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The Multiplicity Function of Mixed Representations on Completely Solvable Lie Groups

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Abstract. We show in this paper that the multiplicities of mixed representations are uniformly infinite or uniformly finite and bounded, in the setting of completely solvable Lie groups extending then the situation of nilpotent Lie groups. Necessary and sufficient conditions for these multiplicities to be finite are provided.

1. Introduction

Recently, it has been shown that mixed representations (up-down and down-up representations) of exponential solvable Lie groups obey the orbital spectrum formula (see [1] and [2]). Explicit formulae of the multiplicity function occurring in the disintegration of such representations were given. The present work is a continuation of the papers [1] and [2] where detailed information about the behavior of the multiplicity function of mixed representations have been given in the setting of nilpotent Lie groups. More precisely, we proved that the multiplicities of such representations are uniformly infinite or finite and bounded. Necessary and sufficient conditions for the multiplicities to be finite were provided. We show in the present work that such results can be extended to encompass completely solvable Lie groups.

2. Generalities and notations

2.1. We begin this section by reviewing some facts about induced and restricted representations of a solvable Lie group. One says that G is an exponential solvable Lie group if the exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Throughout, G always denotes a connected and simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. The group G acts on $\mathfrak{g}^*$ by the coadjoint representation. For any f in $\mathfrak{g}^*$, define the skew-symmetric bilinear form B_f on $\mathfrak{g}$ by the formula $B_f(X,Y) = f([X,Y])$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then we write $\mathfrak{h}(f) = \{X \in \mathfrak{g} \mid B_f(X,\mathfrak{h}) = \{0\}\}$ and $\mathfrak{h}^\perp = \{l \in \mathfrak{g}^* : l_{|\mathfrak{h}} = 0\}$, where $l_{|\mathfrak{h}}$ stands for the restriction of l to $\mathfrak{h}$.

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If $\mathfrak{h} \subset \mathfrak{h}(f)$, then $\mathfrak{h}$ is said to be an isotropic subalgebra of f. Denote by $S(f, \mathfrak{g})$ the set of all isotropic subalgebras of f. If $\mathfrak{h}$ is in $S(f, \mathfrak{g})$, we define the unitary character χ_f of the group $H = \exp(\mathfrak{h})$ by

$$\chi_f(\exp(Y)) = e^{if(Y)}, \quad Y \in \mathfrak{h} .$$

Let σ be a unitary representation of H acting on a Hilbert space $\mathcal{H}_\sigma$, and let db be the left Haar measure on H. The induced representation $\tau(\sigma) = \mathrm{Ind}_H^G \sigma$ acts on the completion of the space

$$C_c^\infty(G, H, \sigma) = \{f : G \to \mathcal{H}_\sigma,\ f \text{ is } C^\infty,\ f(gh) = \Delta_{H,G}^{1/2}(h)\sigma^{-1}(h)f(g),$$
$$h \in H,\ g \in G, \|f\| \text{ compactly supported mod } H\}$$

by the formula

$$\tau(\sigma)(g)f(x) = f(g^{-1}x) . \tag{2.1}$$

It uniquely specifies a G-invariant positive linear form $d\dot{g}$ on G/H which obeys the equality:

$$\int_G f(x)dx = \oint_{G/H} \int_H f(gh)dhd\dot{g} .$$

(see [4]). The action given by formula (2.1) extends to a unitary representation of G on $L^2(G/H, d\dot{g}) = L^2(G, H, \sigma)$.

Let $M(f, \mathfrak{g})$ be the set of elements of $S(f, \mathfrak{g})$ of maximal dimension and $I(f, \mathfrak{g})$ the subset of $S(f, \mathfrak{g})$ consisting of subalgebras $\mathfrak{h}$, such that $\tau(f) = \tau(\chi_f)$ is irreducible. Then, as is known, an element $\mathfrak{h}$ of $M(f, \mathfrak{g})$ is in $I(f, \mathfrak{g})$ if and only if $\mathfrak{h}$ is a Pukanszky polarization, i.e.,

$$H \cdot f = f + \mathfrak{h}^\perp = \Gamma_f . \tag{2.2}$$

This condition holds for all elements of $M(f, \mathfrak{g})$ when G is nilpotent.

The dual space $\hat{G}$ of equivalence classes of irreducible and unitary representations of G is parameterized canonically by the orbit space $\mathfrak{g}^*/G$. More precisely, for $l \in \mathfrak{g}^*$, we may find a real polarization $\mathfrak{b}$ for l which satisfies the Pukanszky condition. The representation $\pi_l = \mathrm{Ind}_B^G \chi_f$ ($B = \exp \mathfrak{b}$) is then irreducible and its class is independent of the choice of $\mathfrak{b}$. The Kirillov-Bernat mapping $\Theta_G : \mathfrak{g}^* \to \hat{G},\ l \mapsto \pi_l$ is surjective and factors to a bijection from $\mathfrak{g}^*/G$ on $\hat{G}$. Given $\pi \in \hat{G}$, we denote by $\Omega_\pi^G \subset \mathfrak{g}^*$ the coadjoint orbit associated to π.

2.2. Let $p_\mathfrak{h} : \mathfrak{g}^* \to \mathfrak{h}^*$ be the canonical projection of $\mathfrak{h}$ and Ω_σ^H the coadjoint orbit associated to the representation σ by the Kirillov-Bernat mapping $\Theta_H : \mathfrak{h}^* \to \hat{H}$, (where $\hat{H}$ is the unitary dual of H). The natural measure on $p_\mathfrak{h}^{-1}(\Omega_\sigma^H)$ is the fiber measure with H-invariant measure in the base Ω_σ^H and the Lebesgue measure on the affine fiber $\mathfrak{h}^\perp$. It follows from Fujiwara's result ([9]) that the representation $\tau(\sigma)$ obeys the orbital spectrum formula, i.e.,

$$\tau(\sigma) \simeq \int_{G\cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H)/G}^{\oplus} n_\phi^\sigma \pi_\phi d\nu_{G,H}^\sigma(\phi)\,, \tag{2.3}$$

where $\nu_{G,H}^\sigma$ is the push-forward of the natural measure on $p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \subset \mathfrak{g}^*$ under the mapping $p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \to G \cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H)/G$, and the value of the multiplicity function n_ϕ^σ is given by the number of H-orbits in $p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \cap G\cdot\phi$, we denote this number by $\#[p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \cap G\cdot\phi/H]$ (For a set E, the symbol $\#E$ means the cardinality of E).

If G is a completely solvable Lie group, then the multiplicity function $\phi \mapsto n_\phi^\sigma$ is either uniformly infinite or uniformly finite and bounded. In particular, these multiplicities are finite if and only if

$$\dim G\cdot\phi - 2\dim H\cdot\phi + \dim \Omega_\sigma^H = 0\,, \tag{2.4}$$

for generic $\phi \in p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H)$ (see [15]).

2.3. Let $A = \exp(\mathfrak{a})$ be an analytic subgroup of G associated to a subalgebra $\mathfrak{a}$ and π a unitary and irreducible representation which is associated to the coadjoint orbit $\Omega_\pi^G \subset \mathfrak{g}^*$. Let also $p_{\mathfrak{a}} : \mathfrak{g}^* \to \mathfrak{a}^*$ be the canonical projection of $\mathfrak{a}$, and let $\pi_{|A}$ be the restriction of π to A. The restriction $\pi_{|A}$ obeys the orbital spectrum formula

$$\pi_{|A} = \int_{p_{\mathfrak{a}}(\Omega_\pi^G)/A}^{\oplus} n_\pi^\psi \sigma_\psi d\mu_{A,G}^\pi(\psi) \tag{2.5}$$

where $n_\pi^\psi = \#[(\Omega_\pi^G \cap p_{\mathfrak{a}}^{-1}(A\cdot\psi))]/A$, and $\mu_{A,G}^\pi$ is the push-forward of the canonical measure on Ω_π^G under the mapping $\Omega_\pi^G \to p_{\mathfrak{a}}(\Omega_\pi^G)/A$ (see [10]).

In the completely solvable case, it is likewise known that the multiplicity function $\psi \mapsto n_\pi^\psi$ is either uniformly infinite or uniformly finite and bounded. These multiplicities are finite if and only if, we have

$$\dim \Omega_\pi^G - 2\dim A\cdot\phi + \dim A\cdot p_{\mathfrak{a}}(\phi) = 0\,, \tag{2.6}$$

generically on Ω_π^G (see [15]).

3. Pseudo-algebraic geometry

3.1. Pseudo-algebraic sets.

DEFINITION 3.1. Let P be a polynomial function with real coefficients on $\boldsymbol{R}^{m+k}$ and $\lambda_i : \boldsymbol{R}^m \to \boldsymbol{R}$, $1 \le i \le k$ are real linear functionals. The function

$$F_P = F : \boldsymbol{R}^m \to \boldsymbol{R},\ x \mapsto P(x, e^{\lambda_1(x)}, \ldots, e^{\lambda_k(x)})\,, \tag{3.1}$$

is called a *pseudo-algebraic* function associated to the polynomial P.

DEFINITION 3.2. A subset V of $\boldsymbol{R}^m$ is called *pseudo-algebraic* if it admits some representation of the form

$$V = \{x \in \boldsymbol{R}^m : F_1(x) = \cdots = F_r(x) = 0\}, \tag{3.2}$$

where $F_i, 1 \le i \le r$ are pseudo-algebraic functions on $\boldsymbol{R}^m$.

DEFINITION 3.3. A subset A of $\boldsymbol{R}^m$ is called *semi-pseudo-algebraic* if it admits some representation of the form

$$A = \{x \in \boldsymbol{R}^m : F_1(x) = \cdots = F_r(x) = 0, \ G_1(x) > 0, \dots, G_l(x) > 0\}, \tag{3.3}$$

where $F_i = F_{P_i}, 1 \le i \le r$ and $G_j = G_{Q_j}, 1 \le j \le l$ are pseudo-algebraic functions on $\boldsymbol{R}^m$ associated respectively to the polynomials P_i and Q_j.

It is so clear that any semi-pseudo-algebraic set of $\boldsymbol{R}^m$ is the intersection of the closed set of common zeros of a finite number of pseudo-algebraic functions on $\boldsymbol{R}^m$ and an open set of $\boldsymbol{R}^m$.

The following theorem plays an important role in the sequel of the paper, as the analysis of the multiplicity function of mixed representations naturally involves the pseudo-algebraic geometry (see next section for details).

THEOREM 3.4. *Let A be a semi-pseudo-algebraic set of $\boldsymbol{R}^m$, defined as in* (3.3). *Then the number of connected components of A in the Euclidean topology is finite and is bounded by a scalar which depends only on m, r, l, k,* $\deg P_i$, *and* $\deg Q_j$, *but not on the coefficients of either the polynomials or the linear functionals.*

PROOF. We argue as in Proposition 4.4.5 of [3]. Let $\pi : \boldsymbol{R}^{m+1} \to \boldsymbol{R}^m$ be the natural continuous projection $(x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_m)$ and let $A_2 \subset \boldsymbol{R}^{m+1}$ be the pseudo-algebraic set defined by

$$\begin{aligned} A_2 = \{(x_1, \dots, x_{m+1}) = (x, x_{m+1}) \in \boldsymbol{R}^{m+1} : F_1(x) = \cdots = F_r(x) = 0 \\ x_{m+1} G_1(x) \cdots G_l(x) = 1\}. \end{aligned}$$

Then the set $A_1 = \pi(A_2)$ is defined by

$$A_1 = \{x \in \boldsymbol{R}^m : F_1(x) = \cdots = F_r(x) = 0, G_1(x) \cdots G_l(x) \neq 0\}.$$

Indeed, suppose $(x, x_{m+1}) \in A_2$, then

$$F_1(x) = \cdots = F_r(x) = 0 \quad \text{and} \quad G_1(x) \cdots G_l(x) \neq 0,$$

so $x \in A_1$ and hence $\pi(A_2) \subset A_1$. Conversely, let $x \in A_1$, since $G_1(x) \cdots G_l(x) \neq 0$, we take $x_{m+1} = (G_1(x) \cdots G_l(x))^{-1}$. Then $(x, x_{m+1}) \in A_2$, it follows that $x \in \pi(A_2)$ and $A_1 \subset \pi(A_2)$. On the other hand, the subset $\{x \in \boldsymbol{R}^m : G_i(x) > 0\}$ is open and closed in $\{x \in \boldsymbol{R}^m : G_i(x) \neq 0\}$, for all $i = 1, \dots, l$. Hence A is open and closed in A_1 and each connected component of A is also a connected component of A_1. Finally, the number of connected components of A is less than or equal to the number of connected components of A_1 which is in turn less than or equal to the number of connected components of A_2, as $A_1 = \pi(A_2)$. The following result, which can be found in [15], enables us to conclude. Every pseudo-algebraic subset V of $\boldsymbol{R}^m$ defined as in (3.2) has only finitely many components in the Euclidean topology. In fact, the number of the connected components of V is bounded by a scalar which depends on m, r, l, k, deg P_i and deg Q_j, but not on the coefficients of either the polynomials or the linear functionals. $\square$

COROLLARY 3.5. *Let $S \subset \boldsymbol{R}^m$ be a semi-pseudo-algebraic set defined as in* (3.3)*. Let $F(x_1, \dots, x_m, y_1, \dots, y_k)$ be a polynomial function with real coefficients. Suppose $\gamma_i : \boldsymbol{R}^m \to \boldsymbol{R}$, $1 \leq i \leq k$ are real linear functionals. Then there is a number N depending only on m, r, l, k,* deg P_i, deg Q_j, *and* deg F, *such that either*

(1) *$F(x, e^{\gamma_1(x)}, \dots, e^{\gamma_k(x)}) = 0$ has an infinite number of solutions in S,*

or

(2) *the number of solutions in S is bounded by N.*

PROOF. Suppose that the number of solutions in S of the equation

$$F(x, e^{\gamma_1(x)}, \dots, e^{\gamma_k(x)}) = 0 \tag{3.4}$$

is finite, so it is equal to the number of connected components of the semi-pseudo-algebraic set $V \cap S$ where

$$V = \{x \in \boldsymbol{R}^m : F(x, e^{\gamma_1(x)}, \dots, e^{\gamma_k(x)}) = 0\},$$

and the corollary follows from the previous theorem. $\square$

3.2. Semi-analytic sets. We now recall some properties of semi-analytic sets in $\boldsymbol{R}^n$. This material is quite standard (see [11], [17]). Given an open neighborhood $U \subset \boldsymbol{R}^n$, let $C^\omega(U, \boldsymbol{R})$ be the set of real analytic functions on U. Denote by $B(C^\omega(U, \boldsymbol{R}))$ the boolean algebra generated by sets of the form

$$\{x \in U : f_1(x) = \cdots = f_p(x) = 0,\ g_1(x) > 0, \dots, g_q(x) > 0\}, \tag{3.5}$$

where $f_1, \dots, f_p, g_1, \dots, g_q$ are in $C^\omega(U, \boldsymbol{R})$.

DEFINITION 3.6. A subset M of $\boldsymbol{R}^n$ is *semi-analytic* if for every $x \in \boldsymbol{R}^n$, there is an open neighborhood U of x such that $M \cap U \in B(C^\omega(U, \boldsymbol{R}))$.

It is well-known that the complement, the finite intersection and the locally finite union of semi-analytic sets is semi-analytic. The inverse image of a semi-analytic by any analytic map is semi-analytic. The closure, the interior and the boundary of any semi-analytic set are semi-analytic.

Let M be a semi-analytic set in $\boldsymbol{R}^n$ and p a positive integer. A point $x \in M$ is said to be p-regular if there is an open neighborhood U of x such that $M \cap U$ is an analytic sub-manifold of dimension p of U; x is 0-regular if it is an isolated point of M. The set of regular points of M (i.e., p-regular points for some p) is dense in M. The dimension $\dim M$ of M is less than or equal to p if there are not q-regular points of M with $q > p$; $\dim M = p$ if $\dim M \leq p$

but not $\dim M \leq p-1$. Let $\dim M = p$; then $\dim \bar{M} = p$ and $\dim(\bar{M} \setminus M) < p$ ($\bar{M}$ denotes the closure of M).

3.3. Structure of coadjoint orbits. The following theorem describes the structure of the coadjoint orbits space of an exponential solvable Lie group and is proved in [8].

THEOREM 3.7. *Let G be a connected and simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Then there is a finite partition $\wp$ of $\mathfrak{g}^*$ such that:*

(1) *each $U \in \wp$ is G-invariant,*

(2) *for a given $U \in \wp$, the dimension of the coadjoint orbits in U is constant,*

(3) *there is a total ordering $U_1 < U_2 < \cdots < U_u$ of $\wp$ such that for each U, $\cup\{U' : U' \leq U\}$ is Zariski open in $\mathfrak{g}^*$.*

Given $U \in \wp$, there is a subspace V in $\mathfrak{g}^*$, and there are associated to U indices $\imath$ and φ, for each $j \in \imath$ a complex valued rational function p_j on $\mathfrak{g}^*$ and for each $j \in \varphi$, a complex valued rational function q_j on $\mathfrak{g}^*$ such that

(4) *the set $\Sigma = \{l \in V \cap U : p_j(l) = 0,\ j \in \imath,\ |q_j(l)|^2 = 1,\ j \in \varphi\}$ is a cross-section for the coadjoint orbits in U.*

Moreover, there is an analytic G-invariant function $P : U \to U$ such that $P(U) = \Sigma$.

REMARK 3.8. If G is a completely solvable Lie group, then φ is empty and for each $j \in \imath$, p_j is a real valued rational function on $\mathfrak{g}^*$ (see [7]).

COROLLARY 3.9. *Let G be a connected and simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Every coadjoint orbit Ω (lying in a layer U) is closed in U and is a semi-analytic set in $\mathfrak{g}^*$. In particular Ω is locally closed in $\mathfrak{g}^*$.*

PROOF. Let U be a layer in $\wp$ such that $\Omega \subset U$ and let $P : U \to U$ be the analytic G-invariant function such that $P(U) = \Sigma$, where Σ is a cross-section for the coadjoint orbits in U, as in Theorem 3.7. So Ω meets Σ in a single point. Let $\{l\} = \Sigma \cap \Omega$. The subset $P^{-1}(\{l\})$ is G-invariant and every orbit $\Omega' \subset P^{-1}(\{l\})$ intersects Ω, so $\Omega = P^{-1}(\{l\})$. This shows that Ω is closed in U and semi-analytic as P is analytic. Finally, as U is semi-algebraic set in $\mathfrak{g}^*$, Ω is semi-analytic set in $\mathfrak{g}^*$. □

4. Multiplicities of up-down representations of completely solvable Lie groups

Let G be a real Lie group, A and H closed connected subgroups of G and σ a unitary representation of H. The representation

$$\rho(G, H, A, \sigma) = \mathrm{Ind}_H^G \sigma|_A , \tag{4.1}$$

of A is called an *up-down* representation, see [1] and [12].

In the context of exponential solvable Lie group, we showed in [1] that the representation $\rho(G, H, A, \sigma)$ obeys the orbital spectrum formula, that is

$$\rho(G, H, A, \sigma) = \int^{\oplus}_{[p_{\mathfrak{a}}(G\cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H))]/A} m(\psi)\rho_\psi \, d\mu^\sigma_{G,H,A}(\psi)\,, \tag{4.2}$$

where $\mu^\sigma_{G,H,A}$ is the push-forward of the measure $(dg \times \mu)$ on $G \times p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H)$ (μ is the natural measure on $p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H)$) under the]mappings

$G \times p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \to G\cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \to p_{\mathfrak{a}}(G\cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H)) \to [p_{\mathfrak{a}}(G\cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H))]/A$ and

$$m(\psi) = \sum_{\Omega \in [G\cdot p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H) \cap G\cdot p_{\mathfrak{a}}^{-1}(\psi)]/G} n^\sigma_\Omega n^\psi_\Omega \tag{4.3}$$

with $n^\sigma_\Omega = \#[(\Omega \cap p_{\mathfrak{h}}^{-1}(\Omega_\sigma^H))]/H$ and $n^\psi_\Omega = \#[(\Omega \cap p_{\mathfrak{a}}^{-1}(A\cdot\psi))]/A$ (see also [12]).

Assume now that G is a completely solvable Lie group with Lie algebra $\mathfrak{g}$. Let χ_f be a unitary character of H. For $\psi \in p_{\mathfrak{a}}(G\cdot \Gamma_f)$, let

$$\mathcal{M}_\psi = G\cdot\Gamma_f \cap G\cdot p_{\mathfrak{a}}^{-1}(\psi)\,. \tag{4.4}$$

Let e be the largest index in $\{1, \ldots, u\}$ such that $U = U_e$, defined as in Theorem 3.7, meets Γ_f in a non-empty Zariski-open subset of Γ_f. Let

$$\Gamma_0 = U \cap \Gamma_f\,. \tag{4.5}$$

For any $\psi \in p_{\mathfrak{a}}(G\cdot\Gamma_0)$, let

$$\mathcal{M}'_\psi = G\cdot\Gamma_0 \cap G\cdot p_{\mathfrak{a}}^{-1}(\psi) = \mathcal{M}_\psi \cap U\,. \tag{4.6}$$

Let d_G be the maximal dimension of G-orbits in $G\cdot\Gamma_f$, d_A the maximal dimension of A-orbits in $G\cdot\Gamma_f$, d'_A the maximal dimension of A-orbits in $p_{\mathfrak{a}}(G\cdot\Gamma_f)$ and d_H the maximal dimension of H-orbits in $G\cdot\Gamma_f$. Then, the set of linear forms ϕ such that $\dim G\cdot\phi = d_G$, $\dim H\cdot\phi = d_H$ and $\dim A\cdot\phi = d_A$ is an open dense co-null set in $G\cdot\Gamma_f$. Likewise, the set of $\psi \in p_{\mathfrak{a}}(G\cdot\Gamma_f)$ such that $\dim A\cdot\psi = d'_A$ is open dense co-null in $p_{\mathfrak{a}}(G\cdot\Gamma_f)$.

For any $\psi \in p_{\mathfrak{a}}(G\cdot\Gamma_0)$, let

$$\mathcal{B}_\psi = \Gamma_f \cap G\cdot p_{\mathfrak{a}}^{-1}(\psi) \text{ and } \mathcal{B}'_\psi = \mathcal{B}_\psi \cap \Gamma_0\,.$$

The set of generic orbits in $\mathcal{M}_\psi$ is then the set

$$\mathcal{M}'_\psi = G\cdot\mathcal{B}'_\psi\,.$$

We prove now our main upshot in this section.

THEOREM 4.1. *Let G be a completely solvable Lie group, A, H analytic subgroups of G and χ_f a unitary character of H. Then*

(1) *the multiplicity function of the representation $\rho(G, H, A, \chi_f)$ is either uniformly infinite or uniformly finite and bounded*;

(2) *the multiplicities of the representation $\rho(G, H, A, \chi_f)$ are uniformly finite if and only if $\mathcal{M}'_\psi$ is a semi-analytic subset of $\mathfrak{g}^*$ and the triple equality*

$$\dim \mathcal{M}'_\psi = 2d_H = 2d_A - d'_A$$

holds generically on $p_{\mathfrak{a}}(G \cdot \Gamma_f)$.

PROOF. The proof of the theorem consists of the following list of lemmas.

LEMMA 4.2. *We keep the same notations and hypotheses. The integer $m(\psi)$ is finite if and only if $\mathcal{M}'_\psi$ is a semi-analytic subset of $\mathfrak{g}^*$ and $\dim \mathcal{M}'_\psi = 2d_A - d'_A = 2d_H$.*

PROOF. If $m(\psi)$ is finite, then $n(\psi) = \#[\mathcal{M}'_\psi]/G < \infty$, and for every G-orbit $\Omega \subset \mathcal{M}'_\psi$, we have

$$n^f_\Omega < \infty \ \text{ and } n^\psi_\Omega < \infty \,. \tag{4.7}$$

Write

$$\mathcal{M}'_\psi = \bigcup_{i=1}^{k} \Omega_i \,,$$

where the $\Omega_i,\ i = 1, ..., k$ are different G-orbits with common dimension, it follows from Corollary 3.9 that $\mathcal{M}'_\psi$ is a semi-analytic set as it is a finite union of semi-analytic sets. Moreover, the dimension of $\mathcal{M}'_\psi$ is equal to $\dim \Omega_i$ for all $i = 1, \dots, k$. The hypothesis (4.7) implies that $2d_H = \dim \Omega_i$ by means of (2.4) and $2d_A - d'_A = \dim \Omega_i$ by formula (2.6) for $i = 1, \dots, k$. Whence,

$$\dim \mathcal{M}'_\psi = 2d_H = 2d_A - d'_A \,.$$

Conversely, write again

$$\mathcal{M}'_\psi = \bigcup_{i \in I} \Omega_i \,.$$

So in this case,

$$\dim \Omega_i \le \dim \mathcal{M}'_\psi = 2d_H \le \dim \Omega_i \,,$$

for any $i \in I$. Hence, $\dim \mathcal{M}'_\psi = \dim \Omega_i$, which means that the G-orbit Ω_i is open in $\mathcal{M}'_\psi$ and, therefore, is closed in $\mathcal{M}'_\psi$. As coadjoint orbits are connected, the orbits $\Omega_i,\ i \in I$ turn out to be the connected components of $\mathcal{M}'_\psi$. Let $\Phi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the mapping defined by $\Phi(g, l) = \mathrm{Ad}^* g\, l$. Since G is completely solvable, the mapping Φ is a pseudo-algebraic. The set

$$\mathcal{D} = \{(g, l) \in G \times p_{\mathfrak{a}}^{-1}(\psi) : \ \Phi(g, l) \in \Gamma_0\}$$

is therefore a semi-pseudo-algebraic set. It is clear that $\mathcal{B}'_\psi = \Phi(\mathcal{D})$. By Theorem 3.4, the set $\mathcal{D}$ has a finite number of connected components. Whence, thanks to the continuity of Φ one obtains that $\mathcal{B}'_\psi$ has a finite number of connected components. Finally, the number of connected components of $\mathcal{M}'_\psi = G \cdot \mathcal{B}'_\psi$ is less than or equal to the number of connected components of $\mathcal{B}'_\psi$ as being the image of $G \times \mathcal{B}'_\psi$ under the mapping Φ and $\#[\mathcal{M}'_\psi]/G < \infty$. The multiplicity functions $n^f_{\Omega_i}$ and $n^\psi_{\Omega_i}$ are then finite by (2.4) and (2.6). □

LEMMA 4.3. *Let G be a completely solvable Lie group, and let H, A be closed connected subgroups of G. Then $\#[\mathcal{M}'_\psi]/G$ is either uniformly infinite or uniformly finite and bounded on $p_\mathfrak{a}(G \cdot \Gamma_f)$.*

PROOF. Let Σ be a cross-section for the G-orbits in U and

$$\mathcal{M} = \mathcal{M}'_\psi \cap \Sigma .$$

Let

$$\mathcal{N} = \{(g_1, g_2, l_1, l_2, l_3) \in G \times G \times \Gamma_f \times p_\mathfrak{a}^{-1}(\psi) \times \Sigma :\ \Phi(g_1, l_1) = l_3,\ \Phi(g_2, l_2) = l_3\},$$

where Φ is defined as in the proof of the previous lemma. As Σ is an algebraic set of U, $\mathcal{N}$ is a semi-pseudo-algebraic set. On the other hand, $\mathcal{M}$ is a cross-section for the G-orbits in $\mathcal{M}'_\psi$ and $\mathcal{M} = q(\mathcal{N})$, where $q : G \times G \times \Gamma_f \times p_\mathfrak{a}^{-1}(\psi) \times \Sigma \to \Sigma$ is the canonical projection. If the number $n(\psi) = \#[\mathcal{M}'_\psi]/G$ is finite then $n(\psi) = \#[q(\mathcal{N})] =$ the number of the connected components of $q(\mathcal{N})$. This number is less than or equal to the number of connected components of $\mathcal{N}$. It follows from Theorem 3.4 that the number of connected components of $\mathcal{N}$ is bounded by a number which does not depend on ψ on $p_\mathfrak{a}(G \cdot \Gamma_f)$ as ψ intervenes only in the expression of the coefficients of the considered pseudo-algebraic functions. Then the number $n(\psi)$ is either uniformly infinite or uniformly finite and bounded by a number which does not depend on ψ on $p_\mathfrak{a}(G \cdot \Gamma_f)$. This completes the proof. □

LEMMA 4.4. *The multiplicity function of $\rho(G, H, A, \chi_f)$ is either uniformly infinite or uniformly finite and bounded on $p_\mathfrak{a}(G \cdot \Gamma_f)$.*

PROOF. We recall that the set of the linear form ϕ such that $\dim G \cdot \phi = d_G$, $\dim H \cdot \phi = d_H$ and $\dim A \cdot \phi = d_A$ is an open dense co-null set in $G \cdot \Gamma_f$, on which $\dim G \cdot \phi - 2 \dim H \cdot \phi$ and $\dim G \cdot \phi - 2 \dim A \cdot \phi$ are constant. It follows from the above lemmas that the multiplicity function $m(\psi)$ is either uniformly infinite or uniformly finite. It remains to prove that if $m(\psi)$ is uniformly finite, then it is bounded. But using the fact that the multiplicity n^ψ_Ω is equal to the number of connected components of the pseudo-algebraic set $\Omega \cap p_\mathfrak{a}^{-1}(A \cdot \psi)$, which does not depend on ψ as we indicated above (see Theorem 3.4). Finally, we use similar arguments as in the nilpotent situation to conclude (see [1] page 190). □

5. On down-up representations

Let G be a Lie group, H a closed subgroup of G and π a unitary representation of G. The unitary representation of G

$$\rho(G, H, \pi) = \mathrm{Ind}_H^G(\pi_{|H}) \tag{5.1}$$

is called a *down-up* representation (see [2]). Assume that G is an exponential solvable Lie group, and π is an irreducible unitary representation of G. Let $d\Omega$ be a canonical measure on the coadjoint orbit Ω_π^G associated to the representation π, and let λ be the Lebesgue measure on $\mathfrak{h}^\perp$. Let $\mu_\pi^{G,H}$ be the push-forward of the measure $(d\Omega \times \lambda)$ on $\Omega_\pi^G \times \mathfrak{h}^\perp$ under the mappings

$$\Omega_\pi^G \times \mathfrak{h}^\perp \to p_{\mathfrak{h}}(\Omega_\pi^G) \times \mathfrak{h}^\perp \to p_{\mathfrak{h}}(\Omega_\pi^G) + \mathfrak{h}^\perp \to G\cdot(p_{\mathfrak{h}}(\Omega_\pi^G) + \mathfrak{h}^\perp)/G = G\cdot(\Omega_\pi^G + \mathfrak{h}^\perp)/G\,.$$

It is established in [2] that the down-up representation $\rho(G, H, \pi)$ obeys the orbital spectrum formula

$$\rho(G, H, \pi) \simeq \int_{G\cdot(\Omega_\pi^G + \mathfrak{h}^\perp)/G}^{\oplus} m_\pi(\phi)\pi_\phi d\mu_\pi^{G,H}(\phi) \tag{5.2}$$

where

$$m_\pi(\phi) = \sum_{\Omega^H \in [p_{\mathfrak{h}}(\Omega_\pi^G) \cap p_{\mathfrak{h}}(\Omega_{\pi_\phi}^G)]/H} n_\pi^{\Omega^H} n_{\pi_\phi}^{\Omega^H} \tag{5.3}$$

with $n_\pi^{\Omega^H} = \#[\Omega_\pi^G \cap p_{\mathfrak{h}}^{-1}(\Omega^H)]/H$ and $n_{\pi_\phi}^{\Omega^H} = \#[\Omega_{\pi_\phi}^G \cap p_{\mathfrak{h}}^{-1}(\Omega^H)]/H$.

Let $V_1 < V_2 < \cdots < V_s$ be the stratification of $\mathfrak{h}^*$ as in Theorem 3.7 and e be the largest index in $\{1, \ldots, s\}$, such that V_e meets $p_{\mathfrak{h}}(\Omega_\pi^G)$. The set $(\Omega_\pi^G + \mathfrak{h}^\perp) \cap p_{\mathfrak{h}}^{-1}(V_e)$ is a non-empty open co-null in $(\Omega_\pi^G + \mathfrak{h}^\perp)$. For any $\phi \in (\Omega_\pi^G + \mathfrak{h}^\perp) \cap p_{\mathfrak{h}}^{-1}(V_e)$, let

$$\mathcal{A}_\phi = p_{\mathfrak{h}}(\Omega_\pi^G) \cap p_{\mathfrak{h}}(G \cdot \phi) \text{ and } \mathcal{A}'_\phi = V_e \cap \mathcal{A}_\phi\,. \tag{5.4}$$

It is clear that the set $\mathcal{A}_\phi$ is H-invariant. The set $\mathcal{A}'_\phi$ is an open dense and H-invariant subset of $\mathcal{A}_\phi$ as $\mathcal{A}'_\phi = (\cup_{e' \leq e} V_{e'}) \cap \mathcal{A}_\phi$ and $(\cup_{e' \leq e} V_{e'})$ is a Zariski open set in $\mathfrak{h}^*$ (see property (3) of Theorem 3.7). It is clear that $\mathcal{A}'_\phi$ is the set of H-orbits of maximal dimension in $\mathcal{A}_\phi$. Recall the notation $\Omega_\pi^G = G \cdot f$.

Let ρ be a unitary and irreducible representation of G. For ϕ in Ω_ρ^G, write

$$d_\rho(\phi) = \dim H \cdot \phi, \quad d_\rho = \max_{\phi \in \Omega_\rho^G} d_\rho(\phi)\,,$$

and

$$d_G(\rho) = 2d_\rho - \dim \Omega_\rho^G\,,$$

the set $Z_\rho = \{\phi \in \Omega_\rho^G : d_\rho(\phi) = d_\rho\}$ is a non-empty co-null set in Ω_ρ^G.

Our main result of this section is the following:

THEOREM 5.1. *Let G be a completely solvable Lie group, $H = \exp\mathfrak{h}$ an analytic subgroup of G and π an irreducible unitary representation of G. Then*

(1) *the multiplicity function of the representation $\rho(G, H, \pi)$ is either uniformly infinite or uniformly finite and bounded*;

(2) *the multiplicities of $\rho(G, H, \pi)$ are finite if and only if $\mathcal{A}'_\phi$ is a semi-analytic subset of $\mathfrak{h}^*$ and the triple equality*

$$\dim \mathcal{A}'_\phi = d_G(\pi_\phi) = d_G(\pi)\,,$$

holds generically on $\Omega^G_\pi + \mathfrak{h}^\perp$.

PROOF. We shall prove in a first time that for ϕ in $(\Omega^G_\pi + \mathfrak{h}^\perp) \cap p_{\mathfrak{h}}^{-1}(V_e)$, the set $\mathcal{A}'_\phi$ has a finite number of connected components in the Euclidean topology. In fact, let $\{X_1, \ldots, X_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$ (i. e. the subspace $\mathfrak{g}_i = \boldsymbol{R} - \operatorname{span}\{X_1, \ldots, X_i\}$ is an ideal of $\mathfrak{g}$ for every $i = 1, \ldots, n$). Since G is completely solvable, the function

$$F : \boldsymbol{R}^n \times \mathfrak{g}^* \to \mathfrak{g}^*$$
$$(x, l) = ((x_1, \ldots, x_n), l) \mapsto \exp x_n X_n \cdots \exp x_1 X_1 \cdot l$$

is a pseudo-algebraic function and for $l \in \mathfrak{g}^*$, one has $G \cdot l = \{F(x, l),\ x \in \boldsymbol{R}^n\}$. We view an element l in $\mathfrak{h}^*$ as an element of $\mathfrak{g}^*$ with a trivial extension on $\mathfrak{g}$. The set

$$\mathcal{A} = \{(x, y, l) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \times \mathfrak{h}^* : l \in V_e,\ F(x, f) - l \in \mathfrak{h}^\perp,\ \ F(y, \phi) - l \in \mathfrak{h}^\perp\}\,,$$

is semi-pseudo-algebraic in $\boldsymbol{R}^n \times \boldsymbol{R}^n \times \mathfrak{h}^*$. Let $q : \boldsymbol{R}^n \times \boldsymbol{R}^n \times \mathfrak{h}^* \to \mathfrak{h}^*$ be the canonical projection, then, it is clear that $\mathcal{A}'_\phi = q(\mathcal{A})$. Therefore the number of connected components of $\mathcal{A}'_\phi$ is less or than equal to the number of connected components of $\mathcal{A}$, so Theorem 3.4 enables us to conclude. With the above in mind, the number of the connected components of $\mathcal{A}'_\phi$ is bounded by a scalar which does not depend on $\phi \in \Omega^G_\pi + \mathfrak{h}^\perp$.

Let now $\phi \in \Omega^G_\pi + \mathfrak{h}^\perp$ and π_ϕ its irreducible and unitary representation. We show that $m_\pi(\phi)$ is finite if and only if $\mathcal{A}'_\phi$ is a semi-analytic set and $\dim \mathcal{A}'_\phi = d_G(\pi_\phi) = d_G(\pi)$. Assume first that $m_\pi(\phi) < \infty$, then $\#[\mathcal{A}'{}_\phi]/H < \infty$ and for all $H \cdot \psi$ in $[\mathcal{A}'{}_\phi]/H$, we have that $n_\pi^{H\cdot\psi} = n_\pi^\psi < \infty$ and $n_{\pi_\phi}^{H\cdot\psi} = n_{\pi_\phi}^\psi < \infty$. Hence, $d_G(\pi) = \dim H \cdot \psi$ and $d_G(\pi_\phi) = \dim H \cdot \psi$. Write

$$\mathcal{A}'{}_\phi = \bigcup_{i=1}^{k} H \cdot \psi_i\,,$$

where the H-orbits, $H \cdot \psi_i$, $i = 1, \ldots, k$ have the same dimension. It follows from Corollary 3.9 that $\mathcal{A}'{}_\phi$ is a finite union of semi-analytic sets, which is in turn a semi-analytic set. Moreover, the dimension of $\mathcal{A}'{}_\phi$ is equal to $\dim H \cdot \psi_i$ for all $i = 1, \ldots, k$. Hence

$\dim \mathcal{A}'_\phi = d_G(\pi_\phi) = d_G(\pi)$. On the other hand, let

$$\mathcal{A}'_\phi = \bigcup_{i \in I} H \cdot \psi_i \,.$$

Then for all $i \in I$, we have:

$$\dim H \cdot \psi_i \geq d_G(\pi_\phi) = d_G(\pi) = \dim \mathcal{A}'_\phi \geq \dim H \cdot \psi_i \,. \tag{5.5}$$

Hence, $\dim \mathcal{A}'_\phi = \dim H \cdot \psi_i$, which means that $H \cdot \psi_i$ is open in $\mathcal{A}'_\phi$ and therefore is closed in $\mathcal{A}'_\phi$. Since H is connected, the orbits $H \cdot \psi_i$, $i \in I$ are the connected components of $\mathcal{A}'_\phi$, which implies that I is finite. Finally, thanks to (5.5), (2.4) and (2.6), we conclude that the multiplicity functions $n_\pi^{\psi_i}$ and $n_{\pi_\phi}^{\psi_i}$ are finite.
Let Σ_e be the cross-section for H-orbits in V_e. For f in Ω_π^G, let

$$\mathbf{U} = \{(x, y, l) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \times \mathfrak{h}^* :\ l \in \Sigma_e, F(x, f) - l \in \mathfrak{h}^\perp,\ F(y, \phi) - l \in \mathfrak{h}^\perp\}\,.$$

As Σ_e is algebraic, $\mathbf{U}$ is pseudo-algebraic. Moreover,

$$\#[\mathcal{A}'_\phi]/H = \#q(\mathbf{U})\,.$$

Whence, thanks to Corollary 3.5, the set $\#[\mathcal{A}'_\phi]/H$ is either uniformly infinite or uniformly finite and bounded on $\Omega_\pi^G + \mathfrak{h}^\perp$ which achieves the proof of the Theorem. $\square$

6. Examples

Let $\mathfrak{g}$ be the 4-dimensional Lie algebra spanned by A, X, Y, and Z with nonzero Lie brackets

$$[A, X] = X,\ [A, Y] = -Y,\ [X, Y] = Z\,.$$

Let $\{A^*, X^*, Y^*, Z^*\}$ be the dual basis of $\mathfrak{g}^*$ and $\varphi = \varphi_{(\alpha,\beta,\gamma,\delta)} = \alpha A^* + \beta X^* + \gamma Y^* + \delta Z^* \in \mathfrak{g}^*$. We denote by $\Omega^G_{(\alpha,\beta,\gamma,\delta)}$ the G-orbit of $\varphi_{(\alpha,\beta,\gamma,\delta)}$. A routine computation gives

$$\begin{aligned}&\mathrm{Ad}^*((\exp aA \exp xX \exp yY \exp zZ)^{-1})\varphi_{(\alpha,\beta,\gamma,\delta)}\\ &= \varphi_{(\alpha-\beta e^a x+\gamma e^{-a} y+\delta xy,\, \beta e^a-\delta y,\, \gamma e^{-a}+\delta x,\, \delta)}\,.\end{aligned}$$

(1) $\mathfrak{h} = \{A, X, Z\}$, $\mathfrak{a} = \{A, Z\}$ and $f = \lambda A^*$. So, $\Gamma_f = f + \mathfrak{h}^\perp = \lambda A^* \oplus \boldsymbol{R}Y^*$ and

$$G \cdot \Gamma_f = \{(\lambda + xy)A^* + yY^*,\ x, y \in \boldsymbol{R}\}\,.$$

Therefore, $p_{\mathfrak{a}}(G \cdot \Gamma_f) = \boldsymbol{R}A^*$. Moreover, for $\psi = \alpha A^* \in p_{\mathfrak{a}}(G \cdot \Gamma_f)$, one has

$$\mathcal{M}_\psi = \begin{cases} \boldsymbol{R}A^* \oplus \boldsymbol{R}^* Y^* & \text{if } \alpha \neq \lambda, \\ G \cdot \Gamma_f & \text{otherwise,} \end{cases}$$

and $\mathcal{M}'_\psi = \Omega^G_{(0,0,1,0)} \cup \Omega^G_{(0,0,-1,0)} = \Omega_+ \cup \Omega_-$. Note that

$$\dim \mathcal{M}'_\psi = 2,\ d_H = 1,\ d_A = 1,\ d'_A = 0\,,$$

and for $\Omega \in [\mathcal{M}'_\psi]/G$,

$$n^f_\Omega = 1 \text{ and } n^\psi_\Omega = 1\,.$$

It follows that $m(\psi) = 2$ and

$$\rho(G, H, A, f) \simeq 2 \int^{\oplus}_{\boldsymbol{R}} \chi_{tA^*} dt\,.$$

(2) $\mathfrak{h} = \{A, X\}$, $\mathfrak{a} = \{A, Y\}$ and $f = \lambda A^*$. We have in this case that $\Gamma_f = \lambda A^* \oplus \boldsymbol{R}Y^* \oplus \boldsymbol{R}Z^*$ and

$$G \cdot \Gamma_f = \{\lambda A^*\} \cup \{aA^* + yY^* :\ a \in \boldsymbol{R}, y \in \boldsymbol{R}^*\} \cup \\ \{(\lambda - \frac{xy}{z})A^* + xX^* + yY^* + zZ^* :\ x, y \in \boldsymbol{R}, z \in \boldsymbol{R}^*\}.$$

So, $p_\mathfrak{a}(G \cdot \Gamma_f) = \{\lambda A^*\} \cup \{xA^* + yY^* :\ x \in \boldsymbol{R}, y \in \boldsymbol{R}^*\}$. The A-orbits in general position in $p_\mathfrak{a}(G \cdot \Gamma_f)$ are parameterized by the forms $\psi = \pm Y^*$. Moreover, for $\psi = \pm Y^* \in p_\mathfrak{a}(G \cdot \Gamma_f)$, we have

$$\mathcal{M}_\psi = \{aA^* + yY^* :\ a \in \boldsymbol{R}, y \in \boldsymbol{R}^*\} \cup \{(\lambda - \frac{xy}{z})A^* + xX^* + yY^* + zZ^* :\ x, y \in \boldsymbol{R}, z \in \boldsymbol{R}^*\}\,,$$

$$\mathcal{M}'_\psi = \{(\lambda - \frac{xy}{z})A^* + xX^* + yY^* + zZ^* :\ x, y \in \boldsymbol{R}, z \in \boldsymbol{R}^*\}$$

and therefore, $\dim \mathcal{M}'_\psi = 3$. It follows that for $\psi \in p_\mathfrak{a}(G \cdot \Gamma_f)$, $m(\psi) = \infty$ and

$$\rho(G, H, A, f) \simeq \infty\sigma_- \oplus \infty\sigma_+\,,$$

where $\sigma_\pm$ is the irreducible unitary representation of A associated to the form $\psi = \pm Y^*$.

(3) We suppose hereafter that $\mathfrak{h} = \{A, Y, Z\}$ and let $\varphi^\circ_{(\alpha,\gamma,\delta)} = \alpha A^* + \gamma Y^* + \delta Z^* \in \mathfrak{h}^*$. as above,

$$\mathrm{Ad}^*((\exp aA \exp yY \exp zZ)^{-1})\varphi^\circ_{(\alpha,\gamma,\delta)} = \varphi^\circ_{(\alpha+\gamma e^{-a}y, \gamma e^{-a}, \delta)}\,.$$

Let $f = \beta X^*,\ \beta > 0$. Then

$$\Omega^G_\pi = G \cdot f = \{aA^* + xX^*,\ \ a \in \boldsymbol{R}, x \in \boldsymbol{R}^*_+\}$$

and

$$G \cdot (\Omega^G_\pi + \mathfrak{h}^\perp) = \boldsymbol{R}A^* \oplus \boldsymbol{R}X^*\,.$$

The generic orbits in $G\cdot(\Omega_\pi^G+\mathfrak{h}^\perp)$ are associated to the linear forms $\pm X^*$. For $\phi=\pm X^*$, we have $\mathcal{A}'_\phi=\mathcal{A}_\phi=\boldsymbol{R}A^*$ and $\#[\mathcal{A}_\phi]/H=\infty$. Hence

$$\rho(G,H,\pi_f)\simeq\infty\pi_{-X^*}\oplus\infty\pi_{X^*}\,.$$

Moreover, we have $\dim\mathcal{A}_\phi=1$ and $d_G(\pi)=0$.

(4) Consider finally the case where $\mathfrak{h}=\{A,X,Z\}$. Let $\varphi^\circ_{(\alpha,\beta,\delta)}=\alpha A^*+\beta X^*+\delta Z^*\in\mathfrak{h}^*$

$$\mathrm{Ad}^*((\exp aA\exp xX\exp zZ)^{-1})\varphi^\circ_{(\alpha,\beta,\delta)}=\varphi^\circ_{(\alpha-\beta e^a x,\beta e^a,\delta)}\,.$$

Let $f=X^*$, we have that

$$\Omega_\pi^G=G\cdot f=\{\alpha A^*+\beta X^*,\ \ \alpha\in\boldsymbol{R},\ \beta\in\boldsymbol{R}_+^*\}\,,$$

$$\Omega_\pi^G+\mathfrak{h}^\perp=\{\alpha A^*+\beta X^*+\gamma Y^*,\ \ \alpha,\gamma\in\boldsymbol{R},\ \beta\in\boldsymbol{R}_+^*\}\,,$$

and

$$G\cdot\left(\Omega_\pi^G+\mathfrak{h}^\perp\right)=\Omega_\pi^G+\mathfrak{h}^\perp\,.$$

Let ϕ be in $\Omega_\pi^G+\mathfrak{h}^\perp$ and we write $\phi=\alpha A^*+\beta X^*+\gamma Y^*$, then

$$G\cdot\phi=\{(\alpha-\beta e^a x+\gamma e^{-a}y)A^*+\beta e^a X^*+\gamma e^{-a}Y^*,\ a,x,y\in\boldsymbol{R}\}\,,$$

and

$$H\cdot\phi=\{(\alpha-\beta e^a x)A^*+\beta e^a X^*+\gamma e^{-a}Y^*,\ a,x\in\boldsymbol{R}\}\,,$$

It follows that $\mathcal{A}_\phi=p_{\mathfrak{h}}(\Omega_\pi^G)\cap p_{\mathfrak{h}}(G\cdot\phi)=\boldsymbol{R}A^*+\boldsymbol{R}_+^*X^*$, and

$$\dim\mathcal{A}_\phi=d_G(\pi)=d_G(\pi_\phi)=2\,.$$

Moreover, for $\varphi^\circ=X^*\in\mathfrak{h}^*$ we have $H\cdot\varphi^\circ=\mathcal{A}_\phi$. Hence $\mathcal{A}'_\phi=\mathcal{A}_\phi$ is a semi-analytic subset of $\mathfrak{h}^*$ and $\#[\mathcal{A}_\phi]/H=1$. Let $\Omega^H_{\sigma(\varphi^\circ)}=H\cdot\varphi^\circ$ where $\sigma(\varphi^\circ)$ is the unitary and irreducible representation associated to φ° by the Kirillov-Bernat mapping Θ_H. It is easy to check that $\Omega_\pi^G\cap p_{\mathfrak{h}}^{-1}(\Omega^H_{\sigma(\varphi^\circ)})=H\cdot\varphi_{(0,1,0,0)}$ and $\Omega_{\pi_\phi}^G\cap p_{\mathfrak{h}}^{-1}(\Omega^H_{\sigma(\varphi^\circ)})=H\cdot\varphi_{(\alpha,1,\gamma,0)}$, therefore $n_\pi^{\Omega^H_{\sigma(\varphi^\circ)}}=n_{\pi_\phi}^{\Omega^H_{\sigma(\varphi^\circ)}}=1$. Finally, we have $m_\pi(\phi)=1$.

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References

[1] A. BAKLOUTI, A. GHORBEL et H. HAMROUNI, Sur les représentations mixtes des groupes de Lie résolubles exponentiels, Publicacions Matemàtiques **46** (2002), 179–199.

[2] A. BAKLOUTI and H. HAMROUNI, On the down-up representations of exponential solvable Lie groups, Russ. J. Math. Phy. **8** (2001), no. 4, 422–432.

[3] R. BENEDETTI and J. J. RISLER, *Real algebraic and semi-algebraic sets*, Hermann, Paris, 1990.

[4] P. BERNAT et al., *Représentations des groupes de Lie résolubles*, Dunod, Paris, 1972.

[5] J. BOCHNAK, M. COSTE and M. F. ROY, *Géométrie algébrique réelle*, Springer-Verlag, 1987.

[6] L. CORWIN, F. P. GREENLEAF and G. GRÉLAUD, Direct integral decompositions and multiplicities for induced representations of nilpotent Lie groups, Trans. Amer. Math. Soc. **304** (1987), 537–547.

[7] B. N. CURREY and R. PENNEY, The structure of the space of coadjoint orbits of a completely solvable Lie groups, Michigan Math. J. **36** (1989), 309–320.

[8] B. N. CURREY, The structure of the space of coadjoint orbits on an exponential solvable Lie group, Trans. Am. Math. Soc. **332** (1992), no. 1, 241–269.

[9] H. FUJIWARA, Représentations monomiales des groupes de Lie résolubles exponentiels, Progress in Math. **82** (1990), 61–84.

[10] H. FUJIWARA, Sur les restrictions des représentations unitaires des groupes de Lie résolubles exponentiels, Inv. Math. **104** (1991), 647–654.

[11] M. E. HERRERA, Integration on a semianalytic set, Bull. Soc. Math. France **94** (1966), 141–180.

[12] R. LIPSMAN, The up-down formula for nil-homogeneous spaces, Annali di Matematica Pura ed Applicata (IV),Vol. **CLXVI** (1994), 291–300.

[13] R. LIPSMAN, Restriction representations of completely solvable Lie groups, Can. J. Math. **42** (1990), 790–824.

[14] R. LIPSMAN, Induced representations of completely solvable Lie groups, Ann. Scuola Norm. Sup. Pisa **17** (1990), 127–164.

[15] R. LIPSMAN, The multiplicity function on exponential and completely solvable homogeneous spaces, Geometriae Dedicata **39** (1991), 155–161.

[16] R. LIPSMAN, Orbital parameters for induced and restricted representations, Trans. Amer. Math. Soc. **313** (1989), 433–473.

[17] S. LOJASIEWICZ, Triangulations of semianalytic sets, Ann. Scuola. Norm. Sup. Pisa **18** (1964), 449–474.

[18] J. RISLER, Some aspects of complexity in real algebraic geometry, J. Symbolic Computation **5** (1988), 109–119.

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