TOKYO J. MATH.
VOL. 30, NO. 2, 2007

Bicomplex Polygamma Function

Ritu GOYAL

University of Rajasthan

(Communicated by K. Uchiyama)

Abstract. The aim of this paper is to extend the domain of polygamma function from the set of complex numbers to the set of bicomplex numbers. We also discuss integral representation, recurrence relation, multiplication formula and reflection formula for this function.

1. Introduction, Definitions and Preliminaries

Corrado Segre published a paper [10] in 1892, in which he studied an infinite set of algebras whose elements, he called bicomplex numbers, tricomplex,..., *n*-complex numbers. The algebras of quarternions and bicomplex numbers were developed by making use of so called complex pairs. The work of Segre remained unnoticed for almost a century. But recently mathematicians have started taking interest in the subject and a new theory of special functions has started coming up.

Recently G. Baley Price [7] and Rönn [9] have developed the bicomplex algebra and function theory while Rochon [8] has developed a bicomplex Riemann zeta function. Recently we have extended the domain of Hurwitz zeta function, Gamma and Beta functions from the set of complex numbers to the set of bicomplex numbers [4], [5]. Motivated by this, in this paper we study the Polygamma function whose domain is the set of bicomplex numbers. Also we discuss integral representation, recurrence relation, multiplication formula and reflection formula for this function.

1.1. The Bicomplex Number.

DEFINITION 1. The bicomplex number ω is defined as follows [8]

$$\omega = a + bi_1 + ci_2 + di_1.i_2\,, \quad (a, b, c, d \in \mathbf{R})\,, \tag{1}$$

(where $i_1^2 = i_2^2 = -1$).
It can be written as

$$\omega = (a + bi_1) + (c + di_1)i_2\,, \quad (a, b, c, d \in \mathbf{R})\,, \tag{2}$$

Received May 25, 2006; revised September 25, 2006; revised October 9, 2006

or

$$\omega = z_1 + z_2 i_2\,, \quad (z_1, z_2 \in \mathbf{C})\,, \tag{3}$$

where $z_1 = a + bi_1$ and $z_2 = c + di_1$ and $\mathbf{C}$ is the set of complex numbers.

Let $\mathbf{C}_2$ denotes the set of bicomplex numbers, then

$$\mathbf{C}_2 = \{a + bi_1 + ci_2 + di_1.i_2 : i_1^2 = i_2^2 = -1,\ a, b, c, d \in \mathbf{R}\}\,. \tag{4}$$

Let $w = z_1 + z_2 i_2 \in \mathbf{C}_2$. Then w is non-invertible iff

$$z_1^2 + z_2^2 = 0 \Rightarrow z_1 + z_2 i_1 = 0 \ \ or \ \ z_1 - z_2 i_1 = 0\,. \tag{5}$$

The set of non-invertible elements in $\mathbf{C}_2$ is denoted by O_2.

1.2. Bicomplex Analysis. (a) The bicomplex functions of interest are holomorphic functions and characterized by the fact that they are differentiable. It is possible to define differentiability of a function at a point of $\mathbf{C}_2$ [8]:

DEFINITION 2. Let U be an open set of $\mathbf{C}_2$ and $w_o \in U$. Then $f : U \subseteq \mathbf{C}_2 \to \mathbf{C}_2$ is said to be $\mathbf{C}_2$-differentiable at w_o with derivative equal to $f'(w_o) \in \mathbf{C}_2$, (with $(w - w_o)$ invertible) if

$$\lim_{w \to w_o} \frac{f(w) - f(w_o)}{w - w_o} = f'(w_o) \tag{6}$$

exists.

We shall say that the function f is holomorphic on an open set U iff f is $\mathbf{C}_2$-differentiable at each point of U.

DEFINITION 3. ([8]). Let U be an open set and $f : U \subseteq \mathbf{C}_2 \to \mathbf{C}_2$. Also suppose that $f(z_1 + z_2 i_2) = f_1(z_1, z_2) + i_2 f_2(z_1, z_2)$. Then f is T-holomorphic on U iff f_1 and f_2 are holomorphic in U and satisfy

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \ \text{ and } \ \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{on } U\,. \tag{7}$$

Equations (7) are known as complexified Cauchy-Riemann (C-R) equations and the function f is said to be in class $TH(U)$, $(U \subseteq \mathbf{C}_2)$ of holomorphic functions.

(b) The *bicomplex integration* of a bicomplex function $\phi(\omega) = \phi_1(z_1, z_2) + i_2\phi_2(z_1, z_2)$ is defined as a line integral, that is evaluated with respect to some four-dimensional curve H in $\mathbf{C}_2$. More specifically, the bicomplex integration (see, e.g. Rönn [9]) is defined as

$$I = \int_H \phi(\omega) \otimes d\omega\,, \quad d\omega = (dz_1, dz_2) \tag{8}$$

where H is a piecewise continuously differentiable curve in $\mathbf{C}_2$ and has the parametric equation $H : P = P(t)$, $P(t) = (a(t), b(t))$, for $r \le t \le s$ and H can be taken as a curve made up of two component curves γ_1 and γ_2 in $\mathbf{C}_1$ i.e. $H = (\gamma_1, \gamma_2)$. Thus

$$I = \int_r^s \phi[\omega(t)] \otimes \omega'(t) \otimes dt\,. \tag{9}$$

1.3. The Idempotent Basis. Every Bicomplex number $w = z_1 + z_2 i_2$ has the following unique idempotent representation

$$w = z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2\,, \tag{10}$$

where $e_1 = (1 + i_1.i_2)/2$, $e_2 = (1 - i_1.i_2)/2$.

This representation is very useful because addition, multiplication and division can be done term by term.

DEFINITION 4. $A \subseteq \mathbf{C}_2$ is said to be a $\mathbf{C}_2$-cartesian set determined by $A_1, A_2 \subseteq \mathbf{C}$, if $A = A_1 \times_e A_2 = \{z_1 + z_2 i_2 \in \mathbf{C}_2 : z_1 + z_2 i_2 = w_1 e_1 + w_2 e_2\,, (w_1, w_2) \in A_1 \times A_2\}$.

Obviously if $A_1, A_2 \subseteq \mathbf{C}$, then $A_1 \times_e A_2 \subseteq \mathbf{C}_2$. Thus if $w \in \mathbf{C}_2$ is in the form $w = w_1 e_1 + w_2 e_2$ and also if $w = z_1 + z_2 i_2$, then

$$w_1 = z_1 - z_2 i_1\,, \quad w_2 = z_1 + z_2 i_1\,, \tag{11}$$

$$z_1 = (w_1 + w_2)/2\,, \quad z_2 = (w_1 - w_2) i_1/2\,. \tag{12}$$

The results contained in the following Lemmas will also be required in the sequel.

LEMMA 1. *If $f_{e_1} : A_1 \to \mathbf{C}_1$ and $f_{e_2} : A_2 \to \mathbf{C}$ are holomorphic (analytic) functions in $\mathbf{C}$ on the domains A_1 and A_2 respectively, Then the function $f : A_1 \times_e A_2 \to \mathbf{C}_2$ defined as*

$$f(z_1 + z_2 i_2) = f_{e_1}(z_1 - z_2 i_1)e_1 + f_{e_2}(z_1 + z_2 i_1)e_2 \tag{13}$$

$\forall z_1 + z_2 i_2 \in A_1 \times_e A_2$, *is T-holomorphic on the domain $A_1 \times_e A_2 \subseteq \mathbf{C}_2$.*

LEMMA 2. *If $f_{e_1} : A_1 \to \mathbf{C}$ and $f_{e_2} : A_2 \to \mathbf{C}$ are holomorphic functions in $\mathbf{C}$ on the domains A_1 and A_2 respectively, Then we can define a function $f : A_1 \times_e A_2 \to \mathbf{C}_2$ as*

$$f(\omega) = f_{e_1}(\omega_1)e_1 + f_{e_2}(\omega_2)e_2 \tag{14}$$

where $\omega = \omega_1 e_1 + \omega_2 e_2 \Rightarrow d\omega = (d\omega_1)e_1 + (d\omega_2)e_2 \;\forall\; \omega \in A_1 \times_e A_2 \subseteq \mathbf{C}_2$. Now we have

$$\int_H f(\omega) \otimes d\omega = \left\{ \int_{\gamma_1} f_{e_1}(\omega_1) \otimes d\omega_1 \right\} e_1 + \left\{ \int_{\gamma_2} f_{e_2}(\omega_2) \otimes d\omega_2 \right\} e_2 \tag{15}$$

where H : $\omega = \omega(t)$, $\omega(t) = \omega_1(t)e_1 + \omega_2(t)e_2$ for $r \le t \le s$.

1.4. Polygamma Function. A special function which is given by the $(n+1)^{th}$ derivative of the logarithm of the gamma function is called the *Polygamma function* and is denoted by $\psi_n(z)$ (see, e.g. [2], [6])

$$\begin{aligned}\psi_n(z) &= \frac{d^{n+1}}{dz^{n+1}}\ln[\Gamma(z)] \\ &= \frac{d^n}{dz^n}\frac{\Gamma'(z)}{\Gamma(z)} \\ &= \frac{d^n}{dz^n}\psi_0(z), n > 0\end{aligned} \tag{16}$$

where $\psi_0(z)$ is *Digamma function*.

Also, Polygamma function is related to Hurwitz Zeta function $\zeta(s,a)$ as

$$\begin{aligned}\psi_n(z) &= (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}} \\ &= (-1)^{n+1}n!\zeta(n+1,z)\,.\end{aligned} \tag{17}$$

The polygamma function obeys the recurrence relation

$$\psi_n(z+1) = \psi_n(z) + (-1)^n n! z^{-n-1}\,, \tag{18}$$

the reflection formula

$$\psi_n(1-z) + (-1)^{n+1}\psi_n(z) = (-1)^n\pi\frac{d^n}{dz^n}\cot(\pi z)\,, \tag{19}$$

and the multiplication formula

$$\psi_n(mz) = \delta_{n0}\ln m + \frac{1}{m^{n+1}}\sum_{k=0}^{m-1}\psi_n\left(z+\frac{k}{m}\right)\,, \tag{20}$$

where δ_{mn} is *Kronecker delta*.

The *Euler Mascheroni constant* is a special value of the Digamma function $\psi_0(z)$ given by

$$\gamma = -\Gamma'(1) = -\psi_0(1)\,. \tag{21}$$

The Polygamma function may be represented in integral form as

$$\psi_n(z) = \int_0^{\infty}\frac{t^m e^{-zt}}{1-e^{-t}}dt \tag{22}$$

which holds for $\mathrm{Re}(z) > 0$.

The Taylor series at $z = 1$ is

$$\psi_n(z+1) = \sum_{k=0}^{\infty}(-1)^{n+k+1}(n+k)!\zeta(n+k+1)\frac{z^k}{k!} \tag{23}$$

which converges for $|z| < 1$. Here $\zeta(n)$ is the Riemann zeta function. This series can be derived easily from the corresponding Taylor series for the Hurwitz zeta function.

The Polygamma functions are analytic everywhere in the complex plane except for the poles (of order $n+1$) at all non-positive integers. The residues at these poles are all given by $(-1)^{n+1}n!$.

2. Bicomplex Polygamma Functions

DEFINITION 5. Let $n \in \mathbf{N}$, $\omega = z_1 + z_2 i_2 \in \mathbf{C}_2$. Then bicomplex polygamma function $\psi_n(\omega)$ is defined by

$$\begin{aligned}\psi_n(\omega) &= (-1)^{n+1} n! \zeta(n+1, \omega) \\ &= \frac{d^{n+1}}{d\omega^{n+1}}[\ln \Gamma(\omega)]\end{aligned} \tag{24}$$

$(m+\omega) \notin O_2$, $m \in \mathbf{N} \cup \{0\}$ and O_2 is the set of non-invertible elements in $\mathbf{C}_2$.

This definition is well justified by the following

THEOREM 2. *Let* $\omega = z_1 + z_2 i_2 \in \mathbf{C}_2$, $m + \omega \notin O_2$, $m \in \mathbf{N} \cup \{0\}$. *Then*

$$\begin{aligned}\psi_n(\omega) &= (-1)^{n+1} n! \left[\sum_{k=0}^{\infty} \frac{1}{(z_1 - z_2 i_1 + k)^{n+1}} e_1 + \sum_{k=0}^{\infty} \frac{1}{(z_1 + z_2 i_1 + k)^{n+1}} e_2\right] \\ &= \psi_n(z_1 - z_2 i_1) e_1 + \psi_n(z_1 + z_2 i_1) e_2 \,.\end{aligned} \tag{25}$$

PROOF. By the definition of Bicomplex Hurwitz Zeta function [4], we have

$$\begin{aligned}\zeta(n+1, \omega) &= \zeta(n+1,\ z_1 - z_2 i_1) e_1 + \zeta(n+1,\ z_1 + z_2 i_1) e_2 \\ &= \sum_{k=0}^{\infty} \frac{1}{(z_1 - z_2 i_1 + k)^{n+1}} e_1 + \sum_{k=0}^{\infty} \frac{1}{(z_1 + z_2 i_1 + k)^{n+1}} e_2 \,.\end{aligned} \tag{26}$$

Thus

$$\begin{aligned}(-1)^{n+1} n! \zeta(n+1, \omega) &= (-1)^{n+1} n! \zeta(n+1,\ z_1 - z_2 i_1) e_1 \\ &\quad + (-1)^{n+1} n! \zeta(n+1,\ z_1 + z_2 i_1) e_2 \\ &= \psi_n(z_1 - z_2 i_1) e_1 + \psi_n(z_1 + z_2 i_1) e_2 \,.\end{aligned} \tag{27}$$

Further, using (16) in (27), we get

$$\begin{aligned}\psi_n(\omega) &= \frac{d^{n+1}}{d(z_1 - z_2 i_1)^{n+1}} \ln[\Gamma(z_1 - z_2 i_1)] e_1 + \frac{d^{n+1}}{d(z_1 + z_2 i_1)^{n+1}} \ln[\Gamma(z_1 + z_2 i_1)] e_2 \\ &= \left(\frac{d^{n+1}}{d(z_1 - z_2 i_1)^{n+1}} e_1 + \frac{d^{n+1}}{d(z_1 + z_2 i_1)^{n+1}} e_2 \right) \\ &\quad \times (\ln\{\Gamma(z_1 - z_2 i_1) e_1 + \Gamma(z_1 + z_2 i_1) e_2\}) \\ &= \frac{d^{n+1}}{d\omega^{n+1}} \ln[\Gamma(\omega)] \qquad (\text{using } [5, \text{ p.136, } (30)]) .\end{aligned} \tag{28}$$

2.1. Domain of Bicomplex Polygamma Functions. If O_2 is set of non-invertible elements as in (5) in $\mathbf{C}_2$, we extend the domain of $\psi_n(\omega)$ on the set $\mathbf{C}_2/\{-m+O_2\}$, $m \in \mathbf{N} \cup \{0\}$ as follows:

We know from definition of Polygamma function that it has poles of order $(n+1)$ with residue $(-1)^{n+1} n!$ when

$$(z_1 - z_2 i_1) = -m \text{ or } (z_1 + z_2 i_1) = -m, \ \ m \in \mathbf{N} \cup \{0\}$$

combining above and using (25), we get poles of bicomplex polygamma function for

$$\omega \in \{-m + O_2\}, \quad m \in \mathbf{N} \cup \{0\} . \tag{29}$$

LEMMA 3. *Bicomplex Polygamma functions are T-holomorphic on* $\mathbf{C}_2/\{-m + O_2\}$, $m \in \mathbf{N} \cup \{0\}$.

PROOF. We define $\mathbf{C}_2$ cartesian set

$$\begin{aligned}A = A_1 \times_e A_2 = \{\omega &= z_1 + z_2 i_2 \in \mathbf{C}_2 : z_1 + z_2 i_2 \\ &= \omega_1 e_1 + \omega_2 e_2, \ \omega_1, \omega_2 \neq (-m), \ (\omega_1, \omega_2) \in A_1 \times A_2\} .\end{aligned} \tag{30}$$

Let us define $f_{e_1} : A_1 \to \mathbf{C}$, $f_{e_1}(x) = \psi_n(x)$ and $f_{e_2} : A_2 \to \mathbf{C}$, $f_{e_2}(x) = \psi_n(x)$. Then the function $f : A_1 \times_e A_2 \to \mathbf{C}_2$ is defined by $f(\omega) = \psi_n(\omega)$, $\forall\ \omega \in A_1 \times_e A_2$.

Now since the function $\psi_n(x)$ is analytic in $\mathbf{C}$ on the domains A_1 and A_2 respectively, by definition (25) and Lemma 1, we find that $\psi_n(\omega)$ is T-holomorphic on the domain $A_1 \times_e A_2 \subseteq \mathbf{C}_2$.

REMARK. Let $\omega_0 \in \{-m + O_2\}$ then for $\omega \notin \{-m + O_2\}$

$$\lim_{\omega \to \omega_0} |\psi_n(\omega)| = \infty . \tag{31}$$

Hence the domain $\mathbf{C}_2/\{-m + O_2\}$ is the best possible.

2.2. Integral Representation. *Integral representation for bicomplex Polygamma function is given by*

$$\psi_n(\omega) = \int_H \frac{e^{-\omega p} \otimes p^n}{1 - e^{-p}} \otimes dp \tag{32}$$

provided that $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$. *Where* $\omega = z_1 + i_2 z_2 \in \mathbf{C}_2$, $p = p_1 e_1 + p_2 e_2 \in \mathbf{C}_2$, $p_1,\ p_2 \in R^+$, *and* $H \equiv (\gamma_1, \gamma_2)$, $\gamma_1 \equiv \gamma_1(p_1)$ *and* $\gamma_2 \equiv \gamma_2(p_2)$.

PROOF. The representation given by (32) is justified since by using (22) and Lemma 2, we have

$$\begin{aligned}\int_H \frac{e^{-\omega p} \otimes p^n}{1-e^{-p}} \otimes dp &= \int_0^\infty \frac{e^{-(z_1 - i_1 z_2)p_1} p_1^n}{1-e^{-p_1}} dp_1 e_1 + \int_0^\infty \frac{e^{-(z_1 + i_1 z_2)p_2} p_2^n}{1-e^{-p_2}} dp_2 e_2 \\ &= \psi_n(z_1 - i_1 z_2)e_1 + \psi_n(z_1 + i_1 z_2)e_2 = \psi_n(\omega)\,.\end{aligned} \tag{33}$$

Also, by the condition for the integral representation (22) for Polygamma function, we have

$$\mathrm{Re}(z_1 - i_1 z_2) > 0 \text{ and } \mathrm{Re}(z_1 + i_1 z_2) > 0$$

Let $z_1 = \sigma_1 + t_1 i_1$ and $z_2 = \sigma_2 + t_2 i_1$, $\sigma_1, \sigma_2, t_1, t_2 \in \mathbf{R}$. Then

$$\sigma_1 + t_2 > 0 \text{ and } \sigma_1 - t_2 > 0$$

$$\Rightarrow \sigma_1 > 0 \text{ and } -\sigma_1 < t_2 < \sigma_1$$

$$\Rightarrow \mathrm{Re}(z_1) > 0 \text{ and } |\mathrm{Im}(z_2)| < \mathrm{Re}(z_1)\,. \tag{34}$$

2.3. Recurrence relation. *Recurrence relation for Polygamma function is given by*

$$\psi_n(\omega + 1) = \psi_n(\omega) + (-1)^n n! \omega^{-(n+1)} \tag{35}$$

PROOF. By the recurrence relation (18) and the result (25), we have

$$\begin{aligned}\psi_n(\omega+1) &= \psi_n(z_1 - z_2 i_1 + 1)e_1 + \psi_n(z_1 + z_2 i_1 + 1)e_2 \\ &= \{\psi_n(z_1 - z_2 i_1)e_1 + \psi_n(z_1 + z_2 i_1)e_2\} \\ &\quad + (-1)^n n! [(z_1 - z_2 i_1)^{-(n+1)} e_1 + (z_1 + z_2 i_1)^{-(n+1)} e_2]\,.\end{aligned} \tag{36}$$

Hence using (25), we get (35).

2.4. Reflection formula and Multiplication formula. Using (19), (20) and (25), we have the Reflection formula for bicomplex Polygamma function given by

$$\psi_n(1-\omega) + (-1)^{n+1}\psi_n(\omega) = (-1)^n \pi \frac{d^n}{d\omega^n} \cot(\pi\omega)\,, \tag{37}$$

and multiplication formula given by

$$\psi_n(m\omega) = \delta_{n0} \ln m + \frac{1}{m^{n+1}} \sum_{k=0}^{m-1} \psi_n\left(\omega + \frac{k}{m}\right)\,. \tag{38}$$

ACKNOWLEDGEMENT. The author is thankful to Council of Scientific and Industrial Research, India for providing financial assistance under research scheme No.

9/149(294)/2K2/ EMR-I. Also she is thankful to Prof. S.P. Goyal, Department of mathematics, University of Rajasthan, Jaipur, for his kind and able guidance during the preparation of this paper.

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Present Address:
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF RAJASTHAN,
JAIPUR-302004, INDIA.
e-mail: ritugoyal_1980@yahoo.co.in