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On q-Analogues of the Barnes Multiple Zeta Functions

Yoshinori YAMASAKI

Kyushu University

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Abstract. In this paper, we introduce q-analogues of the Barnes multiple zeta functions. We show that these functions can be extended meromorphically to the whole plane, and moreover, tend to the Barnes multiple zeta functions when $q \uparrow 1$ for all complex numbers.

1. Introduction

The aim of the present paper is to introduce q-analogues of the Barnes multiple zeta function ([3]);

$$\zeta_r(s,z;\boldsymbol{\omega}) := \sum_{n_1,\dots,n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + z)^{-s} \quad (\mathrm{Re}\,(s) > r)\,,$$

where $\omega_1, \dots, \omega_r$ are complex parameters which lie on some half plane. We study an analytic continuation of the q-analogue of $\zeta_r(s,z;\boldsymbol{\omega})$. We determine especially, *true* q-analogues of the Barnes multiple zeta function when $\omega_i = 1$ $(1 \leq i \leq r)$. Here, by a true q-analogue, we mean when the classical limit $q \uparrow 1$ of the q-analogue reproduces the original zeta function for *all* $s \in \mathbf{C}$. Recall the Hurwitz zeta function's case, that is, the case $r = 1$. Let $0 < q < 1$ and $[z]_q := (1-q^z)/(1-q)$ for $z \in \mathbf{C}$. In [6] (see also [5]) we studied q-analogues of the Hurwitz zeta function $\zeta(s,z) := \sum_{n=0}^{\infty}(n+z)^{-s}$ defined via the q-series with two complex variables $s, t \in \mathbf{C}$;

$$\tilde{\zeta}_q(s,t,z) := \sum_{n=0}^{\infty} \frac{q^{(n+z)t}}{[n+z]_q{}^s} \quad (\mathrm{Re}\,(t) > 0)\,.$$

The function $\tilde{\zeta}_q(s,t,z)$ is continued meromorphically to the whole s,t-plane. We obtained the necessary and sufficient condition for the variable $t \in \mathbf{C}$ so that $\tilde{\zeta}_q(s,t,z)$ is a true q-analogue of $\zeta(s,z)$. Namely, these functions $\tilde{\zeta}_q^{(\nu)}(s,z) := \tilde{\zeta}_q(s,s-\nu,z)$ $(\nu \in \mathbf{N})$ give true q-analogues

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of the Hurwitz zeta function among the functions of the form $\tilde{\zeta}_q(s,\varphi(s),z)$ where $\varphi(s)$ is a meromorphic function on $\mathbf{C}$. The main purpose is to generalize the results in [6] to $r \geq 1$.

The plan of this paper is as follows. In Section 2, we define a q-analogue $\zeta_{q,r}(s,t,z)$ of the Barnes multiple zeta function for $\omega_i = 1$ $(1 \leq i \leq r)$ and give the main theorem (Theorem 2.1). In Section 3, we first study an analytic continuation of the q-analogue $\zeta_{q,r}(s,t,z)$ and then prove the main theorem. In Section 4, we study a q-analogue $\zeta_{q,r}(s,t,z;\boldsymbol{\omega})$ of the multiple zeta functions for general parameters $\boldsymbol{\omega}$. Using the binomial theorem, we give an analytic continuation of the q-analogue (Proposition 4.1). In the appendix, we introduce a q-analogue $\tilde{\Gamma}_q(z)$ of the gamma function $\Gamma(z)$ associated to the q-analogue $\tilde{\zeta}_q(s,t,z)$ of the Hurwitz zeta function. We first observe fundamental properties of $\tilde{\Gamma}_q(z)$. The rest of the appendix is devoted to study q-analogues of the limit formula of Lerch (Proposition A.3) and the Gauss-Legendre formula (Proposition A.5).

Throughout the paper, we assume $0 < q < 1$. We put $[n]_q! := [n]_q[n-1]_q\cdots[1]_q$ for $n \in \mathbf{N}$. Further, for non-negative integers m and n, we define the q-binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}_q$ by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}},$$

where $(a;q)_m := \prod_{l=0}^{m-1}(1-aq^l)$ for $m \geq 1$ and $(a;q)_0 := 1$. We denote the field of complex numbers, the ring of rational integers and the set of positive integers by $\mathbf{C}$, $\mathbf{Z}$ and $\mathbf{N}$ respectively. Also, if Q is a set, Q_P stands for the set of all elements in Q which satisfy the condition P.

2. Definition of q-analogues and the main theorem

Let $s,t \in \mathbf{C}$ and $z \notin -\mathbf{Z}_{\leq 0}$. We study a q-analogue of the Barnes multiple zeta function

$$\zeta_r(s,z) := \sum_{n_1,\ldots,n_r \geq 0} (n_1+\cdots+n_r+z)^{-s}$$

defined by the following q-series;

$$\zeta_{q,r}(s,t,z) := \sum_{n_1,\ldots,n_r \geq 0} \frac{q^{n_1 t + n_2(t-1)+\cdots+n_r(t-r+1)}}{[n_1+\cdots+n_r+z]_q^{\,s}}.$$

The series $\zeta_{q,r}(s,t,z)$ converges absolutely for $\mathrm{Re}\,(t) > r-1$. When $r=1$, we put $\zeta_q(s,t,z) := \zeta_{q,1}(s,t,z)$. In view of the results in [6], we put $\zeta_{q,r}^{(\nu)}(s,z) := \zeta_{q,r}(s,s-\nu,z)$ and $\zeta_q^{(\nu)}(s,z) := \zeta_q(s,s-\nu,z)$ for $\nu \in \mathbf{N}$. The following theorem is the main result of this paper.

THEOREM 2.1. *Let $t=\varphi(s)$ be a meromorphic function on $\mathbf{C}$. Then the formula*

$$\lim_{q\uparrow 1}\zeta_{q,r}(s,\varphi(s),z)=\zeta_r(s,z)\quad(s\in\mathbf{C})$$

holds if and only if the function $\varphi(s)$ can be written as $\varphi(s)=s-\nu$ for some $\nu\in\mathbf{N}$.

REMARK 2.2. (i) By Theorem 2.1, it is clear that the functions of the type $\sum_{\nu\,:\,\text{finite}}a_q^{(\nu)}(s,z)\zeta_{q,r}^{(\nu)}(s,z)$ for some holomorphic functions $a_q^{(\nu)}(s,z)$ satisfying $\lim_{q\uparrow 1}\sum_{\nu\,:\,\text{finite}}a_q^{(\nu)}(s,z)=1$ are also true q-analogues of $\zeta_r(s,z)$. Note that the q-analogue of the Hurwitz zeta function discussed in [6] is given by $\tilde{\zeta}_q^{(\nu)}(s,z)=\zeta_q^{(\nu)}(s,z)\times q^{z(s-\nu)}$.

(ii) The q-analogue of the Hurwitz zeta function studied in [10] is different from ours. It is not of the form of the (q-) Dirichlet series and, in fact, is needed an extra term (precisely, see [6, Corollary 2.4]).

It is easy to see that $\zeta_r(s,z)$ is expressed as

$$\zeta_r(s,z)=\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}(n+z)^{-s}\,. \tag{2.1}$$

To obtain a similar expression for $\zeta_{q,r}(s,t,z)$, we need the following lemma.

LEMMA 2.3. (i) *For $l,m\in\mathbf{Z}_{\geq 0}$, it holds that*

$$\sum_{d=0}^{l}\begin{bmatrix}m-1+d\\ m-1\end{bmatrix}_q q^d=\begin{bmatrix}m+l\\ m\end{bmatrix}_q\,. \tag{2.2}$$

(ii) *For $r\in\mathbf{N}$, it holds that*

$$\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=n}}q^{n_1+2n_2+\cdots+rn_r}=q^n\begin{bmatrix}n+r-1\\ r-1\end{bmatrix}_q\,. \tag{2.3}$$

PROOF. The formula (2.2) is well-known (see [1], also [4]). We show the formula (2.3) by induction on r. It is clear that (2.3) holds for $r=1$. Suppose it holds for $r-1$. Then the left hand side of (2.3) is equal to

$$\sum_{n_1=0}^{n}q^{n_1+(n-n_1)}\sum_{\substack{n_2,\dots,n_r\geq 0\\ n_2+\cdots+n_r=n-n_1}}q^{n_2+2n_3+\cdots+(r-1)n_r}=q^n\sum_{n_1=0}^{n}\begin{bmatrix}n_1+r-2\\ r-2\end{bmatrix}_q q^{n_1}\,.$$

Using the formula (2.2) for $l=n$, $m=r-1$ and $d=n_1$, we obtain the desired formula. □

PROPOSITION 2.4. *It holds that*

$$\zeta_{q,r}(s,t,z)=\sum_{n=0}^{\infty}\begin{bmatrix}n+r-1\\ r-1\end{bmatrix}_q\frac{q^{n(t-r+1)}}{[n+z]_q{}^s}\,. \tag{2.4}$$

PROOF. It is easy to see that

$$\zeta_{q,r}(s,t,z) = \sum_{n=0}^{\infty} \sum_{\substack{n_1,\dots,n_r \geq 0 \\ n_1+\cdots+n_r=n}} \frac{q^{(t+1)(n_1+\cdots+n_r)-(n_1+2n_2+\cdots+rn_r)}}{[n_1+\cdots+n_r+z]_q{}^s}$$

$$= \sum_{n=0}^{\infty} \frac{q^{(t+1)n}}{[n+z]_q{}^s} \sum_{\substack{n_1,\dots,n_r \geq 0 \\ n_1+\cdots+n_r=n}} q^{-(n_1+2n_2+\cdots+rn_r)} .$$

Substituting q^{-1} for q into (2.3) yields

$$\sum_{\substack{n_1,\dots,n_r \geq 0 \\ n_1+\cdots+n_r=n}} q^{-(n_1+2n_2+\cdots+rn_r)} = q^{-n} \begin{bmatrix} n+r-1 \\ r-1 \end{bmatrix}_{q^{-1}} = q^{-nr} \begin{bmatrix} n+r-1 \\ r-1 \end{bmatrix}_q .$$

Hence we obtain the formula (2.4). □

3. Proof of the main theorem

In this section, we give a proof of Theorem 2.1. We first provide analytic continuations of $\zeta_r(s,z)$ with respect to s (see [9]) and study of $\zeta_{q,r}(s,t,z)$ with respect to t. Since we have the following ladder relations

$$\zeta_r(s,z) = \zeta_r(s,z+1) + \zeta_{r-1}(s,z) ,$$

$$\zeta_{q,r}(s,t,z) = q^{t-r+1}\zeta_{q,r}(s,t,z+1) + \zeta_{q,r-1}(s,t,z) , \tag{3.1}$$

it is sufficient to study the analytic continuation when $\mathrm{Re}\,(z) > 0$. Here we understand $\zeta_0(s,z) = z^{-s}$ and $\zeta_{q,0}(s,t,z) = [z]_q{}^{-s}$.

3.1. An analytic continuation of $\zeta_r(s,z)$. For each $l \in \mathbf{Z}_{\geq 0}$, we put $(x)_l := x(x+1)\cdots(x+l-1) = \Gamma(x+l)/\Gamma(x)$. Then $(x)_l$ can be written as $(x)_l = \sum_{j=0}^{l} s(l,j)x^j$ where $s(l,j)$ is the Stirling number of the first kind. Hence we have

$$\binom{n+r-1}{r-1} = \frac{(n)_r}{n(r-1)!} = \frac{1}{(r-1)!}\sum_{j=0}^{r} s(l,j)n^{j-1} = \sum_{l=0}^{r-1} P_r^l(z)(n+z)^l ,$$

where $P_r^l(z)$ $(0 \leq l \leq r-1)$ is a polynomial in z defined by

$$P_r^l(z) := \frac{1}{(r-1)!}\sum_{j=l}^{r-1} \binom{j}{l} s(r,j+1)(-z)^{j-l} .$$

Thus, we have by (2.1)

$$\zeta_r(s,z) = \sum_{l=0}^{r-1} P_r^l(z)\zeta(s-l,z) . \tag{3.2}$$

Recall also the Euler-Maclaurin summation formula (see, e.g., [1, p. 619]): For $a, b \in \mathbf{Z}$ satisfying $a < b$, a C^{∞}-function $f(x)$ on $[a, \infty)$, and an arbitrary integer $M \geq 0$, we have

$$
\begin{aligned}
\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \frac{1}{2}(f(a) + f(b)) \\
+ \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!}(f^{(k)}(b) - f^{(k)}(a)) \\
- \frac{(-1)^{M+1}}{(M+1)!} \int_a^b \tilde{B}_{M+1}(x) f^{(M+1)}(x)dx ,
\end{aligned}
\tag{3.3}
$$

where B_k is the Bernoulli number and $\tilde{B}_k(x)$ is the periodic Bernoulli polynomial defined by $\tilde{B}_k(x) = B_k(x - \lfloor x \rfloor)$ with $\lfloor x \rfloor$ being the largest integer not exceeding x. Putting $f(x) := (x+z)^{-s}$, we obtain

$$
\begin{aligned}
\zeta(s, z) = \frac{1}{s-1} z^{-s+1} + \frac{1}{2} z^{-s} + \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!}(s)_k z^{-s-k} \\
- \frac{(s)_{M+1}}{(M+1)!} \int_0^{\infty} \tilde{B}_{M+1}(x)(x+z)^{-s-M-1} dx .
\end{aligned}
\tag{3.4}
$$

Since $\mathrm{Re}\,(z) > 0$, the equation (3.4) gives an analytic continuation of the Hurwitz zeta function $\zeta(s, z)$ to the region $\mathrm{Re}\,(s) > -M$. Therefore, by (3.2) and (3.4), we obtain the following

PROPOSITION 3.1. *For any integers $M_l \geq 0$ $(0 \leq l \leq r-1)$, we have*

$$
\begin{aligned}
\zeta_r(s, z) = \sum_{l=0}^{r-1} \frac{P_r^l(z)}{s-l-1} z^{-s+l+1} + \frac{1}{2} \sum_{l=0}^{r-1} P_r^l(z) z^{-s+l} \\
+ \sum_{l=0}^{r-1} P_r^l(z) \sum_{k_l=1}^{M_l} \frac{B_{k_l+1}}{(k_l+1)!}(s-l)_{k_l} z^{-s+l-k_l} \\
- \sum_{l=0}^{r-1} \frac{P_r^l(z)(s-l)_{M_l+1}}{(M_l+1)!} \int_0^{\infty} \tilde{B}_{M_l+1}(x)(x+z)^{-s+l-M_l-1} dx .
\end{aligned}
$$

This gives an analytic continuation of $\zeta_r(s, z)$ to the region $\mathrm{Re}\,(s) > M$ *where* $M := \max\{-M_l + l \mid 0 \leq l \leq r-1\}$. □

3.2. An analytic continuation of $\zeta_{q,r}(s, t, z)$. It is easy to see that

$$
\begin{bmatrix} n+r-1 \\ r-1 \end{bmatrix}_q = \frac{1}{[r-1]_q!} \prod_{j=1}^{r-1} \frac{1 - q^{n+z} + q^{n+z} - q^{n+j}}{1-q}
$$

$$= \frac{1}{[r-1]_q!}\prod_{j=1}^{r-1}([n+z]_q - q^{n+j}[z-j]_q) = \sum_{l=0}^{r-1} q^{n(r-1-l)} P^l_{q,r}(z)[n+z]_q{}^l\,,$$

where $P^l_{q,r}(z)$ $(0 \le l \le r-1)$ is a function of z defined by

$$P^l_{q,r}(z) := \frac{(-1)^{r-1-l}}{[r-1]_q!} \sum_{1\le m_1<\cdots<m_{r-1-l}\le r-1} q^{m_1+\cdots+m_{r-1-l}}[z-m_1]_q\cdots[z-m_{r-1-l}]_q$$

for $0 \le l \le r-2$ and $P^{r-1}_{q,r}(z) := 1/[r-1]_q!$. Therefore we have by (2.4)

$$\zeta_{q,r}(s,t,z) = \sum_{l=0}^{r-1} P^l_{q,r}(z)\zeta_q(s-l,t-l,z)\,. \tag{3.5}$$

For example, we have

$$\begin{aligned}\zeta_{q,2}(s,t,z) &= \zeta_q(s-1,t-1,z) - q[z-1]_q\zeta_q(s,t,z)\,,\\ \zeta_{q,3}(s,t,z) &= \frac{1}{1+q}\{\zeta_q(s-2,t-2,z)\\ &\quad - (q[z-1]_q + q^2[z-2]_q)\zeta_q(s-1,t-1,z)\\ &\quad + q^3[z-1]_q[z-2]_q\zeta_q(s,t,z)\}\,.\end{aligned}$$

We now recall the analytic continuation of $\zeta_q(s,t,z)$ proved in [6]. Let $N \in \mathbf{N}$. Put $f_q(x) := q^{xt}(1-q^{x+z})^{-s}$. Define the polynomial $b^\varepsilon_j(s)$ $(0 \le \varepsilon \le j)$ in s by the following equation:

$$\frac{d^j}{dx^j}\{(1-q^{x+z})^{-s}\} = (\log q)^j \sum_{\varepsilon=0}^{j} b^\varepsilon_j(s)(1-q^{x+z})^{-s-\varepsilon}\,.$$

By the Leibniz rule, we have

$$f_q^{(k)}(x) = (\log q)^k q^{xt}\sum_{\varepsilon=0}^{k} c^\varepsilon_k(s,t)(1-q^{x+z})^{-s-\varepsilon}\,,\quad c^\varepsilon_k(s,t) := \sum_{j=\varepsilon}^{k}\binom{k}{j} t^{k-j} b^\varepsilon_j(s)\,.$$

Choosing $f(x) = f_q(x)$ and $M = N$ in (3.3), we have

$$\begin{aligned}\zeta_q(s,t,z) &= \frac{1}{2}\left(\frac{1-q^z}{1-q}\right)^{-s} - \sum_{k=1}^{N}\sum_{\varepsilon=0}^{k}\frac{B_{k+1}}{(k+1)!}c^\varepsilon_k(s,t)\left(\frac{1-q^z}{1-q}\right)^{-s-\varepsilon}\frac{(\log q)^k}{(1-q)^\varepsilon}\\ &\quad + (1-q)^s I^0_{q,0}(s,t,z) + \frac{(-1)^N(\log q)^{N+1}(1-q)^s}{(N+1)!}\\ &\quad \times \sum_{\varepsilon=0}^{N+1} c^\varepsilon_{N+1}(s,t) I^{N+1}_{q,\varepsilon}(s,t,z)\,,\end{aligned} \tag{3.6}$$

where

$$I^m_{q,\varepsilon}(s,t,z) := \int_0^\infty \tilde{B}_m(x) q^{xt}(1-q^{x+z})^{-s-\varepsilon} dx\,.$$

Note that $\tilde{B}_0(x) = 1$. Recall now the Fourier expansion of $\tilde{B}_m(x)$ (see, e.g., [11, p. 191]);

$$\tilde{B}_m(x) = -m! \sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{e^{2\pi\sqrt{-1}nx}}{(2\pi\sqrt{-1}n)^m} \quad (m \geq 2)\,. \tag{3.7}$$

Put $u = q^{x+z}$. Then we have

$$I^0_{q,0}(s,t,z) = -\frac{q^{-zt}}{\log q} b_{q^z}(t, -s+1)\,, \tag{3.8}$$

$$I^m_{q,\varepsilon}(s,t,z) = \sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{m! e^{-2\pi\sqrt{-1}nz}}{(2\pi\sqrt{-1}n)^m} \frac{q^{-zt}}{\log q} b_{q^z}(\delta n + t, -s-\varepsilon+1) \quad (m \geq 2)\,, \tag{3.9}$$

where $\delta = 2\pi\sqrt{-1}/\log q$. Here $b_w(\alpha,\beta)$ is the incomplete beta function defined by the integral

$$b_w(\alpha,\beta) := \int_0^w u^{\alpha-1}(1-u)^{\beta-1} du \quad (0 < \mathrm{Re}\,(w) < 1)\,.$$

This integral converges absolutely for $\mathrm{Re}\,(\alpha) > 0$. Hence the function $b_w(\alpha,\beta)$ is holomorphic for $\mathrm{Re}\,(\alpha) > 0$ and for all $\beta \in \mathbf{C}$. Note that if $\mathrm{Re}\,(\beta) > 0$, we have $\lim_{w\to 1} b_w(\alpha,\beta) = \mathrm{B}(\alpha,\beta)$ where $\mathrm{B}(\alpha,\beta)$ is the beta function. Further, for any integer $N' \geq 2$, repeated use of integration by parts yields

$$\begin{aligned} b_w(\alpha,\beta) = &\sum_{l=1}^{N'-1} (-1)^{l-1} \frac{(1-\beta)_{l-1}}{(\alpha)_l} w^{\alpha+l-1}(1-w)^{\beta-l} \\ &+ (-1)^{N'-1} \frac{(1-\beta)_{N'-1}}{(\alpha)_{N'-1}} b_w(\alpha+N'-1, \beta-N'+1)\,. \end{aligned} \tag{3.10}$$

As a function of α, this expression gives an analytic continuation of $b_w(\alpha,\beta)$ to the region $\mathrm{Re}\,(\alpha) > 1-N'$. Hence the functions $I^0_{q,0}(s,t,z)$ and $I^m_{q,\varepsilon}(s,t,z)$ are meromorphically continued to the region $\mathrm{Re}\,(t) > 1-N'$ for any integer $N' \geq 2$. Let $M \geq 0$ be an arbitrary large integer. Using the expressions (3.8) and (3.9), and applying the formula (3.10) to $I^m_{q,\varepsilon}(s,t,z)$ with $N' := M - N + 1 \geq 2$, we see that the formula (3.6) can be written as

$$\begin{aligned} \zeta_q(s,t,z) = &-\frac{q^{-zt}(1-q)^s}{\log q} b_{q^z}(t,-s+1) + \frac{1}{2}\left(\frac{1-q^z}{1-q}\right)^{-s} \\ &+ D^1_q(s,t,z;N,M) + D^2_q(s,t,z;N,M) + D^3_q(s,t,z;N,M)\,, \end{aligned} \tag{3.11}$$

where

$$D_q^1(s,t,z;N,M) := -\sum_{k=1}^{N}\sum_{\varepsilon=0}^{k}\frac{B_{k+1}}{(k+1)!}c_k^{\varepsilon}(s,t)\left(\frac{1-q^z}{1-q}\right)^{-s-\varepsilon}\frac{(\log q)^k}{(1-q)^{\varepsilon}},$$

$$D_q^2(s,t,z;N,M) := \sum_{\varepsilon=0}^{N+1}\sum_{l=1}^{M-N}\sum_{n\in\mathbf{Z}\setminus\{0\}}\frac{(-1)^{N+l-1}}{(2\pi\sqrt{-1}n)^{N+1}}\frac{c_{N+1}^{\varepsilon}(s,t)(s+\varepsilon)_{l-1}q^{z(l-1)}}{(1-q)^l(\delta n+t)_l}$$
$$\times\left(\frac{1-q^z}{1-q}\right)^{-s-\varepsilon+1-l}\frac{(\log q)^N}{(1-q)^{\varepsilon-1}},$$

$$D_q^3(s,t,z;N,M) := \sum_{\varepsilon=0}^{N+1}\sum_{n\in\mathbf{Z}\setminus\{0\}}\frac{(-1)^{M+1}}{(2\pi\sqrt{-1}n)^{N+1}}\frac{c_{N+1}^{\varepsilon}(s,t)(s+\varepsilon)_{M-N}q^{z(M-N)}}{(1-q)^{M-N}(\delta n+t)_{M-N}}$$
$$\times\frac{(\log q)^{N+1}}{(1-q)^{\varepsilon}}\int_0^{\infty}e^{2\pi\sqrt{-1}nx}q^{x(t+M-N)}\left(\frac{1-q^{x+z}}{1-q}\right)^{-s-\varepsilon-M+N}dx\,.$$

The equation (3.11) gives an analytic continuation of $\zeta_q(s,t,z)$ to the region $\mathrm{Re}\,(t) > 1-N' = N-M$. Note that, by the fact $c_k^k(s,t)=(s)_k$ and (3.7) again, we have

$$\lim_{q\uparrow 1}D_q^1(s,t,z;N,M) = \sum_{k=1}^{N}\frac{B_{k+1}}{(k+1)!}(s)_k z^{-s-k}, \tag{3.12}$$

$$\lim_{q\uparrow 1}D_q^2(s,t,z;N,M) = \sum_{l=N+1}^{M}\frac{B_{l+1}}{(l+1)!}(s)_l z^{-s-l}, \tag{3.13}$$

$$\lim_{q\uparrow 1}D_q^3(s,t,z;N,M) = -\frac{(s)_{M+1}}{(M+1)!}\int_0^{\infty}\tilde{B}_{M+1}(x)(x+z)^{-s-M-1}dx\,. \tag{3.14}$$

Therefore, by (3.5) and (3.11), we obtain the following

PROPOSITION 3.2. *For any integers $N_l \geq 1$ and $M_l \geq N_l+1$ $(0 \leq l \leq r-1)$, we have*

$$\zeta_{q,r}(s,t,z) = -\frac{(1-q)^{s-(r-1)}}{\log q}\sum_{l=0}^{r-1}P_{q,r}^l(z)q^{-z(t-l)}(1-q)^{r-1-l}b_{q^z}(t-l,-s+l+1)$$
$$+\frac{1}{2}\sum_{l=0}^{r-1}P_{q,r}^l(z)\left(\frac{1-q^z}{1-q}\right)^{-s+l}+\sum_{l=0}^{r-1}P_{q,r}^l(z)D_q^1(s-l,t-l,z;N_l,M_l)$$
$$+\sum_{l=0}^{r-1}P_{q,r}^l(z)D_q^2(s-l,t-l,z;N_l,M_l)$$
$$+\sum_{l=0}^{r-1}P_{q,r}^l(z)D_q^3(s-l,t-l,z;N_l,M_l)\,.$$

This gives an analytic continuation of $\zeta_{q,r}(s,t,z)$ *to the region* $\mathrm{Re}\,(t) > M'$ *where* $M' := \max\{N_l - M_l + l \mid 0 \le l \le r-1\}$. □

3.3. Proof of Theorem 2.1. Note the following lemma.

LEMMA 3.3. *It holds that*

$$\lim_{q\uparrow 1} P^l_{q,r}(z) = P^l_r(z)\,.$$

PROOF. By the definition of $P^l_{q,r}(z)$, it is sufficient to show

$$(3.15)\qquad \begin{aligned}(-1)^{r-1-l} &\sum_{1\le m_1<\cdots<m_{r-1-l}\le r-1} (z-m_1)\cdots(z-m_{r-1-l})\\ &= \sum_{k=l}^{r-1}\binom{j}{l} s(r,j+1)(-z)^{j-l}\,.\end{aligned}$$

Notice that the left hand side of (3.15) is equal to the coefficient of x^l in the polynomial $p_r(x) := \prod_{j=1}^{r-1}(x-(z-j))$ in x. Since $p_r(x) = (x-z)_r/(x-z)$, we have

$$p_r(x) = \sum_{j=0}^{r-1} s(r,j+1)(x-z)^j = \sum_{l=0}^{r-1}\left(\sum_{j=l}^{r-1}\binom{j}{l}s(r,j+1)(-z)^{j-l}\right)x^l\,.$$

Hence the desired formula follows. □

We are ready to prove the main theorem.

PROOF OF THEOREM 2.1. We first show the sufficiency. Let $t = s-\nu$ for $\nu \in \mathbf{N}$. Notice that, by (3.5), we have $\zeta^{(\nu)}_{q,r}(s,z) = \sum_{l=0}^{r-1} P^l_{q,r}(z)\zeta^{(\nu)}_q(s-l,z)$. Hence, by [6, Theorem 2.1], Lemma 3.3 and (3.2), we have

$$\lim_{q\uparrow 1}\zeta^{(\nu)}_{q,r}(s,z) = \sum_{l=0}^{r-1} P^l_r(z)\zeta(s-l,z) = \zeta_r(s,z) \quad (s \in \mathbf{C})\,.$$

We next show the necessity. Suppose that $\lim_{q\uparrow 1}\zeta_{q,r}(s,t,z)$ exists and satisfies $\lim_{q\uparrow 1}\zeta_{q,r}(s,t,z) = \zeta_r(s,z)$ for all $s \in \mathbf{C}$ with some meromorphic function $t = \varphi(s)$. Then, by Proposition 3.1, Proposition 3.2, Lemma 3.3, (3.12), (3.13) and (3.14), it is necessary to hold

$$\begin{aligned}-\lim_{q\uparrow 1}\frac{(1-q)^{s-(r-1)}}{\log q}&\sum_{l=0}^{r-1} P^l_{q,r}(z)q^{-z(t-l)}(1-q)^{r-1-l}b_{q^z}(t-l,-s+l+1)\\ &= \sum_{l=0}^{r-1}\frac{P^l_r(z)}{s-l-1}z^{-s+l+1}\,.\end{aligned}$$

Assume $\mathrm{Re}(s) < 1$. Since $\lim_{q\uparrow 1}(1-q)^{s-(r-1)}/\log q$ diverges, it is necessary to hold

$$\lim_{q\uparrow 1}\sum_{l=0}^{r-1} P_{q,r}^{l}(z)q^{-z(t-l)}(1-q)^{r-1-l}b_{q^z}(t-l,-s+l+1)=0\,. \tag{3.16}$$

Notice that $\lim_{q\uparrow 1} b_{q^z}(t-l,-s+l+1)=\mathrm{B}(t-l,-s+l+1)$ for all l $(0\le l\le r-1)$. Further, since the left hand side of (3.16) is equal to $\mathrm{B}(t-r+1,-s+r)=\Gamma(t-r+1)\Gamma(-s+r)/\Gamma(t-s+1)$, we have $t-s+1\in\mathbf{Z}_{\le 0}$, whence $t=\varphi(s)=s-\nu$ for some positive integer $\nu\in\mathbf{N}$ in the region $\mathrm{Re}(s)<1$. Since $\varphi(s)$ is meromorphic on $\mathbf{C}$, we have $\varphi(s)=s-\nu$ for all $s\in\mathbf{C}$. This proves the theorem. □

4. Remarks on q-analogues of $\zeta_r(s,z;\boldsymbol{\omega})$.

We introduce here a q-analogue of the Barnes multiple zeta function $\zeta_r(s,z;\boldsymbol{\omega})$ for a general parameter $\boldsymbol{\omega}:=(\omega_1,\dots,\omega_r)$. Assume $\omega_i>0$ $(1\le i\le r)$ and $\mathrm{Re}(z)>0$. We define a q-analogue of $\zeta_r(s,z;\boldsymbol{\omega})$ by the series

$$\zeta_{q,r}(s,t,z;\boldsymbol{\omega}):=\sum_{n_1,\dots,n_r\ge 0}\frac{q^{n_1\omega_1 t+n_2\omega_2(t-1)+\cdots+n_r\omega_r(t-r+1)}}{[n_1\omega_1+\cdots+n_r\omega_r+z]_q{}^s}\,.$$

We put $\zeta_{q,r}^{(\nu)}(s,z;\boldsymbol{\omega}):=\zeta_{q,r}(s,s-\nu,z;\boldsymbol{\omega})$ for $\nu\in\mathbf{N}$. The series $\zeta_{q,r}(s,t,z;\boldsymbol{\omega})$ converges absolutely for $\mathrm{Re}(t)>r-1$. It is clear that $\zeta_{q,r}(s,t,z)=\zeta_{q,r}(s,t,z;\mathbf{1}_r)$ where $\mathbf{1}_r:=(\underbrace{1,1,\dots,1}_{r})$. By the following proposition, $\zeta_{q,r}(s,t,z;\boldsymbol{\omega})$ is continued meromorphically to the whole s, t-plane. The proof can be obtained by the similar way to [5, Proposition 1] and [6, Proposition 2.9].

PROPOSITION 4.1. (i) *The function $\zeta_{q,r}(s,t,z;\boldsymbol{\omega})$ can be written as*

$$\zeta_{q,r}(s,t,z;\boldsymbol{\omega})=(1-q)^s\sum_{l=0}^{\infty}\binom{s+l-1}{l}q^{lz}\prod_{j=1}^{r}(1-q^{\omega_j(t-j+1+l)})^{-1}\,. \tag{4.1}$$

This gives a meromorphic continuation of $\zeta_{q,r}(s,t,z;\boldsymbol{\omega})$ to the whole s, t-plane with simple poles at $t\in j-1+\mathbf{Z}_{\le 0}+\delta_j\mathbf{Z}$ $(1\le j\le r)$. Here $\delta_j:=2\pi\sqrt{-1}/(\omega_j\log q)$.

(ii) *The function $\zeta_{q,r}^{(\nu)}(s,z;\boldsymbol{\omega})$ can be written as*

$$\zeta_{q,r}^{(\nu)}(s,z;\boldsymbol{\omega})=(1-q)^s\sum_{l=0}^{\infty}\binom{s+l-1}{l}q^{lz}\prod_{j=1}^{r}(1-q^{\omega_j(s-\nu-j+1+l)})^{-1}. \tag{4.2}$$

This gives a meromorphic continuation of $\zeta_{q,r}^{(\nu)}(s,z;\boldsymbol{\omega})$ *to the whole plane* $\mathbf{C}$ *with simple poles at the points in*

$$\begin{cases} j+\delta_i\mathbf{Z}\backslash\{0\} & (j\in\mathbf{Z}_{\le 0},\ 1\le i\le r)\,,\\ j+\delta_i\mathbf{Z} & (1\le j\le \nu,\ 1\le i\le r)\,,\\ \nu+j+\delta_i\mathbf{Z} & (1\le j\le r-1,\ j+1\le i\le r)\,. \end{cases}$$

In particular, the poles of $\zeta_{q,r}^{(\nu)}(s,z;\boldsymbol{\omega})$ *on the real axis are given by* $s=1,2,\ldots,r,r+1,\ldots,r+\nu-1$.

(iii) *Let* $m\in\mathbf{Z}_{\ge 0}$. *Then we have*

$$\begin{aligned} \zeta_{q,r}^{(\nu)}(-m,z;\boldsymbol{\omega}) = (1-q)^{-m}\Bigg\{&\sum_{l=0}^{m}(-1)^l\binom{m}{l}q^{lz}\prod_{j=1}^{r}(1-q^{\omega_j(-m-\nu+l-j+1)})^{-1}\\ &+\frac{q^{(m+\nu-1)z}}{\log q}\sum_{l=1}^{r}\frac{(-1)^{m+1}m!(l+\nu-2)!\,q^{lz}}{(l+m+\nu-1)!\,\omega_l}\prod_{\substack{j=1\\ j\ne l}}^{r}(1-q^{\omega_j(l-j)})^{-1}\Bigg\}\,. \end{aligned} \tag{4.3}$$

PROOF. The formula (4.1) is obtained by the binomial theorem, whence (4.2) immediately follows. The formula (4.3) is derived from the fact $(s+m)/(1-q^{\omega_l(s+m)}) = -1/(\omega_l\log q)+O(s+m)$ as $s\to -m$. □

These facts motivate the

CONJECTURE 4.2. *Let* $t=\varphi(s)$ *be a meromorphic function on* $\mathbf{C}$. *Then the formula*

$$\lim_{q\uparrow 1}\zeta_{q,r}(s,\varphi(s),z;\boldsymbol{\omega})=\zeta_r(s,z;\boldsymbol{\omega})\quad (s\in\mathbf{C})$$

holds if and only if the function $\varphi(s)$ *can be written as* $\varphi(s)=s-\nu$ *for some* $\nu\in\mathbf{N}$.

In fact, since $\zeta_1(s,z;\boldsymbol{\omega})=\omega^{-s}\zeta(s,z/\omega)$ and $\zeta_{q,1}(s,t,z;\boldsymbol{\omega})=[\omega]_q{}^{-s}\zeta_{q^\omega}(s,t,z/\omega)$ for $\omega>0$, Conjecture 4.2 is true for $r=1$ by (3.4) and (3.11).

A. An associated q-analogue of the gamma function

In this appendix, we introduce a q-analogue of the gamma function defined via the q-analogue of the Hurwitz zeta function:

$$\tilde{\zeta}_q(s,z):=\zeta_q^{(1)}(s,z)\times q^{z(s-1)}=\sum_{n=0}^{\infty}\frac{q^{(n+z)(s-1)}}{[n+z]_q{}^s}\quad (\mathrm{Re}\,(s)>1)\,.$$

Note that by (3.1), we have

$$\tilde{\zeta}_q(s,z)=\tilde{\zeta}_q(s,z+1)+\frac{q^{z(s-1)}}{[z]_q{}^s}\,. \tag{A.1}$$

Imitating the Lerch formula [8] (the zeta regularization)

$$\frac{\partial}{\partial s}\zeta(s,z)\bigg|_{s=0} = \log\frac{\Gamma(z)}{\sqrt{2\pi}},$$

we define a q-analogue $\tilde{\Gamma}_q(z)$ of the gamma function by

$$\tilde{\Gamma}_q(z) := \exp\left(\frac{\partial}{\partial s}\tilde{\zeta}_q(s,z)\bigg|_{s=0} - \frac{\partial}{\partial s}\tilde{\zeta}_q(s,1)\bigg|_{s=0}\right).$$

Then the function $\tilde{\Gamma}_q(z)$ is well-defined as a single valued meromorphic function. Indeed, let

$$\tilde{\zeta}_q(s,z) = a_0(z;q) + a_1(z;q)s + a_2(z;q)s^2 + \cdots$$

be the Taylor expansion of $\tilde{\zeta}_q(s,z)$ around $s=0$. Note that $\tilde{\zeta}_q(s,z)$ is holomorphic at $s=0$. Assume Re $(z) > 0$. Then, by Proposition 4.1, $\tilde{\zeta}_q(s,z)$ has the following expression;

$$\text{(A.2)} \qquad \tilde{\zeta}_q(s,z) = (1-q)^s \sum_{n=0}^{\infty} \binom{s+n-1}{n} \frac{q^{z(s-1+n)}}{1-q^{s-1+n}}.$$

Hence one can calculate the coefficient $a_1(z;q)$ by the same manner performed in [7] as

$$\text{(A.3)} \qquad \begin{aligned} a_1(z;q) = &\sum_{n=2}^{\infty} \frac{1}{n} \frac{q^{(n-1)z}}{1-q^{n-1}} - z + \frac{1}{2} + \frac{1-z(1-q)}{(1-q)^2} q^{1-z} \log q \\ &- \left(\frac{q^{1-z}}{1-q} + \frac{1}{\log q}\right) \log(1-q). \end{aligned}$$

Therefore $\tilde{\Gamma}_q(z)$ is meromorphic in the region Re $(z) > 0$. If $-1 <$ Re $(z) < 0$, by the ladder relation (A.1), we have

$$a_1(z;q) = q^{-z}\log q^z - q^{-z}\log\left(\frac{1-q^z}{1-q}\right) + a_1(z+1;q).$$

Hence we have

$$\text{(A.4)} \qquad \tilde{\Gamma}_q(z) = (q^{-z}[z]_q)^{-q^{-z}} \tilde{\Gamma}_q(z+1).$$

This gives a meromorphic continuation of $\tilde{\Gamma}_q(z)$ to the region Re $(z) > -1$. Repeating the same procedure, we see that $\tilde{\Gamma}_q(z)$ can be extended as a meromorphic function on $\mathbf{C}$.

From Theorem 2.1, by the Lerch formula, we have immediately

$$\text{(A.5)} \qquad \lim_{q\uparrow 1} \tilde{\Gamma}_q(z) = \Gamma(z) \quad (z \notin -\mathbf{Z}_{\geq 0}).$$

Moreover, $\tilde{\Gamma}_q(z)$ satisfies the following properties.

PROPOSITION A.1. *We have*

$$\tilde{\Gamma}_q(z+1) = (q^{-z}[z]_q)^{q^{-z}}\tilde{\Gamma}_q(z)\,, \tag{A.6}$$

$$\tilde{\Gamma}_q(1) = 1, \tag{A.7}$$

$$\frac{d^2}{dz^2}\log\tilde{\Gamma}_q(z+1) \geq 0 \quad (z \geq 0)\,. \tag{A.8}$$

In particular, for a positive integer n, we have

$$\tilde{\Gamma}_q(n+1) = q^{-\sum_{k=1}^n kq^{-k}}\prod_{k=1}^n([k]_q)^{q^{-k}}\,. \tag{A.9}$$

PROOF. By the definition of $\tilde{\Gamma}_q(z)$, (A.7) is obvious. The formula (A.6) is clear from (A.4). The assertion (A.9) follows from (A.6) and (A.7) by induction. To show the inequality (A.8), take the logarithm of $\tilde{\Gamma}_q(z)$:

$$\begin{aligned}\log\tilde{\Gamma}_q(z) = &\sum_{n=2}^\infty \frac{1}{n}\frac{q^{z(n-1)}-q^{n-1}}{1-q^{n-1}} - z + 1\\ &+ \frac{q^{-z}(1-(1-q)z)-1}{(1-q)^2}q\log q + \frac{1-q^{1-z}}{1-q}\log(1-q)\,.\end{aligned} \tag{A.10}$$

We calculate as

$$\frac{d^2}{dz^2}\log\tilde{\Gamma}_q(z+1) = (\log q)^2\sum_{n=2}^\infty\frac{(n-1)^2}{n}\frac{q^{(z+1)(n-1)}}{1-q^{n-1}} + \frac{(\log q)^2q^{-z}}{(1-q)^2}\eta_q(z)\,,$$

where $\eta_q(z) := (\log q)(1-(1-q)(z+1)) - (1-q)\log(1-q) + 2(1-q)$. Therefore, it suffices to show that $\eta_q(z) \geq 0$ for all $0 < q < 1$ if $z \geq 0$, and this is indeed true. In fact, since $\frac{d}{dq}\eta_q(z) \leq 0$ for $0 < q < 1$, we conclude that $\eta_q(z) \geq \lim_{q\uparrow 1}\eta_q(z) = 0$. Hence the proposition follows. □

REMARK A.2. One can find the similar formulas to (A.6), (A.7) and (A.8) in the q-analogue of the Bohr-Morellup theorem for the Jackson q-gamma function in [2]. It has not yet been clarified that these properties characterize the function $\tilde{\Gamma}_q(z)$.

By the expression (A.2) again, $\tilde{\zeta}_q(s,z)$ has the following Laurent expansion around $s = 1$:

$$\tilde{\zeta}_q(s,z) = \frac{q-1}{\log q}\frac{1}{s-1} + \gamma_q(z) + O(s-1) \quad (\mathrm{Re}\,(z) > 0)\,, \tag{A.11}$$

where

$$\gamma_q(z) := \sum_{n=1}^\infty\frac{q^{nz}}{[n]_q} + (1-q)\left(-z + \frac{1}{2} - \frac{\log(1-q)}{\log q}\right). \tag{A.12}$$

We next show a q-analogue of the Lerch limit formula [8]:

$$\lim_{s\to 1}\left(\zeta(s,z)-\frac{1}{s-1}\right)=-\frac{\Gamma'}{\Gamma}(z)\,. \tag{A.13}$$

PROPOSITION A.3. *It holds that*

$$\gamma_q(z)=\lim_{s\to 1}\left(\tilde{\zeta}_q(s,z)-\frac{q-1}{\log q}\frac{1}{s-1}\right)=-\frac{q-1}{\log q}\frac{\tilde{\Gamma}_q'}{\tilde{\Gamma}_q}(z)+C_q(z)\quad(\mathrm{Re}\,(z)>0)\,, \tag{A.14}$$

where

$$\begin{aligned}C_q(z):=&\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{q^{nz}}{[n]_q}+q^{1-z}+\frac{\log q}{1-q}(1-(1-q)z)q^{1-z}+\frac{1-q}{\log q}\\&-q^{1-z}\log(1-q)-\frac{1-q}{\log q}\log(1-q)+\left(-z+\frac{1}{2}\right)(1-q)\end{aligned}$$

and $\lim_{q\uparrow 1}C_q(z)=0$. *Put* $\gamma_q:=\gamma_q(1)$. *Then we have, in particular,* $\lim_{q\uparrow 1}\gamma_q=\gamma$ *where* $\gamma=0.577215\ldots$ *denotes the Euler constant.*

PROOF. By (A.10), we have

$$\begin{aligned}\frac{\tilde{\Gamma}_q'}{\tilde{\Gamma}_q}(z)=&\frac{\log q}{1-q}\left(\sum_{n=1}^{\infty}\frac{q^{zn}}{[n]_q}-\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{q^{nz}}{[n]_q}\right)-1\\&-\frac{(1-q)+(1-(1-q)z)\log q}{(1-q)^2}q^{1-z}\log q+\frac{\log q}{1-q}q^{1-z}\log(1-q)\,.\end{aligned} \tag{A.15}$$

Plugging (A.12) into (A.15), we obtain the formula (A.14). It is straightforward to show the fact $\lim_{q\uparrow 1}C_q(z)=0$ when $\mathrm{Re}\,(z)>0$. Hence we have $\lim_{q\uparrow 1}\gamma_q=\gamma$ by the limit formulas (A.5), (A.13) and the facts $\Gamma(1)=1$, $\Gamma'(1)=-\gamma$. This completes the proof. □

REMARK A.4. The q-analogue of the Lerch limit formula obtained in this paper is different from the one given in [7].

As a final remark, we give a q-analogue of the Gauss-Legendre formula.

PROPOSITION A.5. *Let* $N\in\mathbf{N}$. *Then we have*

$$\begin{aligned}&[N]_q^{[1-Nz]_q}\tilde{\Gamma}_{q^N}\left(\frac{1}{N}\right)\cdots\tilde{\Gamma}_{q^N}\left(\frac{N-1}{N}\right)\tilde{\Gamma}_q(Nz)\\&\qquad=\tilde{\Gamma}_{q^N}(z)\tilde{\Gamma}_{q^N}\left(z+\frac{1}{N}\right)\cdots\tilde{\Gamma}_{q^N}\left(z+\frac{N-1}{N}\right).\end{aligned}$$

PROOF. The proof is straightforward from (A.10). □

References

[1] G. E. ANDREWS, R. ASKEY and R. ROY, *Special Functions*, Encyclopedia of math. and appl., vol. 71, Cambridge U.P. (1999).

[2] R. ASKEY, The q-gamma and q-Beta functions, Appl. Anal. **8** (1978), 125–141.

[3] E. BARNES, On the theory of the multiple gamma functions, Trans. Cambridge Philos. Soc. **19** (1904), 374–425.

[4] K. KIMOTO, N. KUROKAWA, S. MATSUMOTO and M. WAKAYAMA, Multiple finite Riemann zeta function, Acta Arith. **116** (2) (2005), 173–187.

[5] M. KANEKO, N. KUROKAWA and M. WAKAYAMA, A variation of Euler's approach to values of the Riemann zeta function, Kyushu J. Math. **57** (2003), 175–192.

[6] K. KAWAGOE, M. WAKAYAMA and Y. YAMASAKI, q-Analogues of the Riemann zeta, the Dirichlet L-functions, and a crystal zeta function, to appear in Forum. Math. (2006).

[7] N. KUROKAWA and M. WAKAYAMA, On q-analogue of the Euler constant and Lerch's limit formula, Proc. Amer. Math. Soc. **132** (2003), 935–943.

[8] M. LERCH., Dalši studie v oboru Malmsténovských řad, Rozpravy České Akad. **3** No. 28 (1894), 1–61.

[9] T. SEO, J. CHOI, J. GANG and B. OK, Note on the analytic continuation of the multiple Hurwitz zeta functions $\zeta_n(s, a)$, Kyungpook Math J. **33**(2) (1993), 127–131.

[10] H. TSUMURA, On modification of the q-L-series and its applications, Nagoya Math. J. **164** (2001), 185–197.

[11] E. T. WHITTAKER and G. N. WATSON, *A Course of Modern Analysis* 4th edn, Cambridge University Press (1927).

Present Address:
GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY,
HAKOZAKI, FUKUOKA, 812–8581 JAPAN.
e-mail: yamasaki@math.kyushu-u.ac.jp