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A Generalization of the Hankel Transform and the Lorentz Multipliers

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Abstract. Let ϕ be a bounded function on $[0,\infty)$ continuous except on a null set, and $\phi_\varepsilon(\xi)=\phi(\varepsilon\xi)$ $(\varepsilon>0)$. Also let $\tilde{T}_\varepsilon$ be the operator on Jacobi series such that $(\tilde{T}_\varepsilon f)^\wedge(n)=\phi_\varepsilon(n)\hat{f}(n)$ $(n\in\mathbf{Z})$, where $\hat{f}(n)$ is the coefficient of Jacobi expanstion of f, and $\mathcal{H}_\alpha(Tf)(\xi)$ be defined by $\phi(\xi)\mathcal{H}_\alpha f(\xi)$ $(\xi\in(0,\infty))$, where $\mathcal{H}_\alpha f$ is the modified Hankel transform of f with order α. Then the author [7] proved that if the operator norm of $\tilde{T}_\varepsilon$ is uniformly bounded for all $\varepsilon>0$, T is a bounded operator on the modified Hankel transforms in the Lorentz spaces, and we have the maximal type theorem in the Lorentz spaces, respectively. In this paper, we give a generalized definition of the modified Hankel transform and the Hankel transform, and prove a generalization of the results in [7].

1. Introduction

First we give some definitions and notations. Let (X,ν) be a measure space, and for any $1\le p\le\infty,\ 1\le q\le\infty,\ L^{p,q}(X)$ define the Lorentz space such that

$$L^{p,q}(X)=\{f:\ f \text{ is measurable},\ \|f\|^*_{p,q}<\infty\}\,,$$

where

$$\|f\|^*_{p,q}=\begin{cases}\{q\int_0^\infty(t\nu(|f|>t\})^{1/p})^q\frac{dt}{t}\}^{1/q} & \text{if } 1\le q<\infty\\ \sup_{t>0}\nu(\{|f|>t\})^{1/p} & \text{if } q=\infty\,.\end{cases}$$

In particular, $L^{p,q}(X)=L^p(X)$ for $p=q$. We call $\|f\|^*_{p,q}$ the Lorentz norm of f in $L^{p,q}(X)$(cf.[4]).

Now let $\alpha,\beta>-1$ and $a,\ b$ be any real number such that $-\frac{1}{2}<a\le\alpha+\frac{1}{2}$ and $-\frac{1}{2}<b\le\beta+\frac{1}{2}$.

Also we denote $d\mu=d\mu_a(y)=y^{2a}dy,\ dm=dm_{a,b}(\theta)=(\sin\frac{\theta}{2})^{2a}(\cos\frac{\theta}{2})^{2b}d\theta$, and $Q_n(\theta)=Q_n^{(\alpha,\beta,a,b)}(\theta)=t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)(\sin\frac{\theta}{2})^{\alpha+1/2-a}(\cos\frac{\theta}{2})^{\beta+1/2-b}$, where

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$P_n^{(\alpha,\beta)}(x)$(the Jacobi polynomial of degree n and order (α,β)) and $t_n^{(\alpha,\beta)}$ are as follows:

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\},$$

$$[t_n^{(\alpha,\beta)}]^{-2}=\int_0^\pi |P_n^{(\alpha,\beta)}(\cos\theta)|^2\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}d\theta$$
$$=\left[\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right]^{-1}.$$

(put $[t_0^{(\alpha,\beta)}]^{-2}=\int_0^\pi(\sin(\theta/2))^{2\alpha+1}(\cos(\theta/2))^{2\beta+1}d\theta$ for $n=0$)
Here, we give a definition of the generalized Hankel transform.

DEFINITION 1. We define $\mathcal{H}=\mathcal{H}_{\alpha,a}$ the generalized Hankel transform such that

$$\mathcal{H}f(x)=\mathcal{H}_{\alpha,a}f(x)=\int_0^\infty f(y)(xy)^{1/2-a}J_\alpha(xy)d\mu_a(y).$$

For $a=\alpha+\frac{1}{2}$, it is called the modified Hankel transform of f, and for $a=0$, it is called the Hankel transform of f. Also we define

$$\hat{h}(n)=\int_0^\pi h(\theta)Q_n(\theta)dm(\theta)\quad(n=0,1,2,\dots),$$

and the generalized Jacobi series of h:

$$h\sim\sum_{n=0}^\infty\hat{h}(n)Q_n(\theta).$$

Next, we give a property of it.

PROPOSITION 1. *$\{Q_n\}_{n=0}^\infty$ is a complete orthonormal system in $L^2((0,\pi),m)$.*
We easily obtain the proof by the property of $\{P_n^{\alpha,\beta}\}$(cf. [1]).

DEFINITION 2. For $\phi\in l^\infty(\{0,1,2,\cdots\})$, we put

$$\tilde{T}_\phi h\sim\sum_{n=0}^\infty\phi(n)\hat{h}(n)Q_n(\theta).$$

Also we denote by $\|h\|_{p,q}^J$ the Lorentz norm of h in $L^{p,q}((0,\pi),m)$,

$$\|\tilde{T}_\phi\|_{M(p,r;p,q)}^J=\sup\{\|\tilde{T}_\phi h\|_{p,q}^J:\|h\|_{p,r}^J\le 1,\ h\in C_c^\infty(0,\pi)\}$$

the operator norm, and $M^J(p,r;p,q)=\{\tilde{T}_\phi:\|\tilde{T}_\phi\|_{M(p,r;p,q)}^J<\infty\}$. For a bounded measurable function ϕ on $[0,\infty)$, we define

$$T_\phi f(x)=\int_0^\infty\phi(y)\mathcal{H}f(y)(xy)^{1/2-a}J_\alpha(xy)d\mu(y).$$

Also we denote by $\|f\|^H_{p,r}$ the Lorentz norm of f in $L^{p,r}((0,\infty),\mu)$,

$$\|T_\phi\|^H_{M(p,r;p,q)} = \sup\{\|T_\phi f\|^H_{p,q} : \|f\|^H_{p,r} \le 1,\ f \in C_c^\infty(0,\infty)\}$$

the operator norm, and $M^H(p,r;p,q) = \{T_\phi : \|T_\phi\|^H_{M(p,r;p,q)} < \infty\}$.

DEFINITION 3. For a bounded a.e. continuous function ϕ on $[0,\infty)$ and $\varepsilon > 0$, we define the following:

$$(\tilde{T}_\varepsilon h)(\theta) = \sum_{n=0}^{\infty} \phi(\varepsilon n)\hat{h}(n)Q_n(\theta)\ (h \in C_c^\infty(0,\pi))\,,$$

$$(\tilde{T}^* h)(\theta) = \sup_{\varepsilon>0} |\tilde{T}_\varepsilon h(\theta)|\,,$$

$$\|\tilde{T}^*\|^J_{M(p,r;p,q)} = \sup\{\|\tilde{T}^* h\|^J_{p,q} : \|h\|^J_{p,r} \le 1,\ h \in C_c^\infty(0,\pi)\}\,,$$

$$(T_\varepsilon f)(x) = \int_0^\infty \phi(\varepsilon y)\mathcal{H}h(y)(xy)^{1/2-a}J_\alpha(xy)d\mu(y)\,,$$

$$T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|\,.$$

$$(\tilde{T}^* h)(\theta) = \sup_{\varepsilon>0} |\tilde{T}_\varepsilon h(\theta)|\,,$$

$$\|T^*\|^H_{M(p,r;p,q)} = \sup\{\|T^* f\|^H_{p,q} : \|f\|^H_{p,r} \le 1,\ f \in C_c^\infty(0,\infty)\}\,.$$

Then, our results are as follows:

THEOREM 1. *Let* $1 \le p < \infty,\ 1 \le q \le \infty,\ 1 \le r < \infty$ *and* $\alpha, \beta > -1$. *Assume that* ϕ *is a bounded continuous function except on a null set and* $\{\|\tilde{T}_\varepsilon\|^J_{M(p,r;p,q)}\}_\varepsilon$ *are bounded. Then* T_ϕ *is in* $M^H(p,r;p,q)$, *where we set* $q = \infty$ *or* $q = r = 1$, *if* $p = 1$.

THEOREM 2. *Let* $1 < p < \infty,\ 1 < q \le \infty,\ 1 \le r < \infty$ *and* $\alpha, \beta > -1$. *Assume that* ϕ *is a bounded continuous function on* $[0,\infty)$ *and* $\|\tilde{T}^*\|^J_{M(p,r;p,q)}$ *is finite. Then*

$$\|T^*\|^H_{M(p,r;p,q)} < \infty\,.$$

In 1972, Igari[5] first showed a theorem of this type. In fact, he proved Theorem 1 for $a = \alpha + 1/2,\ b = \beta + 1/2$, and $1 \le p = q = r < \infty$. After that, Connett-Schwartz[3] proved the weak type version by Igari's method and Kanjin[6] proved the maximal type version. These are the results for the modified Hankel transforms. On the other hand, the generalized Hankel transform for $a = 0$ is well known as the Hankel transform. In the case of the Hankel transform, we have the analogy of Theorem 1 for $1 < p = q = r < \infty$ by the transplantation theorem (cf. [8]). But, it seems that the weak version isn't known in the case of the Hankel transform. Therefore, we prove the results which contains the weak version in Theorem 1 in the case of the Hankel transform.

The aim of our paper is to prove a generalization of [7] (cf. [5], [6]) for the generalized Hankel transform and to have the weak type version in the case of the Hankel transform as a Corollary.

Here, we state some properties of the generalized Hankel transform(cf. [2; Lemma 2.7]).

PROPOSITION 2. *Let f and g be in $C_c^\infty(0,\infty)$. Then, we have the following properties*:

(i) $\mathcal{H}f(x) = O(x^{-a-n})$ *as* $x \to \infty$ *for all* $n = 0, 1, \cdots$, *and* $\mathcal{H}f(x) = O(1)$ *as* $x \to +0$.

(ii) $\mathcal{H}(\mathcal{H}f) = f$.

(iii) $(\mathcal{H}f, \mathcal{H}g) = (f, g)$, *where* $(f, g) = \int_0^\infty f(x)g(x)d\mu(x)$.

We can easily prove these results by the properties of the Bessel function $J_\alpha(x)$:

$$J_\alpha(x) = O(x^\alpha)\ (y \to +0)\,, \quad \text{and} \quad J_\alpha(x) = O(x^{-1/2})\ O(y \to \infty)\,.$$

Throughout this paper, for $s > 0$, we denote by s' the conjugate exponent of s i.e. $1/s + 1/s' = 1$, and by the letter C a positive constant that may vary from line to line.

2. The proof of Theorem 1

We may assume $1 < p < \infty$, since we can show it in the same way as in the case $p = 1$. Let $M > 0$, and ε a positive number such that $\pi/\varepsilon > M$ and N a positive integer. Also let $g \in C_c^\infty(0,\infty)$ with $supp\ g \subset [\eta, M]$ for some $0 < \eta < M$ and $g_\varepsilon(\theta) = g(\theta/\varepsilon)$. We define

$$G(\tau, 1/\varepsilon) = \sum_{n=0}^{\infty} \phi(\varepsilon n)\hat{g}_\varepsilon(n)Q_n(\varepsilon\tau)(= \tilde{T}_\varepsilon g_\varepsilon(\varepsilon\tau))\,,$$

$$G^N(\tau, 1/\varepsilon) = \sum_{n=0}^{N[1/\varepsilon]} \phi(\varepsilon n)\hat{g}_\varepsilon(n)Q_n(\varepsilon\tau)\,,$$

$$H^N(\tau, 1/\varepsilon) = G(\tau, 1/\varepsilon) - G^N(\tau, 1/\varepsilon)\,, \text{ and}$$

$$G(\tau) = \int_0^\infty \phi(y)\mathcal{H}g(y)(xy)^{1/2-a}J_\alpha(xy)d\mu(y)\,.$$

Now let $K > 0$ and $f \in C_c^\infty(0, K)$ be fixed. Then we obtain

$$\int G^N(\tau, 1/\varepsilon)f(\tau)d\mu(\tau) = \int G(\tau, 1/\varepsilon)f(\tau)d\mu(\tau) - \int H^N(\tau, 1/\varepsilon)f(\tau)d\mu(\tau)\,,$$

and

$$\left|\int G^N(\tau, 1/\varepsilon)f(\tau)d\mu(\tau)\right|$$
$$\leq \left|\int_0^K G(\tau, 1/\varepsilon)f(\tau)d\mu\right| + \left|\int_0^K H^N(\tau, 1/\varepsilon)f(\tau)d\mu(\tau)\right|$$

$$\leq C\|\chi_{(0,K)}G(\tau,1/\varepsilon)\|^{H}_{p,q}\|f\|^{H}_{p',q'} + \|H^N(\tau,1/\varepsilon)\|_{L^2((0,K),\mu)}\|f\|_{L^2((0,K),\mu)}\,,$$

where $\chi_{(0,K)}$ is the characteristic function of $(0,K)$.
Here, we estimate $\|\chi_{(0,K)}G(\tau,1/\varepsilon)\|^{H}_{p,q}$.

There exists a positive constant $C > 1$ and $\varepsilon_1 > 0$ such that $\pi/\varepsilon_1 > M + K$ and

$$C^{-1}\left(\frac{\varepsilon\tau}{2}\right)^{2a} \leq \left(\sin\frac{\varepsilon\tau}{2}\right)^{2a} \times \left(\cos\frac{\varepsilon\tau}{2}\right)^{2b} \leq C\left(\frac{\varepsilon\tau}{2}\right)^{2a} (0 < \varepsilon < \varepsilon_1\,, 0 < \tau < K)\,.$$

Hence, we have by $G(\theta/\varepsilon, 1/\varepsilon) = \tilde{T}_\varepsilon g_\varepsilon(\theta)$,

$$\begin{aligned}
&\|\chi_{(0,K)}G(\tau,1/\varepsilon)\|^{H}_{p,q} \\
&\leq \begin{cases} C^{1/p}\varepsilon^{-(2a+1)/p}2^{2a/p}(q\int_0^\infty (tm(\{0\leq\theta\leq\pi : |G(\theta/\varepsilon,1/\varepsilon)| > t\})^{1/p})^q \frac{dt}{t})^{1/q} & \text{if } q < \infty \\ C^{1/p}\varepsilon^{-(2a+1)/p}2^{2a/p}\sup_{t>0} t(m(\{0\leq\theta\leq\pi : |G(\theta/\varepsilon,1/\varepsilon)| > t\})^{1/p} & \text{if } q = \infty \end{cases} \\
&= C^{1/p}\varepsilon^{-(2a+1)/p}2^{2a/p}\|\tilde{T}_\varepsilon g_\varepsilon\|^{J}_{p,q}\,.
\end{aligned}$$

Also in the same way, we have

$$m\left(\left\{\theta \leq \pi : \left|g\left(\frac{\theta}{\varepsilon}\right)\right| > t\right\}\right) \leq C(\varepsilon^{2a+1}2^{-2a})^p\mu(\{\tau \leq M : |g(\tau)| > t\})\,,$$

and

$$\|g_\varepsilon\|^{J}_{p,r} \leq C\varepsilon^{(2a+1)/p}2^{-2a/p}\|g\|^{H}_{p,r}.$$

Therefore,

$$\|\tilde{T}_\varepsilon g_\varepsilon\|^{J}_{p,q} \leq C(\sup_{\varepsilon>0}\|\tilde{T}_\varepsilon\|^{J}_{M(p,r;p,q)}) \times \varepsilon^{(2a+1)/p}2^{-2a/p}\|g\|^{H}_{p,r}\,,$$

since

$$\|\tilde{T}_\varepsilon g_\varepsilon\|^{J}_{p,q} \leq \sup_{\varepsilon>0}\|\tilde{T}_\varepsilon\|^{J}_{M(p,r;p,q)} \times \|g_\varepsilon\|^{J}_{p,r}.$$

Then, we have

$$\|\chi_{(0,K)}G(\tau,1/\varepsilon)\|^{H}_{p,q} \leq C\sup_{\varepsilon>0}\|\tilde{T}_\varepsilon\|^{J}_{M(p,r;p,q)}\|g\|^{H}_{p,r}.$$

So for any $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned}
&\left|\int_0^K G^N(\tau,1/\varepsilon)f(\tau)d\mu(\tau)\right| \\
&\quad\leq C(\sup_{\varepsilon>0}\|\tilde{T}_\varepsilon\|^{J}_{M(p,r;p,q)})\|g\|^{H}_{p,r}\|f\|^{H}_{p',q'} + \|H^N(\tau,1/\varepsilon)\|_{L^2((0,K),\mu)}\|f\|_{L^2((0,K),\mu)}.
\end{aligned}$$

Here, we have two lemmas:

LEMMA 1. (i) *There exists a positive sequence* $\{\varepsilon_j\}_j$, $\varepsilon_j \downarrow 0$ *such that* $G(\tau, 1/\varepsilon_j) \to G(\tau)$ *as* $j \to \infty$ *weakly in* $L^2((0,K),\mu)$ *for all* $K > 0$, *for some* $G(\tau) \in L^2((0,\infty),\mu)$.

(ii)

$$\int_0^K |H^N(\tau, 1/\varepsilon)|^2 d\mu(\tau) \le \frac{B}{N^2}$$

for $0 < \varepsilon < \varepsilon(K)$ *with some* $\varepsilon(K)$, *where* $B = B(\alpha,\beta,a,b,\phi)$ *is a constant.*

PROOF. (i): In the same way as the argument before Lemma 1, we obtain

$$\|\chi_{(0,K)}G(\tau,1/\varepsilon)\|^H_{L^2((0,K),\mu)} \le C\varepsilon^{-(2a+1)/2}2^a\|\tilde{T}_\varepsilon g_\varepsilon\|^J_{2,2}\ (0<\varepsilon<\varepsilon(K))$$

for some $\varepsilon(K)$ with $\pi/\varepsilon(K) > K$, and

$$\|\tilde{T}_\varepsilon g_\varepsilon\|^J_{2,2} \le \sup_{\varepsilon>0}\|\tilde{T}_\varepsilon\|^J_{M(2,2;2,2)} \times \|g_\varepsilon\|^J_{2,2} \le C\|\phi\|_\infty\|g_\varepsilon\|^J_{2,2}\,.$$

Hence, we have

$$\left\|\chi_{(0,K)}G\left(\tau,\frac{1}{\varepsilon}\right)\right\|^H_{L^2((0,K),\mu)} \le C\|\phi\|_\infty\|g\|^H_{2,2}\ (0<\varepsilon<\varepsilon(K))$$

for all $K > 0$. Then by the diagonal argument and Fatou's lemma there exists $\{\varepsilon_j\}(\varepsilon_j \downarrow 0)$ and $G(\tau) \in L^2((0,\infty),\mu)$ such that

$$G\left(\tau,\frac{1}{\varepsilon_j}\right) \to G(\tau)\ (j\to\infty)$$

weakly in $L^2((0,K),\mu)$ for all $K > 0$. We remark $G(\tau)$ depends on the choice of $\{\varepsilon_j\}$.
(ii): First we notice Szegö's formula([9]): for $\alpha,\beta > -1$

$$\frac{d}{d\theta}\left[\left(\sin\frac{\theta}{2}\right)^{2\alpha+2}\left(\cos\frac{\theta}{2}\right)^{2\beta+2}P^{(\alpha+1,\beta+1)}_{n-1}(\cos\theta)\right]
= n\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}P^{(\alpha,\beta)}_n(\cos\theta)\,.$$

By this formula and integration by parts, we estimate $\hat{g}_\varepsilon(n)$:

$$\begin{aligned}
\hat{g}_\varepsilon(n) &= \int_0^\pi g\left(\frac{\theta}{\varepsilon}\right)Q_n(\theta)dm(\theta)\\
&= \int_0^\pi g\left(\frac{\theta}{\varepsilon}\right)t^{(\alpha,\beta)}_n P^{(\alpha,\beta)}_n(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2+a}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2+b}d\theta\\
&= \frac{t^{(\alpha,\beta)}_n}{n}\int_0^\pi \frac{g(\theta/\varepsilon)(\sin\theta/2)^a(\cos\theta/2)^b}{(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}}n\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}P^{(\alpha,\beta)}_n(\cos\theta))d\theta
\end{aligned}$$

$$= \frac{t_n^{(\alpha,\beta)}}{n} \left\{ \left[\frac{g(\theta/\varepsilon)(\sin\theta/2)^a(\cos\theta/2)^b}{(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}} \left(\sin\frac{\theta}{2}\right)^{2\alpha+2} \left(\cos\frac{\theta}{2}\right)^{2\beta+2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta) \right]_0^\pi \right.$$

$$\left. - \int_0^\pi \left(\frac{g(\theta/\varepsilon)(\sin\theta/2)^a(\cos\theta/2)^b}{(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}} \right)' \left(\sin\frac{\theta}{2}\right)^{2\alpha+2} \left(\cos\frac{\theta}{2}\right)^{2\beta+2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta))d\theta \right\}.$$

So by using

$$Q_{n-1}^{(\alpha+1,\beta+1,a,b)}(\theta) = t_{n-1}^{(\alpha+1,\beta+1)} P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+3/2-a} \left(\cos\frac{\theta}{2}\right)^{\beta+3/2-b}$$

and $dm(\theta) = (\sin\frac{\theta}{2})^{2a}(\cos\frac{\theta}{2})^{2b}d\theta$, we have

$$\frac{n}{t_n^{(\alpha,\beta)}} \hat{g}_\varepsilon(n) = -\frac{1}{t_{n-1}^{(\alpha+1,\beta+1)}} \int_0^\pi \frac{1}{\varepsilon} g'\left(\frac{\theta}{\varepsilon}\right) Q_{n-1}^{(\alpha+1,\beta+1,a,b)}(\theta)dm(\theta)$$

$$+ \frac{2\alpha+1-2a}{4} \int_0^\pi g\left(\frac{\theta}{\varepsilon}\right) \cot\frac{\theta}{2} \frac{Q_{n-1}^{(\alpha+1,\beta+1,a,b)}}{t_{n-1}^{(\alpha+1,\beta+1)}} dm(\theta)$$

$$- \frac{2\beta+1-2b}{4} \int_0^\pi g\left(\frac{\theta}{\varepsilon}\right) \tan\frac{\theta}{2} \frac{Q_{n-1}^{(\alpha+1,\beta+1,a,b)}(\theta)}{t_{n-1}^{(\alpha+1,\beta+1)}} dm(\theta)\,.$$

On the other hand, since we have

$$\int_0^K \left| H^N\left(\tau, \frac{1}{\varepsilon}\right) \right|^2 d\mu(\tau) \le C\varepsilon^{-(2a+1)} 2^{2a} \int_0^\pi \left| H^N\left(\frac{\theta}{\varepsilon}, \frac{1}{\varepsilon}\right) \right|^2 dm(\theta)$$

by the change of variable, and by using Parseval's equality

$$\int_0^\pi \left| H^N\left(\frac{\theta}{\varepsilon}, \frac{1}{\varepsilon}\right) \right|^2 dm(\theta)$$

$$= \sum_{N[1/\varepsilon]+1}^{\infty} |\hat{g}_\varepsilon(n)|^2$$

$$= \sum_{N[1/\varepsilon]+1}^{\infty} \frac{1}{n^2} |n\hat{g}_\varepsilon(n)|^2$$

$$\le \frac{A\varepsilon^2}{N^2} \int_0^\pi \left| \frac{1}{\varepsilon} g'\left(\frac{\theta}{\varepsilon}\right) - \frac{2\alpha+1-2a}{4} g\left(\frac{\theta}{\varepsilon}\right) \cot\frac{\theta}{2} + \frac{2\beta+1-2b}{4} g\left(\frac{\theta}{\varepsilon}\right) \tan\frac{\theta}{2} \right|^2 dm(\theta),$$

where A is an absolute constant. Here, we remark that $supp\ g \subset [\eta, M]$. We change the variable of $u = \theta/\varepsilon$, again. Then, we have:

$$\int_0^\pi \left| H^N\left(\frac{\theta}{\varepsilon}, \frac{1}{\varepsilon}\right) \right|^2 dm(\theta)$$

$$
\begin{aligned}
&\le \frac{C\varepsilon^2}{N^2}\frac{1}{\varepsilon}\int_\eta^M \left| g'(u) - \frac{2\alpha+1-2a}{4} g(u)\varepsilon \cot\frac{\varepsilon u}{2} \right.\\
&\quad \left. + \frac{2\beta+1-2b}{4} g(u)\varepsilon \tan\frac{\varepsilon u}{2} \right|^2 \left(\sin\frac{\varepsilon u}{2}\right)^{2a} \left(\cos\frac{\varepsilon u}{2}\right)^{2b} du \\
&\le \frac{C\varepsilon^2}{N^2}\frac{1}{\varepsilon}\int_\eta^M \left| g'(u) - \frac{2\alpha+1-2a}{4} g(u)\frac{\varepsilon u}{2}\left(\cot\frac{\varepsilon u}{2}\right)\frac{2}{u} \right.\\
&\quad \left. + \frac{2\beta+1-2b}{4} g(u)\varepsilon \tan\frac{\varepsilon u}{2} \right|^2 \left(\frac{\varepsilon u}{2}\right)^{2a} du \le \frac{C\varepsilon^2}{N^2}\varepsilon^{2a-1}\,.
\end{aligned}
$$

Then, we obtain

$$
\int_0^K \left| H^N\left(\tau, \frac{1}{\varepsilon}\right)\right|^2 d\mu(\tau) \le C\varepsilon^{-(2a+1)}\varepsilon^{2a-1} \times \frac{\varepsilon^2}{N^2} = CN^{-2}
$$

for $\varepsilon < \varepsilon(K)$ for all $K > 0$, where C depends on g, α, β, a, b. q.e.d.

LEMMA 2. (i) *In the notation of Lemma 1, there exist* $\{\varepsilon_{j_k}\}$ *a subsequence of* $\{\varepsilon_j\}$ *and* $G^N(\tau) \in L^2((0,\infty),\mu)$ *such that*

$$
G^N\left(\tau, \frac{1}{\varepsilon_{j_k}}\right) \to G^N(\tau)
$$

weakly in $L^2((0,K),\mu)$ *as* $k \to \infty$ *for every* $K > 0$.

(ii) $\{G^N(\tau, \frac{1}{\varepsilon})\}_{\varepsilon>0}$ *converges pointwise to a function as* $\varepsilon \to 0$. *Therefore,* $G^N(\tau)$ *is unique, and independent from* $\{\varepsilon_j\}$ *in* (i). *Moreover,* G *is independent from* $\{\varepsilon_j\}$ *in Lemma* 1.

PROOF. (i): By Lemma 1(i), (ii) and the diagonal argument, there exists $\{\varepsilon'_{j_k}\}$ a subsequence of $\{\varepsilon_j\}$ such that

$$
\int_0^K \left| H^N\left(\tau, \frac{1}{\varepsilon'_{j_k}}\right)\right|^2 d\mu(\tau) \le \frac{B}{N^2}
$$

for all $K > 0$. Hence, there exist a function $H^N \in L^2((0,\infty),\mu)$ and $\{\varepsilon_{j_k}\}$ a subsequence of $\{\varepsilon'_{j_k}\}$ such that

$$
H^N\left(\tau, \frac{1}{\varepsilon_{j_k}}\right) \to H^N(\tau)
$$

weakly in $L^2((0,K),\mu)$ $(j_k \to \infty)$, and

$$
\int_0^K |H^N(\tau)|^2 d\mu(\tau) \le \frac{B}{N^2}
$$

for all $K > 0$. Then, by the diagonal argument, there exists $\{N_j\}$ a sequence such that

$$
H^{N_j} \to 0 a.e. \text{ on } (0,\infty) \text{ as } j \to \infty\,.
$$

Here, we define $G^N = G - H^N$. By $G(\tau, \frac{1}{\varepsilon}) = G^N(\tau, \frac{1}{\varepsilon}) + H^N(\tau, \frac{1}{\varepsilon})$, Lemma 1(i), and the choice of $\{\varepsilon_{j_k}\}$, we have

$$G^N\left(\tau, \frac{1}{\varepsilon_{j_k}}\right) \to G^N(\tau)$$

weakly in $L^2((0, K), \mu)$ for all $K > 0$. Also we remark that

$$G^{N_j} \to G \; a.e. \text{ on } (0, \infty) \text{ as } j \to \infty\,,$$

since $H^{N_j} \to 0$ a.e. on $(0, \infty)$.

Next we prove the next Sublemma, before we give the complete proof of (ii):

Sublemma

$$\begin{aligned} &t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} \\ &\quad = \sqrt{2n}\, J_\alpha(n\theta) + \begin{cases} O(\theta^{1/2}) & \text{if } Cn^{-1} \le \theta < \pi - \varepsilon' \\ O(\theta^{\alpha} n^{\alpha - 1/2}) & \text{if } 0 < \theta \le Cn^{-1}\,, \end{cases} \end{aligned}$$

where $C,\ \varepsilon'$ are fixed constants.

PROOF. Stempak [8; p. 486] proved:

$$\begin{aligned} &t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2} \\ &\quad = \sqrt{n\theta}\, J_\alpha(n\theta) + \begin{cases} O(\theta) & \text{if } Cn^{-1} \le \theta < \pi - \varepsilon' \\ O(\theta^{\alpha+1/2} n^{\alpha - 1/2}) & \text{if } 0 < \theta \le Cn^{-1}\,, \end{cases} \end{aligned}$$

where C and $\varepsilon' < \pi$ are fixed positive constants.

By this result, we obtain

$$\begin{aligned} &t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} \\ &\quad = \sqrt{n\theta}\, J_\alpha(n\theta)\left(\sin\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2}\right)^{-1/2} \\ &\qquad + \begin{cases} O\left(\theta\left(\sin\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2}\right)^{-1/2}\right) & \text{if } Cn^{-1} \le \theta < \pi - \varepsilon' \\ O\left(\theta^{\alpha+1/2} n^{\alpha-1/2}\left(\sin\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2}\right)^{-1/2}\right) & \text{if } 0 < \theta \le Cn^{-1}\,, \end{cases} \\ &\quad = \sqrt{2n}\, J_\alpha(n\theta) + \begin{cases} O(\theta^{1/2}) & \text{if } Cn^{-1} \le \theta < \pi - \varepsilon' \\ O(\theta^{\alpha} n^{\alpha - 1/2}) & \text{if } 0 < \theta \le Cn^{-1}, \end{cases} \end{aligned}$$

Therefore, we have Sublemma. q.e.d.

3. The proof of Lemma 2 (ii)

Let η', K', and $\varepsilon_2 = \varepsilon_2(K')$ be positive numbers such that $0 < \eta' < K' < \infty$ and $\pi/\varepsilon_2 > K'$. Since

$$Q_n(\theta) = \left\{ t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \left(\sin\frac{\theta}{2} \right)^{\alpha} \left(\cos\frac{\theta}{2} \right)^{\beta} \right\} \left(\sin\frac{\theta}{2} \right)^{1/2-a} \left(\cos\frac{\theta}{2} \right)^{1/2-b},$$

we have, by Sublemma

$$\begin{aligned} Q_n(\varepsilon\tau) &= \sqrt{2n} J_\alpha(n\varepsilon\tau) \left(\sin\frac{\varepsilon\tau}{2} \right)^{1/2-a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{1/2-b} \\ &+ \begin{cases} O\left((\varepsilon\tau)^{1/2} \left(\sin\frac{\varepsilon\tau}{2} \right)^{1/2-a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{1/2-b} \right) & \text{if } Cn^{-1} \le \varepsilon\tau < \pi - \varepsilon' \\ O\left((\varepsilon\tau)^{\alpha} n^{\alpha-1/2} \left(\sin\frac{\varepsilon\tau}{2} \right)^{1/2-a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{1/2-b} \right) & \text{if } 0 < \varepsilon\tau \le Cn^{-1}, \end{cases} \end{aligned}$$

where $\eta' \le \tau \le K'$ and $\varepsilon < \varepsilon_2$. Therefore, we obtain for $\eta' \le \tau \le K'$ and $\varepsilon < \varepsilon_2$,

$$\varepsilon^{a-1/2} Q_n(\varepsilon\tau) = \varepsilon^{a-1/2} \left(\frac{\varepsilon\tau}{2} \right)^{1/2-a} \sqrt{2n} J_\alpha(n\varepsilon\tau) + \begin{cases} O(\varepsilon^{1/2}) & \text{if } Cn^{-1} \le \varepsilon\tau < \pi - \varepsilon' \\ O(\varepsilon^{\alpha} n^{\alpha-1/2}) & \text{if } 0 < \varepsilon\tau \le Cn^{-1}, \end{cases} \qquad \blacksquare$$

where C and $\varepsilon\prime < \pi$ are fixed positive constants. Next we estimate $\varepsilon^{1/2-a} \hat{g}_\varepsilon(n)$, precisely. Let $0 < \eta \le \tau \le M$ (*i.e. supp* $g \subset [\eta, M]$), $\varepsilon < \varepsilon_3 = \varepsilon_3(\eta, M)$ with $\pi > \varepsilon_3 M$ and $0 \le n \le N[\frac{1}{\varepsilon}]$. By the change of variable,

$$\begin{aligned} \hat{g}_\varepsilon(n) &= \int_0^\pi g\left(\frac{\theta}{\varepsilon}\right) Q_n(\theta) dm(\theta) \\ &= \varepsilon \int_\eta^M g(\tau) Q_n(\varepsilon\tau) \left(\sin\frac{\varepsilon\tau}{2} \right)^{2a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{2b} d\tau \\ &= \varepsilon \int_\eta^M g(\tau) \left\{ t_n^{(\alpha,\beta)} P_n^{\alpha,\beta)}(\cos\varepsilon\tau) \left(\sin\frac{\varepsilon\tau}{2} \right)^{\alpha+1/2+a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{\beta+1/2+b} \right\} d\tau\,. \end{aligned}$$

Here, by Sublemma, $J_\alpha(x) = O(x^\alpha)(x \to 0+0)$, and $J_\alpha(x) = O(x^{-1/2})(x \to \infty)$, we have

$$\begin{aligned} & t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\varepsilon\tau) \left(\sin\frac{\varepsilon\tau}{2} \right)^{\alpha+1/2+a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{\beta+1/2+b} \\ &\quad = \sqrt{2n} J_\alpha(n\varepsilon\tau) \left(\sin\frac{\varepsilon\tau}{2} \right)^{1/2+a} \left(\cos\frac{\varepsilon\tau}{2} \right)^{1/2+b} \end{aligned}$$

$$+\begin{cases} O\left(\left(\sin\frac{\varepsilon\tau}{2}\right)^{1/2+a}\left(\cos\frac{\varepsilon\tau}{2}\right)^{1/2+b}\left(\frac{\varepsilon\tau}{2}\right)^{1/2}\right) & \text{if } Cn^{-1}\le\varepsilon\tau<\pi-\varepsilon' \\ O\left(\left(\sin\frac{\varepsilon\tau}{2}\right)^{1/2+a}\left(\cos\frac{\varepsilon\tau}{2}\right)^{1/2+b}\left(\frac{\varepsilon\tau}{2}\right)^{\alpha}n^{\alpha-1/2}\right) & \text{if } 0<\varepsilon\tau\le Cn^{-1}, \end{cases}$$

$$=\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\varepsilon\tau}{2}\right)^{1/2+a}$$

$$+\sqrt{2n}J_\alpha(n\varepsilon\tau)\left\{\left(\sin\frac{\varepsilon\tau}{2}\right)^{1/2+a}\left(\cos\frac{\varepsilon\tau}{2}\right)^{1/2+b}-\left(\frac{\varepsilon\tau}{2}\right)^{1/2+a}\right\}$$

$$+\begin{cases} O\left(\left(\sin\frac{\varepsilon\tau}{2}\right)^{1/2+a}\left(\cos\frac{\varepsilon\tau}{2}\right)^{1/2+b}\left(\frac{\varepsilon\tau}{2}\right)^{1/2}\right) & \text{if } Cn^{-1}\le\varepsilon\tau<\pi-\varepsilon' \\ O\left(\left(\sin\frac{\varepsilon\tau}{2}\right)^{1/2+a}\left(\cos\frac{\varepsilon\tau}{2}\right)^{1/2+b}\left(\frac{\varepsilon\tau}{2}\right)^{\alpha}n^{\alpha-1/2}\right) & \text{if } 0<\varepsilon\tau\le Cn^{-1}, \end{cases}$$

$$=\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\varepsilon\tau}{2}\right)^{1/2+a}+\sqrt{2n}J_\alpha(n\varepsilon\tau)O((\varepsilon\tau)^{1/2+a+2})$$

$$+\begin{cases} O(\varepsilon^{1/2+a+1/2}) & \text{if } Cn^{-1}\le\varepsilon\tau<\pi-\varepsilon' \\ O(\varepsilon^{1/2+a+\alpha}n^{\alpha-1/2}) & \text{if } 0<\varepsilon\tau\le Cn^{-1}, \end{cases}$$

$$=\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\varepsilon\tau}{2}\right)^{1/2+a}$$

$$+\begin{cases} O(\varepsilon^{2+a}+\varepsilon^{1+a}) & \text{if } Cn^{-1}\le\varepsilon\tau<\pi-\varepsilon' \\ O(\varepsilon^{a+\alpha+3/2}n^{\alpha-1/2}+\varepsilon^{\alpha+a+1/2}n^{\alpha-1/2}) & \text{if } 0<\varepsilon\tau\le Cn^{-1}, \end{cases}$$

$$=\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\varepsilon\tau}{2}\right)^{1/2+a}+\begin{cases} O(\varepsilon^{1+a}) & \text{if } Cn^{-1}\le\varepsilon\tau<\pi-\varepsilon' \\ O(\varepsilon^{\alpha+a+1/2}n^{\alpha-1/2}) & \text{if } 0<\varepsilon\tau\le Cn^{-1}, \end{cases}.$$

∎

Then, we obtain

$$\hat{g}_\varepsilon(n)=\varepsilon\int_\eta^M g(\tau)\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\varepsilon\tau}{2}\right)^{1/2+a}d\tau$$

$$+\begin{cases} O(\varepsilon^{2+a}) & \text{if } Cn^{-1}\le\varepsilon<\pi-\varepsilon' \\ O(\varepsilon^{\alpha+3/2+a}n^{\alpha-1/2}) & \text{if } 0<\varepsilon\le Cn^{-1}, \end{cases} \quad \text{and}$$

$$\varepsilon^{1/2-a}\hat{g}_\varepsilon(n)=2^{-a}n^a\varepsilon^{a+3/2}\mathcal{H}g(n\varepsilon)+\begin{cases} O(\varepsilon^{5/2}) & \text{if } Cn^{-1}\le\varepsilon<\pi-\varepsilon' \\ O(\varepsilon^{\alpha+2}n^{\alpha-1/2}) & \text{if } 0<\varepsilon\le Cn^{-1}. \end{cases}$$

Therefore, for $0<\eta'\le\tau\le K'$ and $0<\varepsilon<\varepsilon_4=\min(\varepsilon_2,\varepsilon_3)$, we have by using the above estimates of $\varepsilon^{a-1/2}Q_n(\varepsilon\tau)$ and $\varepsilon^{1/2-a}\hat{g}_\varepsilon(n)$,

$$\sum_{n=0}^{N[1/\varepsilon]} \hat{g}_\varepsilon(n) Q_n(\varepsilon\tau)\phi(n\varepsilon) = \sum_{n=0}^{N[1/\varepsilon]} (\varepsilon^{1/2-a}\hat{g}_\varepsilon(n))(\varepsilon^{a-1/2}Q_n(\varepsilon\tau))\phi(n\varepsilon)$$

$$= \sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)\left(2^{-a}n^a\varepsilon^{a+3/2}\mathcal{H}g(n\varepsilon) + \begin{Bmatrix} O(\varepsilon^{5/2}) & \text{if } \tilde{C}n^{-1} \le \varepsilon < \pi - \tilde{\varepsilon} \\ O(\varepsilon^{\alpha+2}n^{\alpha-1/2}) & \text{if } 0 < \varepsilon \le \tilde{C}n^{-1}, \end{Bmatrix}\right)(*)$$

$$\times\left(\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\tau}{2}\right)^{1/2-a} + \begin{Bmatrix} O(\varepsilon^{1/2}) & \text{if } \tilde{C}n^{-1} \le \varepsilon < \pi - \tilde{\varepsilon} \\ O(\varepsilon^{\alpha}n^{\alpha-1/2}) & \text{if } 0 < \varepsilon \le \tilde{C}n^{-1}, \end{Bmatrix}\right)$$

where $\tilde{C}$ and $\tilde{\varepsilon} < \pi$ are fixed positive constants which depend on η', K', ϕ, N and g. $(*) = \sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)(A_n + B_n)(C_n + D_n)$, we say.

First we obtain that

$$\begin{aligned}&\sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)A_nC_n \\ &\quad= \sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)(2^{-a}n^a\varepsilon^{a+3/2}\mathcal{H}g(n\varepsilon))\left(\sqrt{2n}J_\alpha(n\varepsilon\tau)\left(\frac{\tau}{2}\right)^{1/2-a}\right) \\ &\quad= \varepsilon\sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)\mathcal{H}g(n\varepsilon)(n\varepsilon\tau)^{1/2-a}J_\alpha(n\varepsilon\tau)(n\varepsilon)^{2a} \\ &\quad\to \int_0^N \phi(y)(\tau y)^{1/2-a}J_\alpha(\tau y)\mathcal{H}g(y)\,d\mu(y)\end{aligned}$$

for all $0 < \eta' \le \tau \le K'$ as $\varepsilon \to 0$, since the function $\phi(y)(\tau y)^{1/2-a}J_\alpha(\tau y)\mathcal{H}g(y)y^{2a}$ is Riemann integrable on $[0, N]$ for any τ $(0 < \eta' \le \tau \le K')$. Then, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)(2^{-a}n^a\varepsilon^{a+3/2}(\mathcal{H}g)(n\varepsilon))(2^a\tau^{1/2-a}n^{1/2}J_\alpha(n\varepsilon\tau)) \\ &\quad= \int_0^N \phi(y)(\tau y)^{1/2-a}J_\alpha(\tau y)\mathcal{H}g(y)\,d\mu(y)\,.\end{aligned}$$

In the same way with the boundedness of $\mathcal{H}g$ (Proposition 2), we get

$$\sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)B_nC_n = O(\omega_1(\varepsilon))\,,$$

$$\sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon)A_nD_n = O(\omega_2(\varepsilon))\,,$$

and

$$\sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon) B_n D_n = O(\omega_3(\varepsilon)),$$

where $\omega_j(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $O(\omega_j(\varepsilon))$ is uniformly bounded on $\tau \in [\eta', K']$ ($\varepsilon < \min(\varepsilon_2, \varepsilon_3)$) ($j = 1, 2, 3$). After all, we have

$$\lim_{\varepsilon \to 0} \sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon) \hat{g}_\varepsilon(n) Q_n(\varepsilon\tau) = \int_0^N \phi(y) \mathcal{H}g(y)(\tau y)^{1/2-a} J_\alpha(\tau y) d\mu_a(y),$$

pointwise on $\tau \in (0, \infty)$.

We proceed the proof of Lemma 2 (ii).

Now we show that

$$G^N(\tau) = \int_0^N \phi(u) \mathcal{H}g(u)(\tau u)^{1/2-a} J_\alpha(\tau u) d\mu(u)$$

and

$$G^N\left(\tau, \frac{1}{\varepsilon}\right) \to \int_0^N \phi(u) \mathcal{H}g(u)(\tau u)^{1/2-a} J_\alpha(\tau u) d\mu(u)$$

pointwise as $\varepsilon \to 0$ ($\tau > 0$).

In fact, we remark by the above argument,

$$\begin{aligned} G^N\left(\tau, \frac{1}{\varepsilon}\right) &= \sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon) \hat{g}_\varepsilon(n) Q_n(\varepsilon\tau) \\ &= \varepsilon \sum_{n=0}^{N[1/\varepsilon]} \phi(n\varepsilon) \mathcal{H}g(n\varepsilon)(n\varepsilon\tau)^{1/2-a} J_\alpha(n\varepsilon\tau)(n\varepsilon)^{2a} + O(\omega(\varepsilon)), \end{aligned}$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $O(\omega(\varepsilon))$ is uniformly bounded on $\tau \in [\eta', K']$ for some fixed numbers $0 < \eta' < K' < \infty$ and for any $\varepsilon < \min(\varepsilon_2, \varepsilon_3)$. Then, by the properties of J_α and $\mathcal{H}g$, we can show that $G^N(\tau, \frac{1}{\varepsilon})$ is uniformly bounded on $\tau \in [\eta', K']$ for some fixed numbers $0 < \eta' < K' < \infty$ and for any $\varepsilon < \min(\varepsilon_2, \varepsilon_3)$. Therefore, for $f \in C_c^\infty(0, \infty)$, we have

$$\int G^N\left(\tau, \frac{1}{\varepsilon}\right) f(\tau) d\mu(\tau) \to \int \left(\int_0^N \phi(u) \mathcal{H}g(u)(\tau u)^{1/2-a} J_\alpha(\tau u) d\mu(u) \right) f(\tau) d\mu(\tau)$$

by Lebesgue's convergence theorem. Hence, we obtain

$$G^N(\tau) = \int_0^N \phi(u) \mathcal{H}g(u)(\tau u)^{1/2-a} J_\alpha(\tau u) d\mu(u).$$

Also we obtain that

$$G(\tau)=\int_0^{\infty}\phi(u)\mathcal{H}g(u)(\tau u)^{1/2-a}J_{\alpha}(\tau u)d\mu(u)\ a.e.\,,$$

since $lim_{j\to\infty}G^{N_j}(\tau)=G(\tau)\ a.e.$ by Lemma 1. Thus, we proved Lemma 2 (ii). q.e.d.

4. The proof of Theorem 1 (continuation)

Let $q=\infty$. Also let s, w, and K be positive numbers. Then,

$$\mu\left(\left\{\tau\le K:\left|G^N\left(\tau,\frac{1}{\varepsilon}\right)\right|>s+w\right\}\right)$$
$$\le\mu\left(\left\{\tau\le K:\left|G^N\left(\tau,\frac{1}{\varepsilon}\right)\right|>s\right\}\right)+\mu\left(\left\{\tau\le K:\left|H^N\left(\tau,\frac{1}{\varepsilon}\right)\right|>w\right\}\right).$$

By the argument of §2, there exists $\varepsilon_1>0$ such that for $0<\varepsilon<\varepsilon_1$,

$$\begin{aligned}&\mu\left(\left\{0\le\tau\le K:\left|G\left(\tau,\frac{1}{\varepsilon}\right)\right|>s\right\}\right)\\&\le Cm\left(\left\{0\le\theta\le\pi:\left|G\left(\frac{\theta}{\varepsilon},\frac{1}{\varepsilon}\right)\right|>s\right\}\right)\varepsilon^{-(2a+1)}2^{2a}\\&\le\left(\frac{C}{s}\sup_{\varepsilon>0}\|\tilde{T}_{\varepsilon}\|^J_{M(p,r;p,\infty)}\|g_{\varepsilon}\|^J_{p,r}\right)^p\varepsilon^{-(2a+1)}2^{2a}\\&\le\left(\frac{C}{s}\sup_{\varepsilon>0}\|\tilde{T}_{\varepsilon}\|^J_{M(p,r;p,\infty)}\|g\|^H_{p,r}\right)^p\end{aligned}$$

for some constant $C>0$. Also by Lemma 1(ii), we have

$$w^2\mu\left(\left\{\tau\le K:\left|H^N\left(\tau,\frac{1}{\varepsilon}\right)\right|>w\right\}\right)\le\int_0^K\left|H^N\left(\tau,\frac{1}{\varepsilon}\right)\right|^2d\mu(\tau)\le\frac{B}{N^2}$$

for $\varepsilon<\varepsilon(K)$, and

$$\mu\left(\left\{\tau\le K:\left|H\left(\tau,\frac{1}{\varepsilon}\right)\right|>w\right\}\right)\le\frac{B}{N^2w^2}.$$

Hence, we have

$$\mu\left(\left\{0\le\tau\le K:\left|G^N\left(\tau,\frac{1}{\varepsilon}\right)\right|>s+w\right\}\right)$$
$$\le\left(\frac{C}{s}\sup_{\varepsilon>0}\|\tilde{T}_{\varepsilon}\|^J_{M(p,r;p,\infty)}\right)^p(\|g\|^H_{p,r})^p+\frac{B}{N^2w^2}\quad\text{for }\ 0<\varepsilon<\min(\varepsilon_1,\varepsilon(K))\,.$$

By Lemmas 1, 2 and Fatou's lemma, we obtain that

$$\mu(\{0\le\tau\le K:|G^N(\tau)|>s+w\})\le\left(\frac{C}{s}\sup_{\varepsilon>0}\|\tilde{T}_{\varepsilon}\|^J_{M(p,r;p,\infty)}\right)^p(\|g\|^H_{p,r})^p+\frac{B}{N^2w^2}\,,$$

and

$$\mu(\{0 \leq \tau \leq K : |G(\tau)| > s + w\}) \leq \left(\frac{C}{s} \sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,\infty)} \right)^p (\|g\|^H_{p,r})^p$$

by Fatou's Lemma for $N = N_j \to \infty$, again. Therefore, by $w \to 0$ and $K \uparrow \infty$, we obtain

$$\mu(\{\tau \in (0,\infty) : |G(\tau)| > s\}) \leq \left(\frac{C}{s} \sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,\infty)} \right)^p (\|g\|^H_{p,r})^p .$$

Let $q < \infty$. Also let $f \in C_c^\infty(0,\infty)$ be fixed with $supp\ f \subset [\eta, K]$ for some $0 < \eta < K < \infty$. By the argument of Lemmas 1 and 2,

$$\left| \int_0^K G^N\left(\tau, \frac{1}{\varepsilon}\right) f(\tau) d\mu(\tau) \right|$$
$$\leq C\|g\|^H_{p,r} (\sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,q)}) \|f\|^H_{p',q'} + \left\| H^N\left(\tau, \frac{1}{\varepsilon}\right) \right\|_{L^2((0,K),\mu)} \|f\|_{L^2((0,K),\mu)},$$

and

$$\left| \int_0^K G^N(\tau) f(\tau) d\mu(\tau) \right|$$
$$\leq C\|g\|^H_{p,r} (\sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,q)}) \|f\|^H_{p',q'} + \frac{B}{N^2} \|f\|_{L^2((0,K),\mu)}.$$

Then, by Lemma 1 we have

$$\left| \int_\eta^K G^N(\tau) f(\tau) d\mu(\tau) - \int_\eta^K G(\tau) f(\tau) d\mu(\tau) \right| \leq \|G^N - G\|_{L^2((\eta,K),\mu)} \|f\|_{L^2((\eta,K),\mu)}$$
$$= \|H^N\|_{L^2((\eta,K),\mu)} \|f\|_{L^2((0,\infty),\mu)} \leq \frac{B}{N^2} \|f\|_{L^2((0,\infty),\mu)} .$$

So, we obtain

$$\left| \int_0^K G(\tau) f(\tau) d\mu(\tau) \right| \leq C\|g\|^H_{p,r} (\sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,q)}) \|f\|^H_{p',q'} + \frac{B}{N^2} \|f\|_{L^2((0,K),\mu)} ,$$

and as $N \to \infty$,

$$\left| \int_0^K G(\tau) f(\tau) d\mu(\tau) \right| \leq C\|g\|^H_{p,r} (\sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,q)}) \|f\|^H_{p',q'} .$$

Therefore,

$$\|T_\phi f\|^H_{p,q} \leq C\|g\|^H_{p,r} (\sup_{\varepsilon > 0} \|\tilde{T}_\varepsilon\|^J_{M(p,r;p,q)}) ,$$

and we get the desired result. q.e.d.

5. The proof of Theorem 2

First, we insist the following:

LEMMA 3 (cf. [7; Lemma]). *Let* $1 \le r < \infty,\ 1 < q \le \infty$, *and* $1 < p < \infty$. *Then we have the following*:

(i) $\|\tilde{T}^*\|^J_{M(p,r;p,q)}$ *is finite, if and only if there exists a positive constant C such that for any positive integer N, any positive numbers* ε_j, *and* $g_j \in C_c^\infty(0,\infty)$ $(j = 1, 2, \cdots, N)$, *we have*

$$\left\| \sum_{j=1}^N \tilde{T}_{\varepsilon_j} g_j \right\|^J_{p',r'} \le C \|\tilde{T}^*\|^J_{M(p,r;p,q)} \left\| \sum_{j=1}^N |g_j| \right\|^J_{p',r'} .$$

(ii) $\|T^*\|^H_{M(p,r;p,q)}$ *is finite, if and only if there exists a positive constant C such that for any positive integer N, positive numbers* ε_j, *and* $f_j \in C_c^\infty(0,\infty)$ $(j = 1, 2, \cdots, N)$, *we have*

$$\left\| \sum_{j=1}^N T_{\varepsilon_j} f_j \right\|^H_{p',r'} \le C \|T^*\|^H_{M(p,q;p,q)} \left\| \sum_{j=1}^N |f_j| \right\|^J_{p',r'} .$$

We can prove Lemma 3 in the same way of [7; Lemma]. We omit the proof.

Now we proceed to the proof of Theorem 2.

Let $1 < p < \infty,\ 1 < q < \infty,\ 1 \le r < \infty,\ M > 0$, and N be a positive integer. Also $\{g_j\}_1^L \subset C_c^\infty(0,\infty)$ with $supp\ g_j \subset (0, M)$, and $g_{j,\varepsilon}(\theta) = g_j(\theta/\varepsilon)$ for any $0 < \varepsilon < \varepsilon_0$ with $\pi/\varepsilon_0 > M$. Moreover, we denote $\phi_j(x) = \phi(\varepsilon_j x)$ $(j = 1, 2, \cdots\cdots)$, where $\{\varepsilon_j\}_{j=1}^\infty$ are all positive rational numbers,

Now we define the following:

$$G_j(\tau, 1/\varepsilon) = \sum_{n=0}^\infty \phi_j(\varepsilon n)\hat{g}_{j,\varepsilon}(n) Q_n(\varepsilon\tau) ,$$

$$G_j^N(\tau, 1/\varepsilon) = \sum_{n=0}^{N[1/\varepsilon]} \phi_j(\varepsilon n)\hat{g}_{j,\varepsilon}(n) Q_n(\varepsilon\tau), \quad \text{and}$$

$$H_j^N(\tau, 1/\varepsilon) = G_j(\tau, 1/\varepsilon) - G_j^N(\tau, 1/\varepsilon) .$$

Also let g be fixed in $C_c^\infty(0,\infty)$ with $supp\ g \subset [\eta, K]$ for some $0 < \eta < K$. Since $G_j^N(\tau, 1/\varepsilon) = G_j(\tau, 1/\varepsilon) - H_j^N(\tau, 1/\varepsilon)$ $(j = 1, \cdots, L)$, we have

$$\int \sum_{j=1}^L G_j^N(\tau, 1/\varepsilon) g(\tau) d\mu(\tau) = \sum_{j=1}^L \left\{ \int G_j(\tau, 1/\varepsilon) g(\tau) d\mu(\tau) - \int H_j^N(\tau, 1/\varepsilon) g(\tau) d\mu(\tau) \right\},$$

and

$$\begin{aligned}
&\left|\sum_{j=1}^{L}\int_0^K G_j^N(\tau,1/\varepsilon)g(\tau)d\mu(\tau)\right| \\
&\quad\le \left|\int_0^K \sum_{j=1}^{L} G_j(\tau,1/\varepsilon)g(\tau)d\mu(\tau)\right| + \left|\int_0^K H_j^N(\tau,1/\varepsilon)g(\tau)d\mu(\tau)\right| \\
&\quad\le C\left\|\chi_{(0,K)}\sum_{j=1}^{L} G_j(\tau,1/\varepsilon)\right\|_{p,q}^H \|g\|_{p',q'}^H \\
&\qquad + \left\|\sum_{j=1}^{L} H_j^N(\tau,1/\varepsilon)\right\|_{L^2((0,K),\mu)} \|f\|_{L^2((0,K),\mu)}.
\end{aligned}$$

Here, we choose a positive number $\sigma_1 = \sigma_1(K)$ with $K < \pi/\sigma_1$ such that

$$\begin{aligned}
&\mu\left(\left\{\tau \le K : \left|\sum_{j=1}^{L} G_j(\tau,1/\varepsilon)\right| > t\right\}\right) \\
&\quad\le C\varepsilon^{-(2a+1)}2^{2a} m\left(\left\{0 \le \theta \le \pi : \left|\sum_{j=1}^{L} G(\theta/\varepsilon,1/\varepsilon)\right| > t\right\}\right)
\end{aligned}$$

for some constant C and $0 < \varepsilon < \sigma_1$. Then, in the same way of the argument in §2, we have

$$\left\|\chi_{(0,K)}\sum_{j=1}^{L} G_j(\tau,1/\varepsilon)\right\|_{p',r'}^H \le C\varepsilon^{-(2a+1)/p'}2^{2a/p'}\left\|\sum_{j=1}^{L}\tilde{T}_{\varepsilon\varepsilon_j} g_{j,\varepsilon}\right\|_{p',r'}^J.$$

Here, by Lemma 3(i), we have

$$\left\|\chi_{(0,K)}\sum_{j=1}^{L} G_j(\tau,1/\varepsilon)\right\|_{p',r'}^H \le C\varepsilon^{-(2a+1)/p'}2^{2a/p'}\|\tilde{T}^*\|_{M(p,r;p,q)}^J\left\|\sum_{1}^{L}|g_{j,\varepsilon}|\right\|_{p',q'}^J.$$

Here, let q' be finite. Since we have

$$\begin{aligned}
&m\left(\left\{0 \le \theta \le \pi : \left|\sum_{j=1}^{L}|g_j(\theta/\varepsilon)| > t\right.\right\}\right)^{1/p'} \\
&\quad\le C\varepsilon^{(2a+1)/p'}2^{-2a/p'}\mu\left(\left\{\tau \le M : \sum_{j=1}^{L}|g_j(\tau)| > t\right\}\right)^{1/p'},
\end{aligned}$$

$$\left\|\sum_{j=1}^{L}|g_{j,\varepsilon}|\right\|_{p',q'}^J \varepsilon^{-(2a+1)/p'}$$

$$\leq C2^{-2a/p'}\left\{q'\int_0^\infty\left(t\mu\left(\left\{0\leq\tau\leq M:\sum_{j=1}^L|g_j(\tau)|>t\right\}\right)^{1/p'}\right)^{q'}\frac{dt}{t}\right\}^{1/q'}$$

$$\leq C\left\|\sum_{j=1}^L|g_j|\right\|_{p',q'}^H$$

for some $C>0$. After all, by Lemma 3(i), we have

$$\left\|\chi_{(0,K)}\sum_{j=1}^L G_j(\tau,1/\varepsilon)\right\|_{p',r'}^H\leq C\|\tilde{T}^*\|_{M(p,r;p,q)}^J\left\|\sum_{j=1}^L|g_j|\right\|_{p',q'}^H\quad(0<\varepsilon<\sigma_1).$$

Hence, we obtain

$$\begin{aligned}&\left|\int_0^K\sum_{j=1}^L G_j^N(\tau,1/\varepsilon)g(\tau)d\mu(\tau)\right|\\&\quad\leq C\|\tilde{T}^*\|_{M(p,r;p,q)}^J\left\|\sum_{j=1}^L|g_j|\right\|_{p',q'}^H\|g\|_{p\cdot r}^H\\&\qquad+\left\|\sum_{j=1}^L H_j^N(\tau,1/\varepsilon)\right\|_{L^2((0,K),\mu)}\|g\|_{L^2((0,K),\mu)}.\end{aligned}$$

Also, by the same way of the proof of Lemmas 1,2 and Theorem 1, there exists $\{\varepsilon_l'\}_l$ $(\varepsilon_l'<\sigma_1)$ such that

$$G_j^N\left(\tau,\frac{1}{\varepsilon_l'}\right)\to G_j^N(\tau)$$

weakly in $L^2((0,K),\mu)$ $(l\to\infty)$,

$$H_j^N\left(\tau,\frac{1}{\varepsilon_l'}\right)\to H_j^N(\tau)$$

weakly in $L^2((0,K),\mu)$ $(l\to\infty)$, and

$$G_j\left(\tau,\frac{1}{\varepsilon_l'}\right)\to G_j(\tau)$$

weakly in $L^2((0,K),\mu)$ $(l\to\infty)$, for some $G_j\in L^2((0,\infty),\mu)$ $(j=1,2,\cdots,L)$. Then, in the same way of Theorem 1 there exists a constant B such that for any $g\in C_c^\infty(0,K)$

$$\left|\int\sum_{j=1}^L G_j^N(\tau)g(\tau)d\mu(\tau)\right|$$

$$\leq C\left\|\sum_{j=1}^{L}|g_j|\right\|_{p',q'}^{H}\|g\|_{p,r}^{H}\|\tilde{T}^*\|_{M(p,r;p,q)}^{J}+\frac{LB}{N^2}\|g\|_{L^2((0,K),\mu)}.$$

Also, by the same way of Lemma 2(ii), we have

$$G_j(\tau)=\int_0^\infty \phi(\varepsilon_j x)\mathcal{H}_{\alpha,a}g_j(x)(\tau x)^{1/2-a}J_\alpha(\tau x)d\mu(x)(=T_{\varepsilon_j}g_j)\ (j=1,\cdots,L).$$

On the other hand, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{L}\int G_j^N(\tau)g(\tau)d\mu(\tau)-\sum_{j=1}^{L}\int G_j(\tau)g(\tau)d\mu(\tau)\right|\\
&\quad\leq\sum_{j=1}^{L}\|G_j^N-G_j\|_{L^2([\eta,K],\mu)}\|g\|_{L^2((0,K),\mu)}\\
&\quad\leq\sum_{j=1}^{L}\|H_j^N\|_{L^2([\eta,K],\mu)}\|g\|_{L^2((0,K),\mu)}\\
&\quad\leq\frac{LB}{N^2}\|g\|_{L^2((0,K),\mu)}\to 0\ (N\to\infty).
\end{aligned}$$

Therefore, we have

$$\left|\int\sum_{j=1}^{L}G_j(\tau)g(\tau)d\mu(\tau)\right|\leq C\|\sum_{j=1}^{L}|g_j|\|_{p',q'}^{H}\|\tilde{T}^*\|_{M(p,r;p,q)}^{J}\|g\|_{p,r}^{H}.$$

Hence,

$$\left\|\sum_{j=1}^{L}T_{\varepsilon_j}g_j\right\|_{p',r'}^{H}\leq C\|\tilde{T}^*\|_{M(p,r;p,q)}^{J}\left\|\sum_{j=1}^{L}|g_j|\right\|_{p',q'}^{H},$$

and by Lemma 3(ii), we obtain

$$\|T^*\|_{M(p,r;p,q)}^{H}\leq C\|\tilde{T}^*\|_{M(p,r;p,q)}^{J}$$

for some $C>0$. q.e.d.

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