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# A Relation on Floer Homology Groups of Homology Handles

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**Abstract.** We prove a relationship on the rank of the Floer homology groups of integral homology handles. Moreover we make a conjecture on a certain difference between these ranks.

## 1. Introduction

Let $\Sigma$ denote a Seifert homology 3-sphere $\Sigma(a_1, \dots, a_n)$. By performing 0-surgery along the $n$-th singular fiber $k_n$ of $\Sigma$, we obtain a homology handle $\Sigma + 0 \cdot k_n$, which is denoted by $N$. The Floer homology group $HF_*(N)$ can be calculated from that of homology 3-spheres exploiting the Floer exact triangle. In this paper we shall prove a relationship among the Floer homology groups of integral homology handles $N$. The main result is the following:

THEOREM 1. *Let $a_i$ $(i = 1, \dots, n)$ be relatively coprime positive integers, and $\Sigma(a_1, \dots, a_n)$ the Seifert homology 3-sphere corresponding to the data $a_i$. We denote by $N$ and $N^*$ the homology handles $\Sigma(a_1, \dots, a_n) + 0 \cdot k_n$ and $\Sigma(a_1, \dots, a_{n-1}, m - a_n) + 0 \cdot k_n$ respectively, where $m = a_1 a_2 \cdots a_{n-1}$. Let $b_i(N)$ denote the rank of the Floer homology group $HF_i(N)$ $(0 \leqslant i \leqslant 7)$. We then obtain the following relationship between $b_i(N)$ and $b_i(N^*)$ :*

$$b_2(N) - b_0(N) = b_2(N^*) - b_0(N^*)\,.$$

REMARK. Here, the grading on the Floer homology group $HF_*(N)$ is assigned from the triple $(\Sigma, \Sigma + (-1) \cdot k_n, N)$, where $\Sigma$ denotes $\Sigma(a_1, \dots, a_n)$; See also Theorem in Section 2.

In the last section we would like to form a conjecture on the number $b_2(N) - b_0(N)$ for $N = \Sigma(a_1, a_2, a_3) + 0 \cdot k_3$ . Finally we provide lists on the ranks of the Floer homology groups of $N$, which support our conjecture.

## 2. Review of floer homology groups

The Floer homology group for integral homology 3-spheres is defined as follows; see

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Floer [4]. Let $M$ be an integral homology 3-sphere and $P$ the trivial $SU(2)$-bundle over $M$. We then obtain a chain complex $C_*(M)$ that is a free $\mathbf{Z}$-module generated by the gauge equivalence classes of flat connections on $P$. The chain complex has a natural $\mathbf{Z}/8$-grading via the index theorem, which is called the Floer index. The homology group of $C_*(M)$ is the Floer homology group of $M$ and denoted by $HF_*(M)$. We shall denote the rank of $HF_i(M)$ by $b_i(M)$ $(0 \leqslant i \leqslant 7)$. For the details of the Floer homology group, we refer to Donaldson [2].

Floer [5] extended the Floer homology groups to integral homology handles, namely 3-manifolds $N$ whose integral homology group is isomorphic to that of $S^2 \times S^1$. In order to define this homology group we need to change some conditions in the setting above. First of all the trivial $SU(2)$-bundle is replaced by a unique non-trivial $SO(3)$-bundle $Q$ on $N$. Hence there does not exist a trivial flat connection $\theta$ on $Q$. As a result, $HF_*(N)$ has no absolute grading of $\mathbf{Z}/8$.

Consider an integral homology 3-sphere $M$ and a knot $k$ in $M$. We obtain another integral homology 3-sphere $M + (-1) \cdot k$ and a homology handle $M + 0 \cdot k$ by $(-1)$- and 0-surgery along $k$ respectively. By handle attaching we also have suitable cobordisms $X, Y, Z$, whose boundary component is either $M$, $M + (-1) \cdot k$ or $M + 0 \cdot k$. These cobordisms give rise to homomorphisms on Floer homology groups. Floer proved the following:

THEOREM 2 (Floer [5], Braam-Donaldson [1]). *Let $M$, $M + (-1) \cdot k$ and $M + 0 \cdot k$ be as above. We then have the following long exact sequence of Floer homology groups*:

$$\cdots \to HF_{*+1}(M + 0 \cdot k) \xrightarrow{Z_*} HF_*(M) \xrightarrow{X_*} HF_*(M + (-1) \cdot k)$$
$$\xrightarrow{Y_*} HF_*(M + 0 \cdot k) \to \cdots .$$

*Here $X_*, Y_*, Z_*$ are homomorphisms induced by $X, Y, Z$; see*[5] *for the details.*

The long exact sequence above is called the Floer exact triangle. Owing to the Floer exact triangle above we can determine a grading of $HF_*(M + 0 \cdot k)$. Then we denote the rank of $HF_i(M + 0 \cdot k)$ by $b_i(M + 0 \cdot k)$ in the same way as in the case of homology 3-spheres.

## 3. Floer homology groups of seifert homology 3-spheres

Let $a_1, \ldots, a_n$ be positive integers which are relatively coprime. Then the Seifert homology 3-sphere $\Sigma(a_1, \ldots, a_n)$ is obtained as a Seifert manifold with the Seifert invariant $(g, (1, b_0), (a_1, b_1), \ldots, (a_n, b_n))$ which satisfies equations $g = 0$ and $b_0 + \sum_{i=1}^n b_i/a_i = 1/a_1 \cdots a_n$. The presentation of the Seifert invariant is same as in Neumann-Raymond [9]. In particular, when $n = 3$, $\Sigma(a_1, a_2, a_3)$ is called a Brieskorn homology 3-sphere and $\Sigma(a_1, a_2, a_3) = \{(z_1, z_2, z_3) \in \mathbf{C}^3 | z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\} \cap S^5$. See [9].

We regard the $i$-th singular fiber corresponding to $a_i$ as a knot, and denote it by $k_i$. By performing the $(-1)$-surgery (0-surgery ), we obtain a Seifert homology 3-sphere

$\Sigma(a_1,\ldots,a_n)+(-1)\cdot k_i$ (a Seifert manifold $\Sigma(a_1,\ldots,a_n)+0\cdot k_i$, resp.). It is easy to prove

$$\Sigma(a_1,\ldots,a_n)+(-1)\cdot k_n=\Sigma(a_1,\ldots,a_{n-1},m+a_n)\,, \tag{1}$$

see Saveliev [10]. Furthermore, we obtain the following:

THEOREM 3. *There is an orientation-reversing diffeomorphism between* $\Sigma(a_1,\ldots,a_n)+0\cdot k_n$ *and* $\Sigma(a_1,\ldots,a_{n-1},m-a_n)+0\cdot k_n$.

PROOF. We first note that $\Sigma = \Sigma(a_1,\ldots,a_n)$ has a Seifert invariant $(g;(1,b_0),(a_1,b_1),\ldots,(a_n,b_n))$. Then the Seifert invariant of $N=\Sigma+0\cdot k_n$ is $(g;(1,b_0),(a_1,b_1),\ldots,(a_{n-1},b_{n-1}),(m,r))$; see Saveliev [10]. Here $r$ is an integer that satisfies

$$\begin{vmatrix} a_n & b_n \\ -m & r \end{vmatrix}=1\,,$$

and hence $r=(1-mb_n)/a_n$. Then as a corollary of Theorem 1.1. in [9], it follows that a Seifert invariant of $\overline{N}$ is $(g;(1,-b_0),(a_1,-b_1),\ldots,(a_{n-1},-b_{n-1}),(m,-r))$, where $\overline{N}$ stands for $N$ with the reversed orientation. Put $s=b_0m+\sum_{i=1}^{n-1}b_ia_1\cdots\check{a}_i\cdots a_{n-1}+b_n$. Then we obtain a Seifert invariant $(g;(1,-b_0),(a_1,-b_1),\ldots,(a_{n-1},-b_{n-1}),(m-a_n,s))$ of $\Sigma^*=\Sigma(a_1,\ldots,a_{n-1},m-a_n)$. In the same way as in the case of $N$, we can calculate the Seifert invariant for $N^*=\Sigma^*+0\cdot k_n$; the Seifert invariant is $(g;(1,-b_0),(a_1,-b_1),\ldots,(a_{n-1},-b_{n-1}),(m,t))$. Here $t$ is an integer that satisfies

$$\begin{vmatrix} m-a_n & s \\ -m & t \end{vmatrix}=1\,,$$

and hence $t=(1-ms)/(m-a_n)$. It then follows that

$$r+t=\frac{m}{a_n(m-a_n)}\left(1-ma_n\left(b_0+\sum_{i=1}^{n}\frac{b_i}{a_i}\right)\right)=0\,.$$

Thus we obtain $t=-r$. Therefore, $N^*$ is diffeomorphic to $\overline{N}$ preserving the orientations. □

Fintushel and Stern [3] proved that the Floer index of every non-trivial flat connection over a Brieskorn homology 3-sphere $\Sigma$ is even. Therefore the boundary operator $\partial$ of $C_*(\Sigma)$ is trivial, so that we have $HF_*(\Sigma)=C_*(\Sigma)$. Kirk and Klassen [7] and Saveliev [10] proved the same fact for every Seifert homology 3-sphere. Moreover, Saveliev proved that it also holds for the homology handle obtained from a Seifert homology 3-sphere by 0-surgery along a singular fiber $k$. He also proved that its Floer exact triangle is a splitting exact sequence:

$$0\to HF_*(\Sigma)\xrightarrow{X_*}HF_*(\Sigma+(-1)\cdot k)\xrightarrow{Y_*}HF_*(\Sigma+0\cdot k)\to 0\,.$$

This gives rise to an equality:

$$b_i(\Sigma+0\cdot k)=b_i(\Sigma+(-1)\cdot k)-b_i(\Sigma)\ \text{for}\ i=0,\ldots,7\,. \tag{2}$$

Frøyshov [6] further proved that $HF_i(\Sigma)$ is isomorphic to $HF_{i+4}(\Sigma)$, so that we obtain $b_0(\Sigma) = b_4(\Sigma)$ and $b_2(\Sigma) = b_6(\Sigma)$. Then Taubes' theorem [13] implies $b_0(\Sigma) + b_2(\Sigma) = b_4(\Sigma) + b_6(\Sigma) = \lambda(\Sigma)$, where $\lambda(\Sigma)$ is the Casson invariant of $\Sigma$. Also exploiting the identities above, Saveliev [11] proved

$$b_2(\Sigma) - b_0(\Sigma) = b_6(\Sigma) - b_4(\Sigma) = \bar{\mu}(\Sigma)\,, \tag{3}$$

where $\bar{\mu}$ is Neumann's $\bar{\mu}$-invariant [8] of $\Sigma$. On the other hand Neumann [8] proved that every Seifert homology 3-sphere $\Sigma(a_1, \dots, a_n)$ satisfies

$$\bar{\mu}(\Sigma(a_1, \dots, a_n)) = \pm\bar{\mu}(\Sigma(a_1, \dots, a_{n-1}, 2m \pm a_n))\,. \tag{4}$$

Let $N$ be $\Sigma(a_1, \dots, a_n) + 0 \cdot k_n$. With $b_i(N)$, we have the following result. When we fix $a_1, a_2, \dots, a_{n-1}$, then $b_0(N) + b_2(N)$ is independent of $a_n$. On the other hand, $b_2(N) - b_0(N)$ depends only on $a_n$; See Saveliev [12] p. 156.

## 4. Proof of main theorem and observation

We shall denote $\Sigma(a_1, \dots, a_n)$ by $\Sigma$ and $\Sigma(a_1, \dots, a_{n-1}, m - a_n)$ by $\Sigma^*$. Also we denote $\Sigma + 0 \cdot k_n$ by $N$ and $\Sigma^* + 0 \cdot k_n$ by $N^*$.

As we observed, the triples $(\Sigma, \Sigma + (-1) \cdot k_n, N)$ and $(\Sigma^*, \Sigma^* + (-1) \cdot k_n, N^*)$ determines an absolute grading of $HF_*(N)$ and $HF_*(N^*)$ respectively.

PROOF OF THEOREM 1. The equations (2) and (1) imply

$$\begin{aligned} b_2(N) - b_0(N) &= b_2(\Sigma + (-1) \cdot k_n) - b_2(\Sigma) - b_0(\Sigma + (-1) \cdot k_n) + b_0(\Sigma) \\ &= b_2(\Sigma(a_1, \dots, a_{n-1}, m + a_n)) - b_2(\Sigma) - b_0(\Sigma(a_1, \dots, a_{n-1}, m + a_n)) \\ &\quad + b_0(\Sigma) \end{aligned}$$

and

$$\begin{aligned} b_2(N^*) - b_0(N^*) &= b_2(\Sigma^* + (-1) \cdot k_n) - b_2(\Sigma^*) - b_0(\Sigma^* + (-1) \cdot k_n) + b_0(\Sigma^*) \\ &= b_2(\Sigma(a_1, \dots, a_{n-1}, 2m - a_n)) - b_2(\Sigma^*) \\ &\quad - b_0(\Sigma(a_1, \dots, a_{n-1}, 2m - a_n)) + b_0(\Sigma^*)\,. \end{aligned}$$

Therefore, by (3) we obtain

$$b_2(N) - b_0(N) = \bar{\mu}(\Sigma(a_1, \dots, a_{n-1}, m + a_n)) - \bar{\mu}(\Sigma)$$

and

$$\begin{aligned} b_2(N^*) - b_0(N^*) &= \bar{\mu}(\Sigma(a_1, \dots, a_{n-1}, 2m - a_n)) - \bar{\mu}(\Sigma^*) \\ &= \bar{\mu}(\Sigma(a_1, \dots, a_{n-1}, 2m - a_n)) - \bar{\mu}(\Sigma(a_1, \dots, a_{n-1}, 2m - (m + a_n))). \end{aligned}$$

Applying the formula (4), we have

$$b_2(N) - b_0(N) = b_2(N^*) - b_0(N^*)\,.$$

□

EXAMPLE 1. We shall denote $\Sigma(4, 5, a_3)$ by $\Sigma$ and $\Sigma + 0 \cdot k_3$ by $N$. The table 1 is about the ranks of $HF_*(\Sigma)$, $HF_*(\Sigma + (-1) \cdot k_3)$, and $HF_*(N)$. In the Table 1 we only list $(b_0, b_2, b_4, b_6)$ since $b_1, b_3, b_5$ and $b_7$ are equal to zero.

TABLE 1. The ranks of $HF_*(\Sigma)$, $HF_*(\Sigma + (-1) \cdot k_3)$, and $HF_*(N)$

| $a$ | $b_i(\Sigma)$ | $b_i(\Sigma + (-1) \cdot k_3)$ | $b_i(N)$ |
|---|---|---|---|
| 1 | (0,0,0,0) | (6,9,6,9) | (6,9,6,9) |
| 3 | (1,1,1,1) | (8,9,8,9) | (7,8,7,8) |
| 7 | (2,3,2,3) | (10,10,10,10) | (8,7,8,7) |
| 9 | (4,3,4,3) | (11,11,11,11) | (7,8,7,8) |
| 11 | (4,4,4,4) | (11,12,11,12) | (7,8,7,8) |
| 13 | (5,5,5,5) | (13,12,13,12) | (8,7,8,7) |
| 17 | (7,6,7,6) | (14,14,14,14) | (7,8,7,8) |
| 19 | (9,6,9,6) | (15,15,15,15) | (6,9,6,9) |

The Table 2 shows the difference $b_2 - b_0$ of $HF_*(N)$. The Theorem 1 says that the second row $b_2 - b_0$ of the table is symmetric with respect to $10 = 4 \cdot 5/2$.

TABLE 2. A difference $b_2(N) - b_0(N)$

| $a_3$ | 1 | 3 | 7 | 9 | 11 | 13 | 17 | 19 |
|---|---|---|---|---|---|---|---|---|
| $b_2(N) - b_0(N)$ | 3 | 1 | −1 | 1 | 1 | −1 | 1 | 3 |

The following conjecture comes from numerous calculations of Floer homology groups based on a computer. The calculations are divided into two parts. The former one is the program to determine the $SU(2)$-representation space of the fundamental group. The latter one is to calculate the Floer index for every representation according to Fintushel-Stern's formula (see [3]) and to apply the Floer exact triangle.

CONJECTURE 1. *Let $a_1, a_2$ be coprime positive odd integers and $a_3$ be the largest integers satisfying $(a_1, a_3) = (a_2, a_3) = 1$ and $a_3 \leqslant (a_1 a_2 - 1)/2$. Put $N = \Sigma(a_1, a_2, a_3) + 0 \cdot k_3$. Then it holds that*

$$b_2(N) - b_0(N) = 0\,.$$

REMARK. We checked that Conjecture 1 is true if $a_1 + a_2 \leqslant 50$. We also see that the assumption in Conjecture 1 is essential. For example, we have $b_2(N) - b_0(N) = 2$ for $N = \Sigma(3, 11, 14) + 0 \cdot k_3$ while $b_2(N') - b_0(N') = 1$ for $N' = \Sigma(2, 5, 3) + 0 \cdot k_3$.

EXAMPLE 2. We shall exhibit a couple of lists below which support Conjecture1. Here $\Sigma = \Sigma(3, 5, a_3)$ and $N = \Sigma + 0 \cdot k_3$. In the Table 3 we list only $(b_0, b_2, b_4, b_6)$ as in the Table 1.

TABLE 3. The ranks of $HF_*(\Sigma)$, $HF_*(\Sigma + (-1) \cdot k_3)$, and $HF_*(N)$

| $a_3$ | $b_i(\Sigma)$ | $b_i(\Sigma + (-1) \cdot k_3)$ | $b_i(N)$ |
|---|---|---|---|
| 1 | (0,0,0,0) | (3,5,3,5) | (3,5,3,5) |
| 2 | (1,0,1,0) | (4,5,4,5) | (3,5,3,5) |
| 4 | (1,1,1,1) | (5,5,5,5) | (4,4,4,4) |
| 7 | (2,2,2,2) | (6,6,6,6) | (4,4,4,4) |
| 8 | (2,2,2,2) | (6,6,6,6) | (4,4,4,4) |
| 11 | (3,3,3,3) | (7,7,7,7) | (4,4,4,4) |
| 13 | (4,3,4,3) | (7,8,7,8) | (3,5,3,5) |
| 14 | (5,3,5,3) | (8,8,8,8) | (3,5,3,5) |

TABLE 4. A difference $b_2(N) - b_0(N)$

| $a_3$ | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|
| $b_2(N) - b_0(N)$ | 2 | 2 | 0 | 0 | 0 | 0 | 2 | 2 |

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