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Mehler Kernel Approach to Tempered Distributions

Bishnu P. DHUNGANA

Tribhuvan University

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Abstract. Using the Mehler kernel $E(x, \xi, t)$, we show that the solution of the Hermite heat equation $(\partial_t - \triangle + |x|^2)U(x, t) = 0$ in $\mathbf{R}^n \times (0, T)$ satisfying $\sup_{x\in\mathbf{R}^n} |U(x, t)| \le C(1 + t^{-N})$ for some constants C and N can be expressed as $U(x, t) = \langle u(\xi), E(x, \xi, t)\rangle$ for unique u in $\mathcal{S}'(\mathbf{R}^n)$. This is a parallel result with the one in (Theorem 1.2, T. Matsuzawa, *A calculus approach to hyperfunctions* III, Nagoya Math. J. **118** (1990), 133–153). Moreover we represent the tempered distributions as initial values of solution of the Hermite heat equation and apply it to generalize a theorem by Strichartz [Theorem 3.2, Trans. Amer. Math. Soc. **338** (1993), 971–979] in the space of tempered distributions.

1. Introduction

We denote by h_k the normalized Hermite function on $\mathbf{R}$ defined by

$$h_k(x) = \frac{(-1)^k\, e^{x^2/2}}{(2^k\, k!\, \pi^{1/2})^{1/2}} \frac{d^k}{dx^k} e^{-x^2}, \quad k = 0, 1, 2, \dots .$$

For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $\mu = (\mu_1, \dots, \mu_n) \in \mathbf{N}_0^n$; we define $\Phi_\mu(x) := \prod_{j=1}^n h_{\mu_j}(x_j)$ and call it the normalized Hermite function on $\mathbf{R}^n$. It is well known that $\{\Phi_\mu\}$ forms a complete orthonormal basis on $L^2(\mathbf{R}^n)$ and solves the eigenvalue problem $(-\triangle + |x|^2)\Psi = \lambda\Psi$ with $\lambda = 2|\mu| + n$. For all $x, \xi \in \mathbf{R}^n$ and $w \in \mathbf{C}$ with $|w| < 1$, the well known Mehler formula (p. 107, [8] & p. 6, [6]) is

$$\sum_\mu w^{|\mu|} \Phi_\mu(x)\Phi_\mu(\xi) = \frac{1}{\pi^{\frac{n}{2}}(1 - w^2)^{\frac{n}{2}}}\, e^{-\frac{1}{2}\,\frac{1+w^2}{1-w^2}(|x|^2+|\xi|^2)+\frac{2w}{1-w^2}x\cdot\xi} \quad (|w| < 1)\,,$$

where the series is uniformly and absolutely convergent on $\{w \in \mathbf{C} : |w| < 1\}$. Then for $t > 0$, it is not difficult to see that

$$\sum_\mu e^{-(2|\mu|+n)t} \Phi_\mu(x)\Phi_\mu(\xi) = \frac{e^{-nt}}{\pi^{\frac{n}{2}}(1 - e^{-4t})^{\frac{n}{2}}} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}|x-\xi|^2 - \frac{1-e^{-2t}}{1+e^{-2t}}x\cdot\xi} . \tag{1.1}$$

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We denote by $E(x,\xi,t)$ the Mehler kernel defined by

$$E(x,\xi,t)=\begin{cases}\sum_\mu e^{-(2|\mu|+n)t}\Phi_\mu(x)\Phi_\mu(\xi)\,, & x,\xi\in\mathbf{R}^n\,, \quad t>0\\ 0\,, & x,\xi\in\mathbf{R}^n\,, \quad t\le 0\,.\end{cases} \tag{1.2}$$

For each $\xi\in\mathbf{R}^n$ and each $t>0$, $E(x,\xi,t)$ converges in $\mathcal{S}(\mathbf{R}^n)$ (see Section 2). Moreover for each $\xi\in\mathbf{R}^n$, it satifies the Hermite heat equation $(\partial_t-\triangle+|x|^2)U(x,t)=0$ for $x\in\mathbf{R}^n$ and $0<t<\infty$. Thus for any u in $\mathcal{S}'(\mathbf{R}^n)$, the pair $\langle u(\cdot),E(x,\cdot,t)\rangle$ is well defined. We then define the function $U(x,t):=\langle u(\xi),E(x,\xi,t)\rangle$ in $\mathbf{R}^n\times(0,T)$ and call it *the defining function* of u.

As a parallel result with the one in [3], the main purpose of this paper is to establish the following characterization:

"For fixed $T>0$, the defining function $U(x,t)=\langle u(\xi),E(x,\xi,t)\rangle$ of any u in $\mathcal{S}'(\mathbf{R}^n)$ is the smooth solution of $(\partial_t-\triangle+|x|^2)U(x,t)=0$ in $\mathbf{R}^n\times(0,T)$ such that

$$\sup_{x\in\mathbf{R}^n}|U(x,t)|\le C(1+t^{-N})\quad\text{for some constants } C,N>0\,.$$

Conversely every smooth function $U(x,t)$ in $\mathbf{R}^n\times(0,T)$ with the above growth and satisfying the Hermite heat equation can be represented as $U(x,t)=\langle u(\xi),E(x,\xi,t)\rangle$ for unique $u\in\mathcal{S}'(\mathbf{R}^n)$."

Furthermore we represent the tempered distributions as initial values of solution of the Hermite heat equation and apply it to provide a generalization in the space $\mathcal{S}'(\mathbf{R}^n)$ of the following theorem by Strichartz:

THEOREM 1.1 (Theorem 3.2, [5]). *If f is a function on $\mathbf{R}^n$ satisfying*

$$\|(-\triangle+|x|^2)^j f\|_\infty\le Mn^j$$

for some constant M and all $j\in\mathbf{N}_0$, then $f(x)=C\,e^{-\frac{|x|^2}{2}}$.

Throughout the paper, we denote by $\mathbf{N}$ the set of positive integers and $\mathbf{N}_0$ the set of non-negative integers. For any $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{N}_0^n$ and any $x=(x_1,\dots,x_n)\in\mathbf{R}^n$, we adopt the standard notations $|\alpha|=\alpha_1+\cdots+\alpha_n$, $x^\alpha={x_1}^{\alpha_1}\cdots{x_n}^{\alpha_n}$ and $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ where $\partial_i=\partial/\partial x_i$ for $i=1,\dots,n$.

2. Characterization of the spaces $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$

We denote by $\mathcal{S}(\mathbf{R}^n)$ the Schwartz space of all $\mathcal{C}^\infty$ functions ϕ on $\mathbf{R}^n$ such that for all $\alpha,\beta\in\mathbf{N}_0^n$

$$\sup_{x\in\mathbf{R}^n}|x^\alpha\partial^\beta\phi(x)|<\infty\,.$$

The topology on $\mathcal{S}(\mathbf{R}^n)$ is generated by the set of seminorms $\|\phi\|_{\alpha,\beta}=\sup_{x\in\mathbf{R}^n}|x^\alpha\partial^\beta\phi(x)|$. A sequence $\{\phi_j\}_{j\in\mathbf{N}}$ is said to converge to zero in $\mathcal{S}(\mathbf{R}^n)$ if $\|\phi_j\|_{\alpha,\beta}\to 0$ as $j\to\infty$ for all

$\alpha, \beta \in \mathbf{N}_0^n$. We denote by $\mathcal{S}'(\mathbf{R}^n)$ the dual space of $\mathcal{S}(\mathbf{R}^n)$ and call it the space of tempered distributions. As remarked in (p. 142, [4]), we devote this section to give the proofs of the characterization of the spaces $\mathcal{S}$ and $\mathcal{S}'$ for n-dimensional case. First we give a lemma.

LEMMA 2.1. *Let Φ_μ be the normalized Hermite function on $\mathbf{R}^n$. Then for any $\alpha, \beta \in \mathbf{N}_0^n$, there exists a positive constant C such that*

$$\| \Phi_\mu \|_{\alpha,\beta} \leq C^n (2\sqrt{e})^{|\alpha|+|\beta|} (|\alpha| + |\beta|)^{\frac{|\alpha|+|\beta|}{2}} (1 + |\mu|)^{\frac{|\alpha|+|\beta|}{2}} .$$

PROOF. First we simply take $n = 1$ and suppose that $k, \alpha, \beta \in \mathbf{N}_0$, $x \in \mathbf{R}$ and $D = d/dx$. It is well known that the normalized Hermite function h_k on $\mathbf{R}$ satisfies

$$\begin{cases} (x + D)h_k = 0\,, & k = 0 \\ (x + D)h_k = \sqrt{2k}\, h_{k-1}\,, & k \geq 1 \\ (x - D)h_k = \sqrt{2(k+1)}\, h_{k+1}\,, & k \geq 0\,. \end{cases} \tag{2.1}$$

Moreover in view of (p. 171, [7]), it is easy to see that there exists a constant $G > 0$ such that

$$|h_k(x)| \leq G \tag{2.2}$$

for all x and all k. Consider the nontrivial case $\alpha + \beta \neq 0$. Then

$$\begin{aligned} x^\alpha D^\beta h_k(x) &= 2^{-\alpha-\beta} \ \{(x + D) + (x - D)\}^\alpha \ \{(x + D) - (x - D)\}^\beta \ h_k(x) \\ &= 2^{-\alpha-\beta} \sum_{\varepsilon \in T} (x + \varepsilon_1 D) \cdots (x + \varepsilon_{\alpha+\beta} D) \ h_k(x) \end{aligned} \tag{2.3}$$

where $T = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\alpha+\beta})\colon \varepsilon_i = +1 \text{ or } -1 \text{ for } i = 1, \dots, \alpha + \beta\}$ and $|T| = 2^{\alpha+\beta}$. It now follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned} |x^\alpha D^\beta h_k(x)| &\leq (\sqrt{2})^{\alpha+\beta} \ \{(k+1) \cdots (k + \alpha + \beta)\}^{1/2} \max_{|j| \leq \alpha+\beta} \{|h_{k+j}(x)|\} \\ &\leq G \ (\sqrt{2})^{\alpha+\beta} \left\{ \frac{(k + \alpha + \beta)!}{k!} \right\}^{1/2} . \end{aligned}$$

With the aid of Stirling's formula, we can find a constant C such that

$$\begin{aligned} |x^\alpha D^\beta h_k(x)| &\leq C(\sqrt{2})^{\alpha+\beta} \left\{ \frac{(k + \alpha + \beta)^{k+\alpha+\beta} \ e^k \ \sqrt{k + \alpha + \beta}}{e^{k+\alpha+\beta} \ k^k \sqrt{k}} \right\}^{1/2} \\ &\leq C(\sqrt{2})^{\alpha+\beta} \left\{ (k + \alpha + \beta)^{\alpha+\beta} \left(1 + \frac{\alpha + \beta}{k} \right)^k \right\}^{1/2} \\ &\leq C(2\sqrt{e})^{\alpha+\beta} \left(k^{\frac{\alpha+\beta}{2}} + (\alpha + \beta)^{\frac{\alpha+\beta}{2}} \right) \\ &\leq C(2\sqrt{e})^{\alpha+\beta} (\alpha + \beta)^{\frac{\alpha+\beta}{2}} (1 + k)^{\frac{\alpha+\beta}{2}} . \end{aligned}$$

Thus for $\mu, \alpha, \beta \in \mathbf{N}_0^n$, we have

$$\| \Phi_\mu \|_{\alpha,\beta} = \sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta \Phi_\mu(x)| \leq C^n (2\sqrt{e})^{|\alpha|+|\beta|} (|\alpha| + |\beta|)^{\frac{|\alpha|+|\beta|}{2}} (1 + |\mu|)^{\frac{|\alpha|+|\beta|}{2}} .$$

THEOREM 2.1. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then $\phi = \sum_\mu \langle \phi, \Phi_\mu \rangle \Phi_\mu$ and for every nonnegative integer M there exists a positive constant $C := C(M)$ such that*

$$|\langle \phi, \Phi_\mu \rangle| \leq C(1 + |\mu|)^{-M} . \tag{2.4}$$

Conversely the series $\sum_\mu a_\mu \Phi_\mu$ converges in $\mathcal{S}(\mathbf{R}^n)$ if the coefficients a_μ satisfy the growth condition (2.4)*.*

PROOF. Since $\phi \in \mathcal{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$, clearly $\phi = \sum_\mu \langle \phi, \Phi_\mu \rangle \Phi_\mu$. For every nonnegative integer M, the operator $(-\triangle + |x|^2)^{2M}$ is self-adjoint. So we have

$$\sum_\mu |\langle \phi, \Phi_\mu \rangle|^2 (2|\mu| + n)^{2M} = \langle \phi, (-\triangle + |x|^2)^{2M} \phi \rangle < \infty .$$

From this the assertion follows. To prove the converse, let $\phi(x) := \sum_\mu a_\mu \Phi_\mu(x)$. For $N \in \mathbf{N}_0$, consider the partial sums $\phi_N(x) = \sum_{|\mu|\leq N} a_\mu \Phi_\mu(x)$. Then for every $\alpha, \beta \in \mathbf{N}_0^n$, we have

$$\|\phi_N - \phi_{N-1}\|_{\alpha,\beta} \leq \sum_{|\mu|=N} |a_\mu| \, \|\Phi_\mu\|_{\alpha,\beta} .$$

Using $\sum_{|\mu|=N} 1 = \binom{N+n-1}{N} \leq (1 + N)^n$, Lemma 2.1 and choosing $M = |\alpha| + |\beta| + n + 2$ in the estimate of a_μ, we have $\|\phi_N - \phi_{N-1}\|_{\alpha,\beta} \leq C' \,(1 + N)^{-2}$ for some positive constant C'. Then for all $\varepsilon > 0$ and $N_2 \geq N_1 \geq P$, we have

$$\|\phi_{N_2} - \phi_{N_1}\|_{\alpha,\beta} \leq \sum_{N=N_1+1}^{N_2} \|\phi_N - \phi_{N-1}\|_{\alpha,\beta} \leq C' \sum_{N=P}^{\infty} (1 + N)^{-2} < \varepsilon$$

for sufficiently large P. It follows that $\{\phi_N\}$ is a Cauchy sequence in $\mathcal{S}(\mathbf{R}^n)$. Since the space $\mathcal{S}(\mathbf{R}^n)$ is complete, the assertion follows. $\square$

REMARK 2.1. For fixed x and t, the Mehler kernel $E(x, \xi, t)$ converges in $\mathcal{S}(\mathbf{R}^n)$ since the Hermite coefficient $e^{-(2|\mu|+n)t} \Phi_\mu(x)$ in (1.2) satisfies the estimate as in Theorem 2.1.

THEOREM 2.2. *Let $u \in \mathcal{S}'(\mathbf{R}^n)$. Then there exist positive constants C and M such that*

$$|\langle u, \Phi_\mu \rangle| \leq C(1 + |\mu|)^M . \tag{2.5}$$

Conversely the series $\sum_\mu b_\mu \Phi_\mu$ converges in $\mathcal{S}'(\mathbf{R}^n)$ if the coefficients b_μ satisfy the growth condition (2.5)*. Moreover if $u \in \mathcal{S}'(\mathbf{R}^n)$, then $u = \sum_\mu \langle u, \Phi_\mu \rangle \Phi_\mu$ in the sense that $\langle u, \phi \rangle = \sum_\mu \langle u, \Phi_\mu \rangle \, \langle \phi, \Phi_\mu \rangle$ for every $\phi \in \mathcal{S}(\mathbf{R}^n)$.*

PROOF. Since $u \in \mathcal{S}'(\mathbf{R}^n)$, there exist a constant $C_1 > 0$ and $\alpha, \beta \in \mathbf{N}_0^n$ such that

$$|\langle u, \Phi_\mu \rangle| \le C_1 \, \|\Phi_\mu\|_{\alpha,\beta} \, .$$

By Lemma 2.1, we see that $|\langle u, \Phi_\mu \rangle| \le C_2(1+|\mu|)^M$ where $M := (|\alpha|+|\beta|)/2$, $C_2 := C_1 \, C^n \, (2\sqrt{e})^{2M} \, (2M)^M$ are positive constants.

For the converse, let $u := \sum_\mu b_\mu \Phi_\mu$ and define $\langle u, \phi \rangle = \sum_\mu b_\mu \, \langle \phi, \Phi_\mu \rangle$ for every $\phi \in \mathcal{S}(\mathbf{R}^n)$. It is well-defined because of the estimates of b_μ and $\langle \phi, \Phi_\mu \rangle$. For $N \in \mathbf{N}_0$, consider the partial sums $u_N := \sum_{|\mu| \le N} b_\mu \Phi_\mu$. We show that $u_N \to u$ in $\mathcal{S}'(\mathbf{R}^n)$ as $N \to \infty$. So let $\phi \in \mathcal{S}(\mathbf{R}^n)$ and $a_\mu := \langle \phi, \Phi_\mu \rangle$. Then from the hypothesis and the estimate of a_μ in Theorem 2.1, there exists a positive constant C' and an integer $M_1 > M$ such that

$$|\langle u_N - u, \phi \rangle| \le \sum_{|\mu|>N} |b_\mu||a_\mu| \le C' \sum_{|\mu|>N} (1+|\mu|)^{M_1 - M_1 - 2} \le C' \sum_{|\mu|>N} (1+|\mu|)^{-2}$$

which tends to zero as $N \to \infty$. If the series $\sum_\mu b_\mu \Phi_\mu$ converges to, say v, in $\mathcal{S}'(\mathbf{R}^n)$, then u and v both have the same Hermite coefficients and hence are the same. Last part is obvious from the first part. □

3. Mehler Kernel Approach

In view of (1.1), it is easy to see that $E(x, \xi, t) = \tilde{\eta}(x, t) \; \tilde{E}(x, \xi, t)$ where

$$\tilde{\eta}(x,t) = \frac{2^{\frac{n}{2}} \, e^{-nt}}{(1+e^{-4t})^{\frac{n}{2}}} \, e^{-\frac{1}{2} \, \frac{1-e^{-4t}}{1+e^{-4t}} |x|^2} \, , \tag{3.1}$$

$$\tilde{E}(x,\xi,t) = (2\pi)^{-\frac{n}{2}} \left(\frac{1+e^{-4t}}{1-e^{-4t}} \right)^{\frac{n}{2}} e^{-\frac{1}{2} \, \frac{1+e^{-4t}}{1-e^{-4t}} \left| \xi - \frac{2e^{-2t}}{1+e^{-4t}} x \right|^2} \tag{3.2}$$

for $x, \xi \in \mathbf{R}^n$ and $t > 0$. With this decomposition, we give some lemmas.

LEMMA 3.1. *For any* $\delta > 0$

$$\int_{\mathbf{R}^n} \tilde{E}(x,\xi,t) d\xi = 1 \, , \tag{3.3}$$

$$\int_{\left|\xi - \frac{2e^{-2t}}{1+e^{-4t}} x\right| \ge \delta} \tilde{E}(x,\xi,t) d\xi \to 0 \text{ uniformly for } x \in \mathbf{R}^n \text{ as } t \to 0^+ \, . \tag{3.4}$$

PROOF. It is immediate to derive (3.3) from (3.2). We now prove (3.4). Under the change of variable $\sqrt{\frac{1+e^{-4t}}{2(1-e^{-4t})}} \left(\xi - \frac{2e^{-2t}}{1+e^{-4t}} x \right) = s$, we have

$$\int_{\left|\xi - \frac{2e^{-2t}}{1+e^{-4t}} x\right| \ge \delta} \tilde{E}(x,\xi,t) d\xi = \pi^{-n/2} \int_{|s| \ge \delta \sqrt{\frac{1+e^{-4t}}{2(1-e^{-4t})}}} e^{-|s|^2} ds \, .$$

Thus the above integral converges to 0 uniformly for $x \in \mathbf{R}^n$ as $t \to 0^+$. □

For a continuous and bounded function h on $\mathbf{R}^n$, consider the following Cauchy problem

(3.5)
$$\begin{cases} \frac{\partial U}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 U}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i(x,t)\frac{\partial U}{\partial x_i} - c(x,t)U = 0\,, & (x,t) \in \mathbf{R}^n \times (0,T) \\ U(x,0) = h(x)\,, & x \in \mathbf{R}^n \end{cases}$$

where $|a_{ij}(x,t)| \le M(|x|^2+1)\,,\ |b_i(x,t)| \le M\sqrt{|x|^2+1}\,,\ c(x,t) \le M$ for some constant $M > 0$.

Ilin–Kalašnikov–Olejnik (p. 14, [2]) have shown that the solution of (3.5) is unique in the class of bounded functions in $\mathbf{R}^n \times [0,T]$. With (a_{ij}) = the $n \times n$ identity matrix, $b_i = 0$, $c = -|x|^2$ and $M = 1$, the following theorem is a particular case of [2].

THEOREM 3.1. *For h as in* (3.5)*, the solution of the Cauchy problem*

$$\begin{cases} (\partial_t - \triangle + |x|^2)U(x,t) = 0\,, & (x,t) \in \mathbf{R}^n \times (0,T) \\ U(x,0) = h(x)\,, & x \in \mathbf{R}^n\,. \end{cases} \tag{3.6}$$

is unique in the class of bounded functions in $\mathbf{R}^n \times [0,T]$.

LEMMA 3.2. *Let $E(x,\xi,t)$ be the Mehler kernel and h a continuous and bounded function on $\mathbf{R}^n$. Let $U(x,t) := \int_{\mathbf{R}^n} E(x,\xi,t)h(\xi)d\xi$. Then it is a well-defined $\mathcal{C}^\infty$ function in $\mathbf{R}^n \times (0,T]$ and satisfies that*

(i) $(\partial_t - \triangle + |x|^2)U(x,t) = 0$ *in* $\mathbf{R}^n \times (0,T)$,

(ii) $U(x,t) \to h(x)$ *uniformly on each compact subset of* $\mathbf{R}^n$ *as* $t \to 0^+$.

(iii) $U(x,t)$ *is bounded in* $\mathbf{R}^n \times [0,T]$.

PROOF. The proof of (i) is obvious. To prove (ii), let $\delta > 0$ be arbitrary. Then

$$\begin{aligned} &|U(x,t) - h(x)| \\ &\le \tilde{\eta}(x,t)\int_{\mathbf{R}^n} |h(\xi)-h(x)|\tilde{E}(x,\xi,t)d\xi + |\tilde{\eta}(x,t)-1|\,|h(x)| \\ &\le \tilde{\eta}(x,t) \sup_{\left|\xi - \frac{2e^{-2t}}{1+e^{-4t}}x\right| < \delta} |h(\xi)-h(x)| \int_{\left|\xi - \frac{2e^{-2t}}{1+e^{-4t}}x\right| < \delta} \tilde{E}(x,\xi,t)d\xi \\ &\quad + \tilde{\eta}(x,t)\, 2\|h\|_\infty \int_{\left|\xi - \frac{2e^{-2t}}{1+e^{-4t}}x\right| \ge \delta} \tilde{E}(x,\xi,t)d\xi + |\tilde{\eta}(x,t)-1|\,|h(x)| \\ &= I_1 + I_2 + I_3\,. \end{aligned}$$

Let K be a compact subset of $\mathbf{R}^n$. Since $h(x)$ is uniformly continuous on a δ-neighborhood K_δ of K, it follows that for any $\varepsilon > 0$, $|\xi - x| < \delta$ implies $|h(\xi) - h(x)| < \varepsilon$ for $\xi, x \in K_\delta$. Let $|h(x)| \le C(K)$ for every $x \in K$. We note that $\tilde{\eta}(x,t) \to 1$ in view of (3.1) as $t \to 0^+$. Then clearly I_3 tends to zero as $t \to 0^+$. Furthermore for every $x \in K$, I_1 tends to zero as

$t \to 0^+$ since

$$\left|\xi - \frac{2e^{-2t}}{1+e^{-4t}}x\right| < \delta \Rightarrow |\xi - x| < \delta \text{ as } t \to 0^+$$

and hence applying the uniform continuity of h on K_δ. In view of Lemma 3.1, I_2 tends to zero as $t \to 0^+$. This proves (ii).

Now we prove (iii). Since h is bounded, so is $U(\cdot\,, 0)$. By Lemma 3.1 and boundedness of $\tilde{\eta}(x,t)$, there exists a constant $C > 0$ such that

$$|U(x,t)| \le \|h\|_\infty\, \tilde{\eta}(x,t) \int_{\mathbf{R}^n} \tilde{E}(x,\xi,t)d\xi \le C$$

for all $(x,t) \in \mathbf{R}^n \times (0,T]$. Thus $U(x,t)$ is bounded in $\mathbf{R}^n \times [0,T]$ which proves the assertion. □

4. Main Results

THEOREM 4.1. *For fixed $T > 0$, the defining function $U(x,t) = \langle u(\xi), E(x,\xi,t)\rangle$ of any u in $\mathcal{S}'(\mathbf{R}^n)$ is the smooth solution of the Hermite heat equation $(\partial_t - \triangle + |x|^2)U(x,t) = 0$ in $\mathbf{R}^n \times (0,T)$ such that for some positive constants C and N*

$$\sup_{x\in\mathbf{R}^n} |U(x,t)| \le C(1+t^{-N})\,. \tag{4.1}$$

Conversely every smooth function $U(x,t)$ in $\mathbf{R}^n \times (0,T)$ with the growth of type (4.1) *and satisfying the Hermite heat equation can be represented as $U(x,t) = \langle u(\xi), E(x,\xi,t)\rangle$ for unique $u \in \mathcal{S}'(\mathbf{R}^n)$ and moreover*

$$U(x,t) = \sum_\mu c_\mu\, e^{-(2|\mu|+n)t}\, \Phi_\mu(x)\,, \quad U(\cdot\,, 0^+) = u \tag{4.2}$$

where $|c_\mu| \le C(1+|\mu|)^M$ for some positive constants C and $M := M(N)$.

PROOF. We easily see that the defining function

$$U(x,t) = \langle u(\xi), E(x,\xi,t)\rangle = \sum_\mu e^{-(2|\mu|+n)t}\, \langle u(\xi), \Phi_\mu(\xi)\rangle\, \Phi_\mu(x)$$

satisfies the Hermite heat equation. As such it is smooth in $\mathbf{R}^n \times (0,T)$ by the hypoelliptic property of the operator $\partial_t - \triangle + |x|^2$ (see p. 168, [1]). By Theorem 2.2 and (2.2), there exist a positive integer M and a constant $C_1 > 0$ such that

$$\begin{aligned}|U(x,t)| &\le G\, C_1 \sum_{k=0}^\infty \sum_{|\mu|=k} e^{-2t|\mu|}(1+|\mu|)^M \\ &= G\, C_1\left(1 + \sum_{k=1}^\infty \binom{k+n-1}{k}\frac{(1+k)^M}{e^{2tk}}\right)\end{aligned}$$

$$\leq G\,C_1\Big(1+\sum_{k=1}^{\infty}\frac{2^{n+M}\;k^{n+M}\;(n+M+2)!}{(2tk)^{n+M+2}}\Big)$$
$$= G\,C_1\Big(1+\frac{\pi^2\;(n+M+2)!}{24}\;\frac{1}{t^{n+M+2}}\Big)$$
$$\leq C\;(1+t^{-N})$$

where $N := n+M+2$ and $C := \frac{G\,C_1\,\pi^2\,N!}{24}$ are positive constants.
Conversely for a positive integer m, let

$$f(t)=\begin{cases}0\,, & t\leq 0\,,\\ t^{m-1}/(m-1)!\,, & t>0\end{cases}$$

Multiplying f by a suitable $\mathcal{C}_0^\infty$ function, we obtain functions $v(t)$ and $w(t)$ with

$$v(t)=\begin{cases}f(t)\,, & t\leq T/4\,,\\ 0\,, & t\geq T/2\end{cases}$$

and the support of $w\subset[T/4,T/2]$ such that

$$\Big(\frac{\partial}{\partial t}\Big)^m v(t)=\delta(t)+w(t) \tag{4.3}$$

where δ is the Dirac measure. Now take the integer $m=\lceil N\rceil+2$ where N is the constant in the condition (4.1) and $\lceil N\rceil$ is the least integer greater than N. Consider the following functions in $\mathbf{R}^n\times(0,T/2)$

$$L(x,t)=\int_0^T U(x,t+s)v(s)ds\,,\quad H(x,t)=\int_0^T U(x,t+s)w(s)ds\,.$$

In view of (4.3) it is easy to see that

$$U(x,t)=\Big(-\frac{\partial}{\partial t}\Big)^m L(x,t)-H(x,t)\,. \tag{4.4}$$

By hypothesis, L and H are bounded solutions of the Hermite heat equation in $\mathbf{R}^n\times(0,T/2)$ and can be continuously extended to $\mathbf{R}^n\times[0,T/2]$ for $m=\lceil N\rceil+2$. Define $L(x,0)=:l(x)$ and $H(x,0)=:h(x)$. Then clearly l and h are continuous and bounded functions on $\mathbf{R}^n$. Hence L and H are bounded in $\mathbf{R}^n\times[0,T/2]$. By Theorem 3.1 and Lemma 3.2, we have

$$L(x,t)=\int_{\mathbf{R}^n} l(\xi)E(x,\xi,t)d\xi\,,\quad H(x,t)=\int_{\mathbf{R}^n} h(\xi)E(x,\xi,t)d\xi$$

in $\mathbf{R}^n\times[0,T/2]$ and hence (4.4) reduces to

$$U(x,t)=\Big(-\frac{\partial}{\partial t}\Big)^m\int_{\mathbf{R}^n} l(\xi)E(x,\xi,t)d\xi-\int_{\mathbf{R}^n} h(\xi)E(x,\xi,t)d\xi$$

$$= \left(-\frac{\partial}{\partial t}\right)^m \sum_\mu e^{-(2|\mu|+n)t} \langle l, \Phi_\mu\rangle \, \Phi_\mu(x) - \sum_\mu e^{-(2|\mu|+n)t} \langle h, \Phi_\mu\rangle \, \Phi_\mu(x)$$

$$= \sum_\mu e^{-(2|\mu|+n)t} \left\{(2|\mu|+n)^m \left\langle l, \Phi_\mu\right\rangle - \left\langle h, \Phi_\mu\right\rangle\right\} \Phi_\mu(x) . \tag{4.5}$$

Put $c_\mu := (2|\mu|+n)^m \langle l, \Phi_\mu\rangle - \langle h, \Phi_\mu\rangle$. By Theorem 2.2, we can find some positive constants M' and C' such that

$$|c_\mu| \le 2C'(1+|\mu|)^{M'} (2|\mu|+n)^m \le 2C' n^m (1+|\mu|)^{M'+m} = C(1+|\mu|)^M \tag{4.6}$$

where $m = \lceil N \rceil + 2$, $C := 2C' n^m > 0$ and $M := M' + m > 0$. Define $u := \sum_\mu c_\mu \Phi_\mu$. Then u belongs to $\mathcal{S}'(\mathbf{R}^n)$ by Theorem 2.2 and $\langle u, \Phi_\mu\rangle = c_\mu$. Hence (4.5) takes the form

$$U(x,t) = \sum_\mu e^{-(2|\mu|+n)t} \langle u, \Phi_\mu\rangle \Phi_\mu(x) = \langle u(\xi), E(x,\xi,t)\rangle \quad \text{in } \mathbf{R}^n \times (0, T/2) .$$

Uniqueness of u follows from the uniqueness of the coefficient of the Hermite series. Moreover (4.2) is obvious in view of (4.5) and (4.6). Furthermore

$$\lim_{t\to o^+} \langle U(\cdot, t), \phi\rangle = \lim_{t\to o^+} \sum_\mu e^{-(2|\mu|+n)t} \langle u, \Phi_\mu\rangle \langle \phi, \Phi_\mu\rangle = \langle u, \phi\rangle, \quad \phi \in \mathcal{S}(\mathbf{R}^n) \tag{4.7}$$

from the uniform convergence of the series $\sum_\mu e^{-(2|\mu|+n)t} \langle u, \Phi_\mu\rangle \langle \phi, \Phi_\mu\rangle$ in $(0, T/2)$. □

THEOREM 4.2. *Let $u \in \mathcal{S}'(\mathbf{R}^n)$. Suppose that there exist a constant $L > 0$ and $\alpha, \beta \in \mathbf{N}_0^n$ such that*

$$|\langle(-\triangle + |x|^2)^j u(x), \phi(x)\rangle| \le L\, n^j \, \|\phi\|_{\alpha,\beta} \tag{4.8}$$

for all $j \in \mathbf{N}_0$ and all $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then $u(x) = C\, e^{-\frac{|x|^2}{2}}$ for some constant C.

PROOF. For each $t > 0$, the defining function $U(x,t) = \langle u(\xi), E(x,\xi,t)\rangle$ is a $\mathcal{C}^\infty$ function in $\mathbf{R}^n$. It follows from (4.8) and (2.2) that

$$|(-\triangle + |x|^2)^j U(x,t)| \le \sum_\mu e^{-2t|\mu|} |\langle(-\triangle + |\xi|^2)^j u(\xi), \Phi_\mu(\xi)\rangle| \, |\Phi_\mu(x)|$$

$$\le GL\, n^j \sum_\mu e^{-2t|\mu|} \, \|\Phi_\mu\|_{\alpha,\beta} \tag{4.9}$$

By Lemma 2.1, (4.9) yields that

$$|(-\triangle + |x|^2)^j U(x,t)| \le GLC^n (2\sqrt{e})^{|\alpha|+|\beta|} (|\alpha|+|\beta|)^{\frac{|\alpha|+|\beta|}{2}} \, n^j$$

$$\times \sum_\mu e^{-2t|\mu|} \, (1+|\mu|)^{|\alpha|+|\beta|} . \tag{4.10}$$

But

$$\sum_{\mu} e^{-2t|\mu|}\,(1+|\mu|)^{|\alpha|+|\beta|} = \sum_{k=0}^{\infty}\sum_{|\mu|=k} e^{-2t|\mu|}\,(1+|\mu|)^{|\alpha|+|\beta|}$$
$$= 1+\sum_{k=1}^{\infty}\binom{k+n-1}{k}\frac{(1+k)^{|\alpha|+|\beta|}}{e^{2tk}}$$
$$\le 1+\sum_{k=1}^{\infty}\frac{(1+k)^{|\alpha|+|\beta|+n}}{e^{2tk}}$$
$$\le 1+\sum_{k=1}^{\infty}\frac{2^{|\alpha|+|\beta|+n}\;k^{|\alpha|+|\beta|+n}}{e^{2tk}}$$
$$\le 1+\frac{C_1}{t^{|\alpha|+|\beta|+n+2}} \tag{4.11}$$

where $C_1 = \frac{\pi^2\ (|\alpha|+|\beta|+n+2)!}{24}$ is a positive constant. So from (4.10) and (4.11), we have

$$|(-\triangle+|x|^2)^j U(x,t)| \le GLC^n(2\sqrt{e})^{|\alpha|+|\beta|}(|\alpha|+|\beta|)^{\frac{|\alpha|+|\beta|}{2}}$$
$$\times\left(1+\frac{C_1}{t^{|\alpha|+|\beta|+n+2}}\right)n^j \tag{4.12}$$

Since for each $t > 0$, $GLC^n(2\sqrt{e})^{|\alpha|+|\beta|}(|\alpha|+|\beta|)^{\frac{|\alpha|+|\beta|}{2}}\left(1+\frac{C_1}{t^{|\alpha|+|\beta|+n+2}}\right)$ in (4.12) is a positive constant and independent of j, it follows from Theorem 1.1 that

$$U(x,t) = C_t\ e^{-\frac{|x|^2}{2}} \tag{4.13}$$

for some constant C_t depending on t. Since the defining function $U(x,t)$ satisfies the Hermite heat equation, we have

$$(\partial_t - \triangle + |x|^2)C_t\ e^{-\frac{|x|^2}{2}} = 0\,. \tag{4.14}$$

Using $(-\triangle+|x|^2)e^{-\frac{|x|^2}{2}} = ne^{-\frac{|x|^2}{2}}$ in (4.14), we have $C_t' + nC_t = 0$ so that

$$C_t = C\ e^{-nt} \tag{4.15}$$

for some constant C. Then for every $\phi \in \mathcal{S}(\mathbf{R}^n)$, it follows from (4.7), (4.13) and (4.15) that

$$\langle u(x),\phi(x)\rangle = \lim_{t\to 0^+}\langle U(x,t),\phi(x)\rangle = \langle C\ e^{-\frac{|x|^2}{2}},\phi(x)\rangle\,.$$

This completes the proof. □

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Present Address:
DEPARTMENT OF MATHEMATICS, MAHENDRA RATNA CAMPUS,
TRIBHUVAN UNIVERSITY, KATHMANDU, NEPAL.
DEPARTMENT OF MATHEMATICS, MEIJO UNIVERSITY, NAGOYA, 468–8502 JAPAN.
e-mail: bishnupd2001@yahoo.com