^{k-2}B)(X_1, \dots, X_k)$ is

$$(D^kB)(X, Y, X_1, \dots, X_k) - (D^kB)(Y, X, X_1, \dots, X_k)$$

$$= R^{\perp}(X,Y)(D^{k-2}B)(X_1,\dots,X_k) - \sum_{i=1}^{k}(D^{k-2}B)(X_1,\dots,R(X,Y)X_i,\dots,X_k)\,,$$

where R (resp. $R^{\perp}$) is the curvature tensor of ∇ (resp. $\nabla^{\perp}$). We denote $(D^kB)(X,\dots,X)$ by $(D^kB)(X^{k+2})$ for short.

The equations of Gauss, Codazzi and Ricci are given by

$$R(X,Y)Z = \bar{c}\,(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + A_{B(Y,Z)}X - A_{B(X,Z)}Y\,,$$
$$(DB)(X,Y,Z) = (DB)(Y,X,Z)\,,$$
$$R^{\perp}(X,Y)\xi = B(X,A_\xi Y) - B(A_\xi X,Y)\,,$$

respectively, where $\bar{c}$ is the constant sectional curvature of $\bar{M}$.

Let L be a submanifold of $\bar{M}$. Hereafter, we say that L is *totally geodesic* in $\bar{M}$, if $\bar{\nabla}_X Y$ is tangent to L for any tangent vector fields X and Y of L. We note that the induced tensor field on L from the semi-Riemannian metric on $\bar{M}$ is not necessarily non-degenerate. Since $\bar{M}$ has constant sectional curvature, if, for any $p\in\bar{M}$ and subspace V of $T_p\bar{M}$, there exists a totally geodesic submanifold L containing p such that $T_pL = V$.

For any $p\in M$ and $v\in T_pM$ $(v\neq 0)$, let $E(p,v)$ be the vector subspace of $T_p\bar{M}$ spanned by v and $T_p^{\perp}M$. $E(p,v)$ determines a $(q+1)$-dimensional totally geodesic submanifold $\widehat{E}(p,v)$ of $\bar{M}$ such that $p\in\widehat{E}(p,v)$ and $T_p\widehat{E}(p,v) = E(p,v)$. The intersection of M and $\widehat{E}(p,v)$ gives rise to a regular curve β_v such that $\beta_v(0) = p$ and $\beta_v'(0) = v$ in a neighborhood of p in $\bar{M}$, which is called a *normal section* of f at (p,v). An isometric immersion f has *geodesic normal sections* if β_v is pregeodesic on M in a neighborhood at 0 for any $p\in M$ and $v\in T_pM$ $(v\neq 0)$ (or, equivalently, each geodesic of M is locally a normal section of f). For example, semi-Riemannian spheres and their Veronese immersions ([2, Examples 1.1 and 1.2]) have geodesic normal sections.

EXAMPLE 2.1. (cf. [2], [6]) For real-valued smooth functions $f_1,\dots,f_l\in C^\infty(\mathbf{R}^n)$,

$$\mathbf{R}^n_t \ni p \mapsto (f_1(p),\dots,f_l(p),p,f_1(p),\dots,f_l(p)) \in \mathbf{R}^{n+2l}_{t+l} = \mathbf{R}^l_l\times\mathbf{R}^n_t\times\mathbf{R}^l_0$$

is an isometric immersion with geodesic normal sections, where $\mathbf{R}^n_t$ is the n-dimensional semi-Euclidean space with index t.

In this paper, we say that a curve in $\bar{M}$ is *of proper order e* if the image is contained in some e-dimensional totally geodesic submanifold of $\bar{M}$ and is not contained in any $(e-1)$-dimensional ones. We recall the notion of Frenet curves in semi-Riemannian geometry. Let $c: I\to\bar{M}$ a unit speed curve on $\bar{M}$, that is, $|\langle c',c'\rangle|\equiv 1$. We put for $k\in\mathbf{N}$ and $s\in I$,

$$(G_kc)(s) := \det(\langle c^{(i)}(s), c^{(j)}(s)\rangle)_{1\le i,j\le k}\,.$$

In contrast to the Riemannian case, we note that the equations $G_kc\neq 0$ $(1\le k\le d)$, $G_{d+1}c\equiv 0$ on I do not necessarily mean that $c^{(d+1)}$ is linearly dependent on $c', c'',\dots,c^{(d)}$.

We assume that $G_k c \neq 0$ on I for any $1 \leq k \leq d$. So we have $d \leq e$ in general, where e is the proper order of c. Then, we can apply the Gram-Schmidt orthonormalization process to $c'(s), c''(s), \dots, c^{(d)}(s)$ at each point $c(s)$. Consequently we have the Frenet d-frame field $c_1, c_2, \dots, c_d$ of c, which satisfies

$$c_i' = -\varepsilon_{i-1}\varepsilon_i\lambda_{i-1}c_{i-1} + \lambda_i c_{i+1} \quad \text{for } 1 \leq i \leq d-1\,, \tag{1}$$

where $\varepsilon_0 = \lambda_0 = 0$, $c_0 = 0$, $\varepsilon_i = \langle c_i, c_i\rangle \in \{-1, +1\}$ $(1 \leq i \leq d)$ and λ_i $(1 \leq i \leq d-1)$ are functions on I satisfying the following formulas:

$$\varepsilon_i = \frac{\operatorname{sgn} G_i c}{\operatorname{sgn} G_{i-1} c}\,, \quad \lambda_i = \frac{|G_{i-1}c|^{1/2}|G_{i+1}c|^{1/2}}{|G_i c|}\,, \tag{2}$$

where $G_0 c = 1$. (See [4] for a computational algorithm, the d-Frenet frame, and the curvatures of a Euclidean curve.) Conversely if a curve $c : I \to \bar{M}$ satisfies Equation (1), then $G_k c \neq 0$ on I holds for any $1 \leq k \leq d$. For such a curve c, we say that *c has the Frenet curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $\varepsilon_1, \dots, \varepsilon_d$*. Furthermore, when $c^{(d+1)}$ is linearly dependent on $c', c'', \dots, c^{(d)}$ (hence the proper order is equal to d), the curve c satisfies the Frenet formula: Equation (1) and

$$c_d' = -\varepsilon_{d-1}\varepsilon_d\lambda_{d-1}c_{d-1}\,.$$

Then we call the curve c a *Frenet curve of order d, curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $\varepsilon_1, \dots, \varepsilon_d$*. If all curvatures are constant, the curve c is called a *helix of order d*.

An isometric immersion f is called a *helical space-like geodesic immersion of order d*, for any unit speed space-like geodesic γ of M, the curve $f \circ \gamma$ in $\bar{M}$ is a helix of order d, curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $\varepsilon_1 = +1, \varepsilon_2, \dots, \varepsilon_d$, which are independent of the choice of γ. We define a helical *time-like* geodesic immersion in a similar way.

In [7], we obtained that f is a helical space-like geodesic immersion of order d if and only if f is a helical time-like geodesic immersion of order d. Hence we may call these *helical geodesic immersions*. We also showed that, in semi-Riemannian geometry, a helical geodesic immersion has geodesic normal sections.

In Riemannian geometry, an isometric immersion $f : M \to \bar{M}$ with geodesic normal sections is helical ([5], [11]), in particular, every geodesic of M has the same proper order in $\bar{M}$. Also in the case where $\bar{M}$ is indefinite, semi-Riemannian spheres and their Veronese immersions have this properties for *non-null* geodesics. However, for an isometric immersion in Example 2.1, the proper order in $\mathbf{R}^{n+2l}_{t+l}$ of a non-null geodesic γ of $\mathbf{R}^n_t$ depends on the initial velocity $\gamma'(0)$ in general.

3. Isometric immersions with geodesic normal sections

Let $f : M \to \bar{M}$ be an isometric immersion of a connected semi-Riemannian manifold M into a semi-Riemannian manifold $\bar{M}$ of constant sectional curvature and $\dim M \geq 2$. We denote by U^+M (resp. U^-M) the space-like (resp. time-like) unit tangent bundle of M. Whenever $x \in U_pM := U_p^+M \cup U_p^-M$ $(p \in M)$, we give the normal section β_x of f at (p, x) the arc-length parameter with $\beta_x(0) = p$ and $\beta_x'(0) = x$. Then we temporarily say that f has *space-like* or $(+1)$- (resp. *time-like* or (-1)-) geodesic normal sections if β_x is geodesic in M for any $p \in M$ and $x \in U_p^+M$ (resp. U_p^-M). For $p \in M$ and $v \in T_pM$, we let γ_v stand for a geodesic of M such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. We put $\sigma_v = f \circ \gamma_v$ and $V = \sigma_v'$. The uniqueness theorem for geodesics implies the following lemma ([3, Lemma 1]).

LEMMA 3.1. *If f has ε-geodesic normal sections* $(\varepsilon \in \{-1, +1\})$, *then, for any $x \in U^\varepsilon M$, after a suitable reparametrization, σ_x locally remains a normal section of f at $(\gamma_x(s), \gamma_x'(s))$ for all $s \in \operatorname{dom}\gamma_x$. In particular, the component of $\sigma_x^{(k)}(s)$ tangent to M is proportional to $\sigma_x'(s)$ for any $s \in \operatorname{dom}\gamma_x$ and $k \in \mathbf{N}$.*

From now on, for convenience, we write $(D^{-1}B)(v)$ instead of $v \in TM$. The following arguments are analogous to those of Verheyen [11].

LEMMA 3.2. *If f has ε-geodesic normal sections* $(\varepsilon \in \{-1, +1\})$, *then the following property (F_k) holds for any $x \in U^\varepsilon M$ and $k \in \mathbf{N}$:*
(F_1) $\sigma_x' = X$, $\quad(F_2)$ $\sigma_x'' = B(X^2)$, *and for $k \geq 3$,*
(F_k) *there exist smooth functions $C_{k,l}$ $(1 \leq l \leq k)$ on $\operatorname{dom}\gamma_x$ such that $C_{k,k} = 1$, $C_{k,k-1} = 0$ and*

$$\sigma_x^{(k)} = \sum_{l=1}^{k} C_{k,l}\,(D^{l-2}B)(X^l)\,, \tag{3}$$

and $A_{(D^{k-3}B)(x^{k-1})}x = \varepsilon\langle(D^{k-3}B)(x^{k-1}), B(x^2)\rangle x$ holds.

PROOF. We have $\sigma_x' = X$, $\sigma_x'' = B(X^2)$ and $\sigma_x^{(3)} = -A_{B(X^2)}X + (DB)(X^3)$. According to Lemma 3.1, we get $A_{B(x^2)}x = \varepsilon\langle B(x^2), B(x^2)\rangle x$ for any $x \in U^\varepsilon M$. Hence (F_1), (F_2) and (F_3) hold, where $C_{3,3} = 1$, $C_{3,2} = 0$ and $C_{3,1} = -\varepsilon\langle B(X^2), B(X^2)\rangle$. By induction on k, we shall show that the property (F_k) holds for any $k \geq 3$. We already have (F_3). We verify that (F_{k+1}) holds when (F_l) is true for any $3 \leq l \leq k$. From the induction hypothesis, it follows that

$$\begin{aligned}\sigma_x^{(k+1)} = &\Bigl((XC_{k,1}) - \sum_{l=2}^{k-1} C_{k,l}\,\varepsilon\langle(D^{l-2}B)(X^l), B(X^2)\rangle\Bigr)X - A_{(D^{k-2}B)(X^k)}X \\ &+ \sum_{l=2}^{k}\bigl((XC_{k,l} + C_{k,l-1})\bigr)(D^{l-2}B)(X^l) + (D^{k-1}B)(X^{k+1})\,.\end{aligned}$$

According to Lemma 3.1 again, we have for any $x \in U^\varepsilon M$,

$$A_{(D^{k-2}B)(x^k)}x = \varepsilon\langle(D^{k-2}B)(x^k), B(x^2)\rangle x\,.$$

Therefore we put

$$
\begin{cases} C_{k+1,k+1} := 1\,, \quad C_{k+1,l} := (XC_{k,l}) + C_{k,l-1} \text{ for } 2 \le l \le k\,, \\ C_{k+1,1} := (XC_{k,1}) - \sum_{l=2}^{k} C_{k,l}\, \varepsilon \langle (D^{l-2}B)(X^l), B(X^2)\rangle\,. \end{cases} \tag{4}
$$

Since we obtain (F_{k+1}), the proof is complete. □

LEMMA 3.3. *Let ε be (-1) or $(+1)$. The following conditions are equivalent*:

(i) *f has ε-geodesic normal sections*,

(ii) *f has geodesic normal sections*,

(iii) *For any $x \in U^{\varepsilon}M$ and $k \ge 2$,*

$$
A_{(D^{k-2}B)(x^k)}x = \varepsilon \langle (D^{k-2}B)(x^k), B(x^2)\rangle x\,,
$$

(iv) *For any $v \in TM$ and $k \ge 2$,*

$$
\langle v, v\rangle\, A_{(D^{k-2}B)(v^k)}v = \langle (D^{k-2}B)(v^k), B(v^2)\rangle v\,.
$$

PROOF. On account of Lemma 3.2, (i) implies (iii). It is obvious that (iii) and (iv) are equivalent. Suppose that (iv) holds. Then we have $A_{(D^{k-2}B)(v^k)}v \wedge v = 0$ for any non-null vector v. Moreover because of the continuity of the map $TM \ni v \mapsto A_{(D^{k-2}B)(v^k)}v \wedge v \in \bigwedge^2 TM$, $A_{(D^{k-2}B)(v^k)}v \wedge v$ is identically vanishing on TM for any $k \in \mathbf{N}$. So, for $p \in M$, $v \in T_pM (v \neq 0)$, we can see that $\mathrm{Sp}\{\sigma_v^{(k)}(s) \mid k \in \mathbf{N}\} \subset E(\gamma_v(s), \gamma_v'(s))$ $(s \in \mathrm{dom}\, \gamma_v)$ is locally parallel along σ_v with respect to $\bar{\nabla}$, where Sp denotes the linear span. Therefore the image of σ_v is locally contained in $\widehat{E}(p, v)$, which is a totally geodesic submanifold in $\bar{M}$ such that $T_p\widehat{E}(p, v) = E(p, v)$. Thus f has geodesic normal sections. It is trivial that (ii) implies (i). □

Since we see that f has space-like geodesic normal sections if and only if f has time-like geodesic normal sections, hereafter, we assume that f has space-like geodesic normal sections, hence $\mathrm{ind}\, M < \dim M$. Then, by virtue of Lemma 3.3, we can see that f has *null* geodesic normal sections, that is, each normal section β_v of f is locally pregeodesic for any null vector $v \in \Lambda = \bigcup_{p \in M} \Lambda_p$, where Λ_p is the nullcone of T_pM.

For $v \in TM$, $k, l \in \mathbf{N}$, we put $\nu_{k,l}(v) := \langle (D^{k-2}B)(v^k), (D^{l-2}B)(v^l)\rangle$ and $(G_kB)(v) := \det(\nu_{i,j}(v))_{1 \le i,j \le k}$, and $(G_0B)(v) := 1$. From Equation (3) and $C_{k,k} = 1$ of Lemma 3.2 and Lemma 3.3, we have $\bigwedge_{i=1}^{k} \sigma_x^{(i)} = \bigwedge_{i=1}^{k} (D^{i-2}B)(X^i)$ for $x \in UM$. So we have for any $x \in UM$,

$$
G_k\sigma_x = (G_kB)(X)\,. \tag{5}
$$

LEMMA 3.4. *The following property (E_i) holds for $i \ge 2$:*
For any $h, j \in \mathbf{N}_0$ $(j \ge 2,\ 0 \le h \le i-1)$, $p \in M$, $x \in U_p^+M$, $y \in U_pM$ $(\langle x, y\rangle = 0)$,

$$
\langle (D^{i-2}B)(x^h, y, x^{i-h-1}), (D^{j-2}B)(x^j)\rangle = 0\,.
$$

PROOF. The property (E_2) follows from Lemma 3.2. Suppose that (E_m) holds for any $2 \le m \le i$. We prove that (E_{i+1}) is satisfied ($i \ge 2$). For $1 \le h \le i$, we have from (E_i),

$$\begin{aligned}&\langle (D^{i-1}B)(x^h, y, x^{i-h}), (D^{j-2}B)(x^j)\rangle \\ &\quad = x \cdot \langle (D^{i-2}B)(X^{h-1}, Y, X^{i-h}), (D^{j-2}B)(X^j)\rangle \\ &\quad\quad - \langle (D^{i-2}B)(x^{h-1}, y, x^{i-h}), (D^{j-1}B)(x^{j+1})\rangle = 0\,,\end{aligned}$$

where X and Y are extensions along $\sigma_x = f \circ \gamma_x$ of x and y respectively satisfying $\bar{\nabla}_x X = \bar{\nabla}_x Y = 0$ and $\langle X, Y\rangle = 0$. Hence we only need to prove (E_{i+1}) for $h = 0$. Using the Codazzi equation, we see that (E_3) for $h = 0$ holds. So we shall prove the case $i \ge 3$. Applying Ricci identity for $(D^{i-3}B)(x^{i-1})$, we have

$$\begin{aligned}&\langle (D^{i-1}B)(x, y, x^{i-1}), (D^{j-2}B)(x^j)\rangle - \langle (D^{i-1}B)(y, x^i), (D^{j-2}B)(x^j)\rangle \\ &\quad = \langle [A_{(D^{i-3}B)(x^{i-1})}, A_{(D^{j-2}B)(x^j)}]x, y\rangle \\ &\quad\quad + \sum_{h=0}^{i-4} \langle (D^{i-3}B)(x^h, R(x, y)x, x), (D^{j-2}B)(x^j)\rangle.\end{aligned}$$

By $\langle x, R(x, y)x\rangle = 0$ and the induction hypothesis, we obtain (E_{i+1}) for $h = 0$. Consequently (E_i) is true for any $i \ge 2$. □

LEMMA 3.5. *If $k + l$ is even, then $\nu_{k,l}$ is constant on U_p^+M for each $p \in M$. If $k + l$ is odd, then $|\nu_{k,l}|$ is constant on U_p^+M for each $p \in M$. In this case, if U_p^+M is connected, then $\nu_{k,l} = 0$ on U_p^+M.*

PROOF. It is clear when $k = 1$ or $l = 1$. So we only need to verify this lemma for $k, l \ge 2$. Let $c_\varepsilon(\theta)$ (resp. $s_\varepsilon(\theta)$) be $\cos\theta$ or $\cosh\theta$ (resp. $\sin\theta$ or $\sinh\theta$) according to whether ε is equal to $(+1)$ or (-1). For any $x \in U_p^+M$ $(p \in M)$ and $y \in U_pM$ such that $\langle x, y\rangle = 0$, $\langle y, y\rangle = \varepsilon \in \{-1, +1\}$, we put $z_\varepsilon(\theta) = c_\varepsilon(\theta)x + s_\varepsilon(\theta)y$, hence $z_\varepsilon(\theta) \in U_p^+M$ for any $\theta \in \mathbf{R}$. Using Lemma 3.4, we have $\frac{d}{d\theta}\big|_{\theta=0}\nu_{k,l}(z_\varepsilon(\theta)) = 0$. Thus $\nu_{k,l}$ is constant on a component of U_p^+M. On the other hand, for any $x \in U_p^+M$, $\nu_{k,l}(-x) = (-1)^{k+l}\nu_{k,l}(x)$. If U_p^+M is non-connected (hence it has two components), then the vector $(-x)$ is in the component which does not contain $x \in U_p^+M$. Therefore this lemma is proved. □

LEMMA 3.6. *Each $\nu_{k,l}$ is constant on U^+M for any $k, l \in \mathbf{N}$. In particular, if $k + l$ is odd, then $\nu_{k,l}$ vanishes.*

PROOF. Since it is clear when $k = 1$ or $l = 1$, we assume that $k, l \ge 2$. Because $\dim M \ge 2$, for any $p \in M$, $y \in U_pM$, there exists $x \in U_p^+M$ such that $\langle x, y\rangle = 0$. Using Lemma 3.4, we have

$$y \cdot \nu_{k,l}(X') = \langle (D^{k-2}B)(y, x^k), (D^{l-2}B)(x^l)\rangle + \langle (D^{k-2}B)(x^k), (D^{l-2}B)(y, x^l)\rangle = 0\,,$$

where X' is an extension of x along σ_x such that $\bar{\nabla}_y X' = 0$. Since the equation above holds for any $p \in M$, $y \in U_p M$, using Lemma 3.5, we can obtain this lemma in the cases where $k + l$ is even. When $k + l$ is odd and $U_p^+ M$ is connected, this lemma is clear by Lemma 3.5. In the case where $k + l$ is odd and $U_p^+ M$ is non-connected,

$$\begin{aligned} \nu_{k,l}(x) &= x \cdot \nu_{k-1,l}(X) - \nu_{k-1,l+1}(x) = -\nu_{k-1,l+1}(x) = \cdots \\ &= ((-1)^{(k-l-1)/2}/2)\, x \cdot \nu_{(k+l-1)/2,(k+l-1)/2}(X) = 0\,, \end{aligned}$$

where $x \in U_p^+ M$ ($p \in M$). Thus we have finished the proof of this lemma. □

Since f has null geodesic normal sections, we can see that $\sigma_v^{(k)}$ is a linear combination of $V, B(V^2), \dots, (D^{k-2}B)(V^k)$, where $v \in \Lambda$. Using Lemma 3.6, in the case where $k + l$ is even, there exists $d_{k,l} \in \mathbf{R}$ such that $\nu_{k,l}(v) = d_{k,l}\langle v, v\rangle^{(k+l)/2}$ for any $v \in TM$. Consequently $\nu_{k,l}(v) = 0$ for any $k, l \in \mathbf{N}$ and null vector $v \in \Lambda$. So, it is obvious that the scalar product at $\sigma_v(s) \in \bar{M}$ is vanishing on $\mathrm{Sp}\,\{\sigma_v^{(k)}(s) \mid k \in \mathbf{N}\}$ which is a subspace of $E(\gamma_v(s), \gamma_v'(s))$. Therefore we conclude

COROLLARY 3.7. *If f has space-like geodesic normal sections, then f has null geodesic normal sections. In particular, for any null vector $v \in \Lambda$, each normal section β_v is locally contained in a totally geodesic submanifold of $\bar{M}$ whose induced metric is identically vanishing.*

Lemma 3.6 implies that $G_k B$ is constant on $U^+ M$ for any $k \in \mathbf{N}$. Thus there uniquely exists $d \in \mathbf{N}$ such that $G_k B \neq 0$, $1 \leq k \leq d$, and $G_{d+1} B \equiv 0$. Then we call this natural number d the *geodesic non-degeneracy order* of f.

THEOREM 3.8. *Assume that $f : M \to \bar{M}$ is an isometric immersion with space-like geodesic normal sections and its geodesic non-degeneracy order is equal to d. Then, for any space-like* (*resp. time-like*) *unit speed geodesics γ of M, the curve $f \circ \gamma$ has constant Frenet curvatures $\lambda_1, \dots, \lambda_{d-1}$, signatures $\varepsilon_1, \dots, \varepsilon_d$* (*resp.* $(-1)^1\varepsilon_1, \dots, (-1)^d\varepsilon_d$) *in $\bar{M}$, where for $x \in U^+ M$,*

$$\varepsilon_k = \frac{\mathrm{sgn}\,(G_k B)(x)}{\mathrm{sgn}\,(G_{k-1} B)(x)}\,, \quad \lambda_k = \frac{|(G_{k-1}B)(x)|^{1/2}\,|(G_{k+1}B)(x)|^{1/2}}{|(G_k B)(x)|}\,.$$

PROOF. Using the formula (2) and Equation (5), we obtain this theorem for space-like geodesics of M. Since $G_k B$ is constant, say b_k, on $U^+ M$, we obtain $(G_k B)(v) = b_k\langle v, v\rangle^{k(k+1)/2}$ for any $v \in TM$. Thus we have $\mathrm{sgn}\,(G_k B)(x)/\mathrm{sgn}\,(G_{k-1}B)(x) = (-1)^k\varepsilon_k$ for $x \in U^- M$. Hence we can prove the statement for time-like geodesics in a similar way. □

From the definition of geodesic non-degeneracy order d, we can see that even if $f \circ \gamma$ is of proper order $> d$, it never has the d-th Frenet curvature. For convenience, we consider the following property:

(P_e) All geodesics of M are of proper order $\leq e$ in $\bar{M}$ and there exists a geodesic which is of proper order e in $\bar{M}$.

From the definition of Frenet curves, we obtain the following corollary.

COROLLARY 3.9. *Under the same assumption as in Theorem* 3.8, *we see that if* (P_d) *holds*, *then* f *is a helical geodesic immersion of order* d.

We showed in [7] that helical geodesic immersions have geodesic normal sections in semi-Riemannian geometry. For a helical geodesic immersion of order d, it is obvious that the helical order is equal to the geodesic non-degeneracy order, and (P_d) holds. Consequently the next corollary follows.

COROLLARY 3.10. *f is a helical geodesic immersion of order d if and only if f has geodesic normal sections with geodesic non-degeneracy order d and the property (P_d) holds.*

REMARK 3.11. In [8] we constructed helical geodesic immersions of arbitrary order d between semi-Riemannian spheres. Using these immersions, we can obtain an isometric immersion with geodesic normal sections between semi-Riemannian spheres such that its geodesic non-degeneracy order is equal to d and the property (P_e) holds, where d and e are any natural numbers with $d \leq e$.

Put $N := \sharp\{i \mid \varepsilon_i = -1, 1 \leq i \leq d\}$ and $\bar{N} := \sharp\{i \mid (-1)^i \varepsilon_i = -1, 1 \leq i \leq d\}$. If M is indefinite, then it is clear that $N, \bar{N} - 1 \leq \operatorname{ind} \bar{M} - \operatorname{ind} M$ and $d - N - 1, d - \bar{N} \leq \operatorname{codim} M - (\operatorname{ind} \bar{M} - \operatorname{ind} M)$.

COROLLARY 3.12. *Under the same assumption as in Theorem* 3.8, *we can see that if M is indefinite and either* $N = \operatorname{ind} \bar{M} - \operatorname{ind} M$, $\bar{N} - 1 = \operatorname{ind} \bar{M} - \operatorname{ind} M$, $d - N - 1 = \operatorname{codim} M - (\operatorname{ind} \bar{M} - \operatorname{ind} M)$ *or* $d - \bar{N} = \operatorname{codim} M - (\operatorname{ind} \bar{M} - \operatorname{ind} M)$ *holds*, *then* f *is a helical geodesic immersion of order* d.

PROOF. From Lemma 3.6 and the definition (4) of $C_{k,l}$, by induction, we can prove that each $C_{k,l}$ in Equation (3) is constant and that if $k + l$ is odd and $l \geq 2$, then $C_{k,l} = 0$. In the case where $N = \operatorname{ind} \bar{M} - \operatorname{ind} M$, the orthogonal complement of $\operatorname{Sp}\{\sigma_x^{(k)}(0) \mid 1 \leq k \leq d\}$ in $E(\gamma_x(0), \gamma_x'(0))$ is positive definite for any $x \in U^+M$. Then the definition of the geodesic non-degeneracy order and Equations (3) and (5) imply that $(D^{d-1}B)(x^{d+1})$ is a linear combination of $(D^{l-2}B)(x^l)$, $l \in \{2, 4, \ldots, d-1\}$ or $l \in \{3, 5, \ldots, d-1\}$ according as $d + 1$ is even or odd. Therefore we can obtain the following equation for $x \in U^+M$:

$$(D^{d-1}B)(x^{d+1}) = \sum_l c_l \, \langle x, x\rangle^{(d+1-l)/2} (D^{l-2}B)(x^l)\,, \tag{6}$$

where $c_l \in \mathbf{R}$. It is obvious that Equation (6) holds for any $x = v \in TM$. Hence any geodesics of M are of proper order $\leq d$ in $\bar{M}$, that is, (P_d) holds. It follows from Corollary 3.9 that f is helical. Also using a similar argument to the other cases, we obtain this corollary. $\square$

COROLLARY 3.13. *Under the same assumption as in Theorem* 3.8, *we can see that if* M *is space-like and either* $N = \operatorname{ind} \bar{M} - \operatorname{ind} M$ *or* $d - N - 1 = \operatorname{codim} M - (\operatorname{ind} \bar{M} - \operatorname{ind} M)$ *holds*, *then* f *is a helical geodesic immersion of order* d.

When $\bar{M}$ is space-like, that is, Riemannian, the equation $N = \operatorname{ind} \bar{M} - \operatorname{ind} M = 0$ holds. Thus the last corollary recovers results of Verheyen [11], and Hong and Houh [5].

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
TOKYO UNIVERSITY OF SCIENCE,
WAKAMIYA-CHO, SHINJUKU-KU, TOKYO, JAPAN 162–0827.
e-mail: miura@ma.kagu.tus.ac.jp