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On DS-diagrams for 3-manifolds of Heegaard Genus 2

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Abstract. The block number $Bl(M)$ introduced in our previous paper is a new topological invariant of a closed orientable 3-manifold M which estimates a combinatorial complexity of M just like the Heegaard genus $HG(M)$. In our previous paper, we have shown an inequality $HG(M) \leq Bl(M)$ for any $M \neq S^2 \times S^1$. In this paper, we will show that $Bl(M) = HG(M)$ for any M with $HG(M) = 2$ and moreover that $Bl(M) \leq 4$ for any M with $HG(M) = 3$.

1. Introduction

As a new topological invariant, we have introduced the *block number* for an orientable closed 3-manifold M, denoted by $Bl(M)$, in [2]. The block number is defined through a DS-diagram with E-cycle (see [5]). In [2], we have shown that the block number $Bl(M)$ dominates the Heegaard genus $HG(M)$ for any $M \neq S^2 \times S^1$. Besides this, we have exhibited some other properties of the block number, for examples, $Bl(S^2 \times S^1) = 0$, $Bl(S^3) = 1$, $Bl(L(p,q)) = HG(L(p,q)) = 1$, $Bl(M) = HG(M) = 2$ for a Seifert fibered space M having the 2-sphere S^2 as its base manifold and three exceptional fibers. An interesting problem that remains to be considered is whether $HG(M) = Bl(M)$ for any $M \neq S^2 \times S^1,\ S^3$.

In this paper, we focus on the relation between the block number and the Heegaard genus for an oriented closed 3-manifolds. We will give a sufficient condition for $Bl(M) = HG(M)$ in §3. However in the case of $HG(M) = g \geq 3$, it seems to be difficult to see if any manifold M have a Heegaard diagram of genus g which satisfies our sufficient condition. In §4, by a technique of DS-diagrams which does not need the sufficient condition in §3, we make sure that a DS-diagram induced from any Heegaard-diagram with genus 2 can be deformed into the one with E-cycle whose block number is 2, that is, we will show the following main theorem of this paper.

THEOREM 1. *$Bl(M) = HG(M) = 2$ for any orientable closed 3-manifold M with $HG(M) = 2$.*

Applying the same method as our proof of Theorem 1, we obtain $Bl(M) \leq 4$ for any M with $HG(M) = 3$ in §5. However it seems to be difficult to show whether $Bl(M) = HG(M)$

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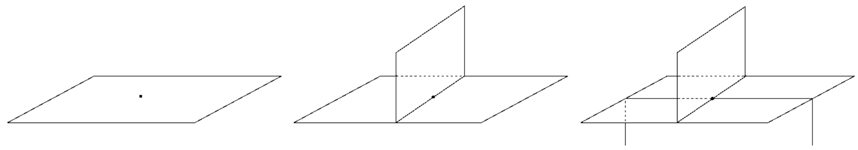

FIGURE 1.

for any M with $HG(M) \geq 3$. We will investigate what is the difficulty for proving $Bl(M) = HG(M) = 3$.

The next section, §2, is devoted to review some definitions and some basic properties of the block number which we need in this paper.

2. Preliminaries and definitions

The notion of *fake surfaces*, *DS-diagrams*, and *DS-diagrams with E-cycle* were introduced by H. Ikeda ([3]–[5]). In this section, we shall review these concepts, and define the *block number* for an orientable closed 3-manifold.

2.1. Fake surfaces, simple spines and DS-diagrams. A finite 2-dimensional polyhedron P is called a *closed fake surface*, if any point has a neighborhood homeomorphic to an open subset of

$$(\mathbf{R}^2 \times \{0\}) \cup (\mathbf{R} \times \{0\} \times \mathbf{R}_+) \cup (\{0\} \times \mathbf{R} \times \mathbf{R}_-) \subset \mathbf{R}^3,$$

that is, the shape of a neighborhood of a point $x \in P$ is one of the three types shown in Figure 1.

A closed fake surface P is naturally stratified as $V(P) \subset S(P) \subset P$, where $V(P)$ is a finite set of points, called vertices, which play the role of 0 in the above subset of $\mathbf{R}^3$, and $S(P)$ is the singular set which consists of the points playing the role of $(t, 0, 0)$ or $(0, t, 0)$. A connected component of $P - S(P)$ is called a *face* of P, and a connected component of $S(P) - V(P)$ is called an *edge* of P. In this paper, we assume that a face of a closed fake surface is an open 2-disk, and an edge is an open arc.

If a closed fake surface P is embedded in a closed 3-manifold M, so that $M - P$ is homeomorphic to an open 3-ball $\mathbf{B}^3$, then we call P a *simple spine* of M. For a simple spine P of a closed 3-manifold M, cutting out along P, we obtain a 3-ball $\mathbf{B}^3$ with a gluing map on its boundary $S^2 = \partial\mathbf{B}^3$. We can formulate this situation as the following definition of a DS-diagram.

DEFINITION 2.1.1. A triple $\Delta = (G, f, P)$ is called a *DS-diagram* if

(1) G is a 3-regular graph embedded in the 2-sphere S^2,

(2) P is a closed fake surface,

(3) f is a local homeomorphism from S^2 onto P such that $f^{-1}(S(P)) = G$ and $f^{-1}(V(P)) = V_G$, where V_G is the set of the vertices of the graph G, and

(4) $\#f^{-1}(x) = 4$ for any $x \in V(P)$, $\#f^{-1}(x) = 3$ for any $x \in S(P) - V(P)$, and $\#f^{-1}(x) = 2$ for any $x \in P - S(P)$.

Let $\mathbf{B}^3$ be a 3-ball with the boundary $\partial\mathbf{B}^3 = S^2$. For a DS-diagram $\Delta = (G, f, P)$, gluing $\partial\mathbf{B}^3$ by the map f, we get a closed 3-manifold $M(\Delta) = \mathbf{B}^3/f$ with a simple spine $P = S^2/f$ of $M(\Delta)$. An element of V_G is called a vertex of Δ, a connected component of $S^2 - G$ is called a face of Δ and a connected component of $G - V_G$ is called an edge of Δ. For a face σ of P, $f^{-1}(\sigma)$ consist of exactly two faces σ^+ and σ^- of Δ such that $f(\sigma^+) = f(\sigma^-) = \sigma$. We say that $\sigma^\pm$ is a *spouse* of $\sigma^\mp$, respectively.

We always assume that the closed 3-manifold $M = M(\Delta)$ represented by a DS-diagram Δ is orientable.

2.2. DS-diagrams with E-cycle and E-data. A *cycle* $c = \{E_1, E_2, \cdots, E_v\}$ of a graph G is a sequence of edges E_j ($E_i \neq E_j$ if $i \neq j$) such that the closure of $\bigcup_j E_j$ is a simple closed curve.

DEFINITION 2.2.1. Let $\Delta = (G, f, P)$ be a DS-diagram, e be a cycle of G (the underlying space of the closure of the cycle e is denoted by the same letter e), and H^+ and H^- be the connected components of $S^2 - e$. The cycle e of the graph G is said to be an *E-cycle*, if it satisfies that

(1) $\#(e \cap f^{-1}(x)) = 2$ for any $x \in V(P)$,
(2) $\#(e \cap f^{-1}(x)) = 1$ for any $x \in S(P) - V(P)$, and
(3) the restriction of f onto H^+ (or H^-) is a bijection.

There exists a DS-diagram which admits some different E-cycles. When we consider a DS-diagram with E-cycle, we always assume that an E-cycle e is specified, and has a fixed orientation. Moreover we assume that S^2 on which the graph G is drawn is oriented, and the component H^+ of $S^2 - e$ of which the restricted orientation is compatible with e, that is $\partial H^+ = e$ (in the oriented sense), is called the *positive region*. We represent a DS-diagram with E-cycle by a four-tuple $\Delta = (G, f, P; e)$. It is known that any closed 3-manifold can be represented by a DS-diagram with E-cycle ([5], [8]), and moreover it is known that, in the orientable case, a DS-diagram with E-cycle defines a closed manifold (see [8] and [9]).

Let $\Delta = (G, f, P; e)$ be a DS-diagram with E-cycle, which is assumed to be given on the unit sphere in $\mathbf{R}^3$. For each vertex $v \in V(P)$ of the spine $P = \partial\mathbf{B}^3/f$, there are exactly two vertices v^+ and v^- of the graph G such that $f(v^+) = f(v^-) = v$, and v^+ and v^- are both on the E-cycle e. These two vertices are characterized by the condition

(∗) $U \cap (G - e) \subset H^+$ (or H^-) for sufficiently small neighborhood U (in S^2) of v^+ (respectively v^-).

Each vertex $v \in V(P)$ is classified into the two cases (ℓ) or (r) shown in Figure 2 (cf. [8] and [9]). We define the *code* $\phi(v)$ so that $\phi(v) = \ell$ (or r) if v is the vertex of type (ℓ) (respectively (r)).

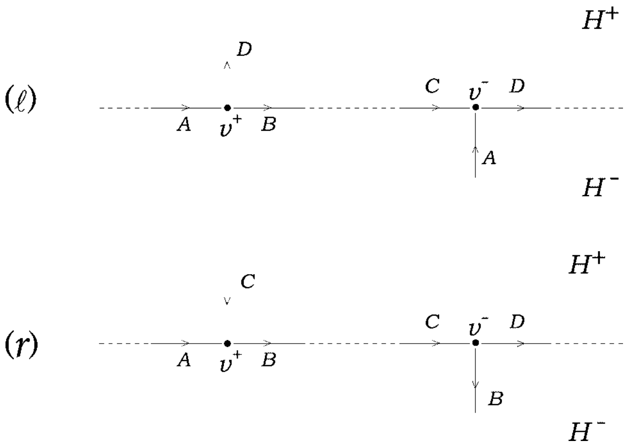

FIGURE 2.

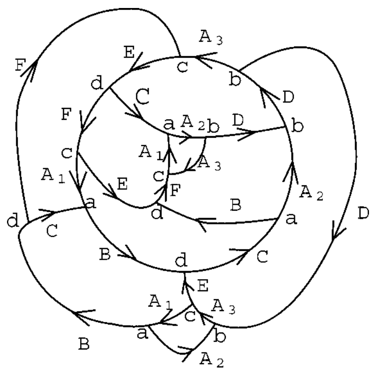

FIGURE 3.

Suppose that $V(P)$ consists of n points $v_1, v_2, \cdots, v_n$. An *E-data*

$$\mathcal{E}(\Delta) = (\phi, \mathcal{A}(\Delta))$$

of $\Delta = (G, f, P; e)$ is a pair of the code $\phi(v_k)$ for each $v_k \in V(P)$ and the arrangement $\mathcal{A}(\Delta)$ of $2n$ points $v_k^{\pm}$ on the oriented circle $S^1 \equiv e$.

It is known that an E-data completely determines a DS-diagram with E-cycle, and there are several methods for representing an E-data (see [8], [9], [10], and see also [1]). In this paper, we use a representation of an E-data by a coded sequence which is introduced in [2].

EXAMPLE 1. Let $\Delta = (G, f, P; e)$ be the DS-diagram drawn in Figure 3, which has four vertices $V(P) = \{a, b, c, d\}$. The codes of these vertices are $\phi(a) = \phi(c) = \phi(d) = \ell$

and $\phi(b) = r$, and the arrangement $\mathcal{A}(\Delta)$ is given by a sequence

$$\mathcal{A}(\Delta) = \{a^+b^+b^-c^-d^+c^+a^-d^-\},$$

which indicates the cyclical order of the points in $V(G) \cap e$. Of course we may employ a cyclically permuted sequence for representing the arrangement. The E-data $\mathcal{E}(\Delta)$ can be represented by the following coded sequence:

$$\mathcal{E}(\Delta) = \{a^{+\ell}b^{+r}b^{-r}c^{-\ell}d^{+\ell}c^{+\ell}a^{-\ell}d^{-\ell}\}.$$

We call a consecutive (in the sense of cyclical order) sequence of symbols in $\mathcal{A}(\Delta)$ a *subword* of an arrangement $\mathcal{A}(\Delta)$. If any symbol in a subword W has the signature $+$ (or $-$), then we call W a *positive subword* (respectively a *negative subword*).

DEFINITION 2.2.2. A maximal positive (or negative) subword of an arrangement $\mathcal{A}(\Delta)$ is called a *positive block* (respectively a *negative block*) of $\mathcal{A}(\Delta)$.

For example, the arrangement $\mathcal{A}(\Delta)$ in Example 1 has two positive blocks $W_1^+ = a^+b^+$, $W_2^+ = d^+c^+$ and two negative blocks $W_3^- = b^-c^-$, $W_4^- = a^-d^-$, and we can represent $\mathcal{A}(\Delta)$ as $\mathcal{A}(\Delta) = W_1^+W_3^-W_2^+W_4^-$. Also for a coded sequence we can define the notion of a “subword”, a “positive subword” and a “negative subword”.

The block number of a DS-diagram with E-cycle of an orientable closed 3-manifold are defined as follow.

DEFINITION 2.2.3. The *block number* $bl(\Delta)$ of a DS-diagram Δ with E-cycle is defined to be the number of positive blocks included in the arrangement $\mathcal{A}(\Delta)$.

2.3. Moves for E-data and block number of a 3-manifold. The moves for DS-diagrams with E-cycle, generators of the deformations of DS-diagrams with E-cycle which preserve the represented manifold, were introduces in [9]. Here we will review those moves in terms of the coded sequences for E-data.

DEFINITION 2.3.1 (The first regular move R_1). Let Δ be a DS-diagram with E-cycle whose E-data $\mathcal{E}(\Delta)$ includes three subwords $W_1 = a^{-\ell}b^{+\ell}$, $W_2 = a^{+\ell}x^{+\ell}$ and $W_3 = x^{-\ell}b^{-\ell}$. Then R_1 is defined to be a deformation changing Δ into Δ' with the E-data $\mathcal{E}(\Delta')$ in which the subwords W_k ($k = 1, 2, 3$) are replaced by $W_1' = b^{+\ell}a^{-\ell}$, $W_2' = a^{+\ell}$ and $W_3' = b^{-\ell}$ respectively. By this move, $a^{-\ell}$ and $b^{+\ell}$ mutually exchange their places in the coded sequence, and $x^{\pm\ell}$ are eliminated. Hence, for the spines P represented by Δ and P' represented by Δ', we have $\#V(P) = \#V(P') + 1$.

DEFINITION 2.3.2 (The second regular move R_2). Let Δ be a DS-diagram with E-cycle whose E-data $\mathcal{E}(\Delta)$ includes two subwords $W_1 = x^{-\ell}y^{-r}$ (or $y^{-r}x^{-\ell}$) and $W_2 = x^{+\ell}y^{+r}$ (or $y^{+r}x^{+\ell}$). Then R_2 is defined to be a deformation eliminating these two subwords W_1 and W_2. Hence, for two spines P represented by Δ and P' represented by $\Delta' = R_2(\Delta)$ obtained by the move R_2, we have $\#V(P) = \#V(P') + 2$.

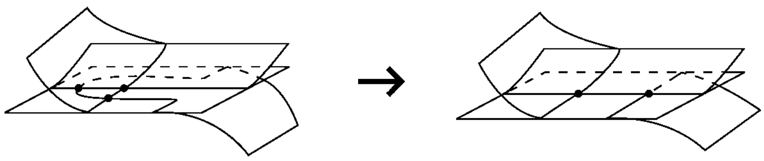

FIGURE 4. The first regular move R_1.

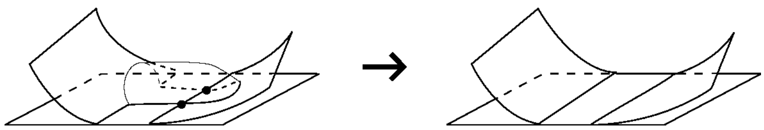

FIGURE 5. The second regular move R_2.

The inverses of the above moves R_j are denoted by R_j^{-1}. If there is a sequence $\Delta = \Delta_0, \Delta_1, \Delta_2, \cdots \Delta_n = \Delta'$ of DS-diagrams with E-cycle such that Δ_k is obtained from Δ_{k-1} by one of the moves $R_1^{\pm 1}$ and $R_2^{\pm 1}$, then Δ' is said to be *regularly equivalent* to Δ, denoted by $\Delta' \cong \Delta$.

Another move, called the *surgery move*, is defined as follows.

DEFINITION 2.3.3 (The surgery move S). If Δ is a DS-diagram with E-cycle whose E-data $\mathcal{E}(\Delta)$ includes a subword $W = x^{-\ell} y^{+\ell} z^{+\ell} z^{-\ell} w^{-r} y^{-\ell} w^{+r} x^{+\ell}$, then the surgery move S gives a DS-diagram $\Delta' = S(\Delta)$ with the E-data $\mathcal{E}(\Delta')$ obtained from $\mathcal{E}(\Delta)$ by eliminating the subword W.

If there is a sequence $\Delta = \Delta_0, \Delta_1, \Delta_2, \cdots \Delta_n = \Delta'$ of DS-diagrams with E-cycle such that Δ_k is obtained from Δ_{k-1} by one of the moves $R_1^{\pm 1}$, $R_2^{\pm 1}$ and $S^{\pm 1}$, then Δ' is said to be *equivalent* to Δ, denoted by $\Delta' \sim \Delta$.

The following theorem was shown in [9].

THEOREM 2. *Let Δ and Δ' be DS-diagrams with E-cycle. Then there is orientation preserving homeomorphism from $M(\Delta)$ onto $M(\Delta')$ if and only if $\Delta \sim \Delta'$.*

DEFINITION 2.3.4. Let M be a orientable closed 3-manifold. Let Δ be a DS-diagram with E-cycle representing $M(\Delta) = M$. The *block number* $Bl(M)$ of an orientable closed 3-manifold M is defined by

$$Bl(M) = \min\{bl(\Delta') | \Delta' \sim \Delta\}\,.$$

Obviously $Bl(M)$ is a topological invariant for closed orientable 3-manifolds. The next proposition was shown in [2].

PROPOSITION 1. *$HG(M) \leq Bl(M)$ for any closed orientable 3-manifold M except for $S^2 \times S^1$.*

3. A sufficient condition for $Bl(M) = HG(M)$

In this section we will give a sufficient condition for a Heegaard diagram of a closed 3-manifold M with genus g which guarantees the existence of a DS-diagram Δ with E-cycle such that $M(\Delta) \cong M$ and $bl(\Delta) = g$.

Let M be a closed 3-manifold having a Heegaard diagram $(M_1, M_2; \vec{D}_1, \vec{D}_2)$ with genus g, where M_1 and M_2 are handlebodies such that $\partial M_1 = \partial M_2$ and $M = M_1 \cup M_2$ with a gluing map $h : \partial M_2 \to \partial M_1$, and $\vec{D}_1 = \{\alpha_1, \alpha_2, \cdots \alpha_g\}$ and $\vec{D}_2 = \{\beta_1, \beta_2, \cdots \beta_g\}$ are complete meridian disk systems of M_1 and M_2 respectively. Deforming the meridian disks $\vec{D}_1 \cup \vec{D}_2$, if necessary, we may assume that $\partial \vec{D}_1 \cup \partial \vec{D}_2$ is connected and each connected components of $\partial M_i - (\partial \vec{D}_1 \cup \partial \vec{D}_2)$ $(i = 1, 2)$ is an open 2-cell. Then the union $\tilde{P} = F \cup \vec{D}_1 \cup \vec{D}_2$ of the Heegaard surface $F \equiv \partial M_i$ and the meridian disks in the systems $\vec{D}_i$ ($i = 1,\ 2$) forms a closed fake surface embedded in M, whose singularity $S(\tilde{P})$ consists of meridian curves $\partial \vec{D}_1 \equiv \bigcup \partial \alpha_i$ and $\partial \vec{D}_2 \equiv \bigcup \partial \beta_i$, and whose vertices $V(\tilde{P})$ consists of $\vec{D}_1 \cap \vec{D}_2$. This fake surface $\tilde{P}$ is not a simple spine because its complement is a union of two open 3-balls $M_1 - \vec{D}_1$ and $M_2 - \vec{D}_2$. A face of the fake surface $\tilde{P}$ is either one of meridian disks α_i and β_i or a connected component of $F - (\partial \vec{D}_1 \cup \partial \vec{D}_2)$. Removing a face λ of $\tilde{P}$ which is on F and whose closure is a compact 2-disk, we can obtain a simple spine $P = \tilde{P} - \lambda$ which we call a *derived spine*. However the DS-diagram of a derived spine is not necessarily one with E-cycle. In what follows we will show that we obtain a DS-diagram with E-cycle if the face to be removed has some good property.

DEFINITION 3.0.1 (A good $2g$-gon). A face λ_0 of the fake surface $\tilde{P}$ is said to be a *good* $2g$*-gon* if

(1) the closure of λ_0 is a compact 2-disk,

(2) $V(\tilde{P}) \cap \partial \lambda_0$ consists of exactly $2g$ points, and

(3) any two edges on $\partial \lambda_0$ are on mutually different meridian, that is, $\partial \lambda_0 \cap \partial \alpha_i \neq \emptyset$ and $\partial \lambda_0 \cap \partial \beta_i \neq \emptyset$ for any $i = 1, 2, \cdots, g$.

THEOREM 3. *If the closed fake surface $\tilde{P} = F \cup \vec{D}_1 \cup \vec{D}_2$ given by a Heegaard diagram of genus g admits a good $2g$-gon λ_0, then the derived simple spine $P \equiv \tilde{P} - \lambda_0$ has an E-cycle, and its DS-diagram Δ satisfies $bl(\Delta) = g$.*

PROOF. For simplicity, we will give the proof only for the case $HG(M) = 2$. We can prove the lemma for general cases by a quite similar method.

Let $(M_1, M_2; \vec{D}_1, \vec{D}_2)$ be a genus-2 Heegaard diagram of M, where $\vec{D}_1 = \{\alpha_1, \alpha_2\}$ and $\vec{D}_2 = \{\beta_1, \beta_2\}$ are complete meridian disk systems for the handlebodies M_1 and M_2 respectively. By h we denote the gluing map $h : \partial M_2 \to \partial M_1$.

Assume λ_0 to be a good 4-gon on the closed fake surface $\tilde{P} = F \cup \vec{D}_1 \cup \vec{D}_2$, and consider the derived spine $P = \tilde{P} - \lambda_0$. The DS-diagram $\Delta = (G, f, P)$ of the derived

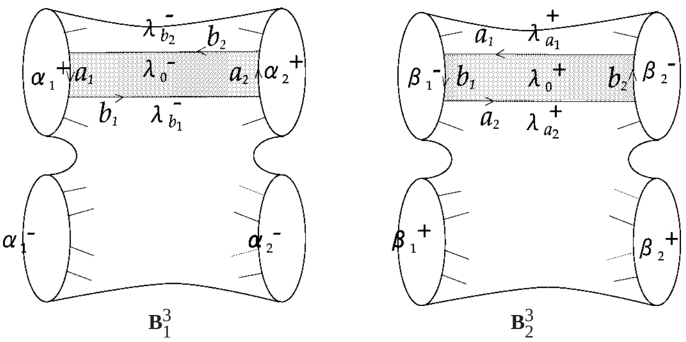

FIGURE 6. $cl(M - N(\tilde{P}))$.

spine P can be obtained in the following manner. Let $\mathbf{B}_i^3$ $(i = 1, 2)$ be the 3-ball obtained by cutting the handlebody M_i along the meridian disks in the system $\vec{D}_i$. The gluing map h and the cutting off operation naturally define the identification map $\tilde{h} : \partial\mathbf{B}_1^3 \cup \partial\mathbf{B}_2^3 \to \tilde{P}$. For a face λ of $\tilde{P}$ which is on the Heegaard surface F, one of the components of $\tilde{h}^{-1}(\lambda)$ is on $\partial\mathbf{B}_1^3$, denoted by λ^-, and the other is on $\partial\mathbf{B}_2^3$, denoted by λ^+. The two components of $\tilde{h}^{-1}(\alpha_i)$ which we denote by α_i^- and α_i^+, are both on $\partial\mathbf{B}_1^3$, and the two components of $\tilde{h}^{-1}(\beta_i)$, which we denote by β_i^- and β_i^+, are both on $\partial\mathbf{B}_2^3$. Since λ_0 is a good 4-gon, we may assume that the inverse images λ_0^- and λ_0^+ are situated on $\partial\mathbf{B}_i^3$ just as in Figure 6. That is, λ_0^- is adjacent to four faces α_1^+, $\lambda_{b_1}^-$, α_2^+, and $\lambda_{b_2}^-$ where $\partial\lambda_0^- \cap \partial\alpha_1^+ = a_1$, $\partial\lambda_0^- \cap \partial\lambda_{b_1}^- = b_1$, $\partial\lambda_0^- \cap \partial\alpha_2^+ = a_2$ and $\partial\lambda_0^- \cap \partial\lambda_{b_2}^- = b_2$. And λ_0^+ is adjacent to four faces $\lambda_{a_1}^+$, β_1^-, $\lambda_{a_2}^+$ and β_2^- where $\partial\lambda_0^+ \cap \partial\lambda_{a_1}^+ = a_1$, $\partial\lambda_0^+ \cap \partial\beta_1^- = b_1$, $\partial\lambda_0^+ \cap \partial\lambda_{a_2}^+ = a_2$ and $\partial\lambda_0^+ \cap \partial\beta_2^- = b_2$. Gluing λ_0^- and λ_0^+, we obtain a 3-ball $\mathbf{B}^3$ (see Figure 7 and 8), on whose boundary $\partial\mathbf{B}^3$ the identification map $f : \partial B^3 \to P = \tilde{P} - \lambda_0$ is naturally defined. Together with the graph $G = f^{-1}(S(P))$ on $\partial\mathbf{B}^3$, the triple $\Delta = (G, f, P)$ becomes the DS-diagram for the derived spine P.

After gluing λ_0^- and λ_0^+, the edges a_1, b_1, a_2 and b_2 of $\tilde{P}$ are not edges of P any more. Two faces α_1^+ of $\mathbf{B}_1^3$ and $\lambda_{a_1}^+$ of $\mathbf{B}_2^3$ become one face of Δ. We denote this face by α_1^+. The faces of Δ which result from $\alpha_i^\pm$ and $\lambda_{a_i}^\pm$ (or $\beta_i^\pm$ and $\lambda_{b_i}^\pm$) of $\mathbf{B}_1^3 \cup \mathbf{B}_2^3$ are denoted by $\alpha_i^\pm$ (or respectively $\beta_i^\pm$) $(i = 1, 2)$. Consider the cycle $e = (\partial\alpha_1^+ - a_1) \cup (\partial\beta_1^- - b_1) \cup (\partial\alpha_2^+ - a_2) \cup (\partial\beta_2^- - b_2)$ of the graph G. The component of $S^2 - e$ which includes the faces α_i^-, β_i^- is denoted by H^-.The other component, which consists of the faces α_i^+ and β_i^+ and λ_j^+, is denoted by H^+. The restriction of f onto H^+ (or H^-) is a bijection. Since the singularity $S(P)$ consists of $\partial\vec{D}_1 \cup \partial\vec{D}_2 - \partial\lambda$, each edge X on P belongs to $(\partial\alpha_i - a_i)$ or $(\partial\beta_j - b_j)$. Three edges $f^{-1}(X)$ of G appear on e, H^- and H^+. Since the vertices $V(P)$ consists of $\partial\vec{D}_1 \cap \partial\vec{D}_2 - \lambda$, each vertex v of P belongs to $(\partial\alpha_i - a_i) \cap (\partial\beta_j - b_j)$. Four vertices $f^{-1}(v)$

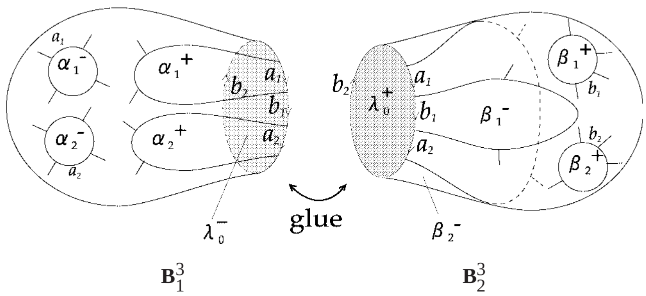

FIGURE 7. gluing $\mathbf{B}_1^3$ and $\mathbf{B}_2^3$.

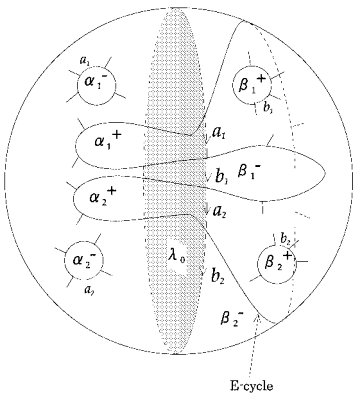

FIGURE 8. A DS-diagram Δ with E-cycle.

of G appear on $\partial\alpha_i^+ - a_i \subset e$, $\partial\beta_j^- - b_j \subset e$, $\partial\alpha_i^- - a_i \subset H^-$ and $\partial\beta_j^+ - b_j \subset H^+$. So the cycle e turns out to be an E-cycle of Δ.

We can see that $bl(\Delta) = 2$ because, for any vertices v^+ and v^- on e, v^+ is included in $\partial\beta_1^- - b_1$ or $\partial\beta_2^- - b_2$, v^- is included in $\partial\alpha_1^+ - a_1$ or $\partial\alpha_2^+ - a_2$. □

Maybe our main theorem can be proved by applying the above theorem, that is, we can deform the meridian disk systems of any Heegaard diagram with genus 2 so that the deformed diagram has a good 4-gon. However it seems to be hard to find a good $2g$-gon for a general Heegaard diagram with genus $g \geq 3$. So, in order to see what is the obstruction for getting a DS-diagram with the block number g by a Heegaard diagram with genus g, we will try to prove the main theorem by using the *remodeling algorithm* in [7], which gives an algorithm for obtaining a DS-diagram with E-cycle from a given DS-diagram without E-cycle. In §5, we will show that a similar method gives an estimate $Bl(M) \leq 4$ for any M with $HG(M) = 3$, and moreover we will explain what is the difficulty for proving $Bl(M) = HG(M)$ in the case of $HG(M) = 3$.

4. Proof of the main theorem

In this chapter, M is assumed to be an orientable closed 3-manifold with genus 2.

The proof of the main theorem is divided into the following three steps:

step 1; defining a derived spine $P_0 \equiv \tilde{P} - \lambda_0$,
step 2; remodeling Δ_0 of P_0 into another DS-diagram Δ_1 having an E-cycle,
step 3; reducing the block number of Δ_1 to 2 by applications of moves.

The remodeling algorithm in step 2 needs *mark lines*, each of which is a proper arc on a face of the spine P_0 (see §4.1 below and also [7]), and, for the successful operation in step 3, these mark lines must be carefully chosen. Also the face λ_0 of the closed fake surface $\tilde{P}$ is so chosen that we can take desirable mark lines on the derived spine $P_0 \equiv \tilde{P} - \lambda_0$.

4.1. A DS-diagram of a derived spine P**.** Let $(M_1, M_2; \vec{D}_1, \vec{D}_2)$ be a Heegaard diagram with genus 2 giving a closed 3-manifold M. Deforming the meridian disks $\vec{D}_1 \cup \vec{D}_2$ if necessary, we may assume that $\partial\vec{D}_1 \cup \partial\vec{D}_2$ is connected and each connected components of $\partial M_i - (\partial\vec{D}_1 \cup \partial\vec{D}_2)$ $(i = 1, 2)$ is an open 2-cell. The notations M_i, $\vec{D}_i$, $\mathbf{B}_i^3$, F, $\tilde{P}$ and $\tilde{h}$ follow ones in §3. Let λ_0, λ_1 and λ_1' be three faces of the fake surface $\tilde{P}$ such that

(1) three faces λ_0, λ_1 and λ_1' are included in $F \subset \tilde{P}$,

(2) a face λ_0 is a 2-gon face whose inverse image λ_0^- is adjacent to α_2^+, and the other inverse image λ_0^+ is adjacent to β_2^-,

(3) the inverse image λ_1^- is adjacent to both λ_0^- and α_1^+,

(4) the inverse image $\lambda_1'^+$ is adjacent to β_1^- and β_2^-.

The inverse images λ_0^-, λ_1^- and $\lambda_1'^-$ on $\partial\mathbf{B}_1^3$ and λ_0^+, λ_1^+ and $\lambda_1'^+$ on $\partial\mathbf{B}_2^3$ are situated just as in Figure 9. Let x, y, A, B_1, B_2, C_1, C_2, A' and B' be edges of $\tilde{P}$ where $y = \partial\lambda_0 \cap \partial\alpha_2$, $x = \partial\lambda_0 \cap \partial\beta_2$, $A \subset \tilde{h}(\partial\lambda_1^- \cap \partial\alpha_1^+)$, $A' \subset \tilde{h}(\partial\lambda_1'^+ \cap \partial\beta_1^-)$, $B' \subset \tilde{h}(\partial\lambda_1'^+ \cap \partial\beta_2^-)$, $\partial\alpha_2 = \cdots B_1 y B_2 \cdots$ and $\partial\beta_2 = \cdots C_1 x C_2 \cdots B' \cdots$.

Let $p \in A$, $q \in B_2$, $p' \in A'$ and $q' \in B'$ be four points. We call a pair of simple arcs γ_1^- on $\partial\mathbf{B}_1^3$ and γ_2^+ on $\partial\mathbf{B}_2^3$ *pre mark-lines*, if they satisfy that

(1) γ_1^- is a directed arc on λ_1^- such that $\tilde{h}(\gamma_1^-)$ goes from p to q,

(2) γ_2^+ is a directed arc on $\lambda_1'^+$ such that $\tilde{h}(\gamma_2^+)$ goes from p' to q' and

(3) $\tilde{h}(\gamma_1^-) \cap \tilde{h}(\gamma_2^+) = \emptyset$.

These pre mark-lines become mark-lines in step 2.

Note that, for γ_1^- (or γ_2^+), there is the spouse γ_1^+ (or γ_2^-) such that $\tilde{h}(\gamma_1^-) = \tilde{h}(\gamma_1^+) = \gamma_1$ (respectively $\tilde{h}(\gamma_2^+) = \tilde{h}(\gamma_2^-) = \gamma_2$), see Figure 9.

LEMMA 4.1.1. *There exists a Heegaard diagram of* M *which gives a closed fake surface admitting a 2-gon* λ_0 *and pre mark-lines* γ_1 *and* γ_2.

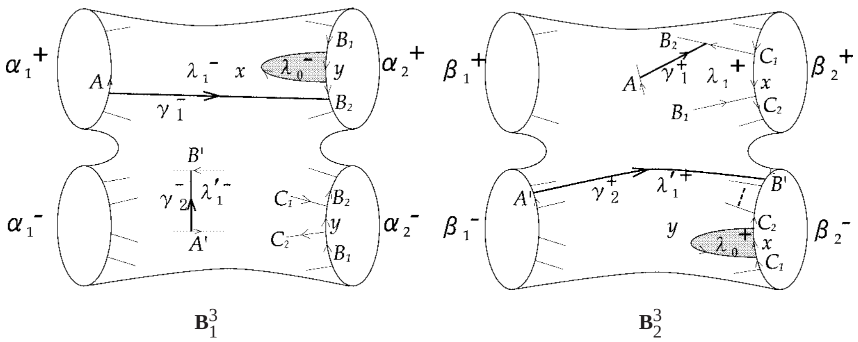

FIGURE 9. The pre mark-lines γ_1^- and γ_2^- on $\partial\mathbf{B}_1^3$, and γ_1^+ and γ_2^+ on $\partial\mathbf{B}_2^3$.

The proof of Lemma 4.1.1 will be shown in §4.5.1. We obtain a derived simple spine $P_0 \equiv \tilde{P} - \lambda_0$ whose singularity $S(P_0)$ consists of $\partial\alpha_1 \cup (\partial\alpha_2 - y)$ and $\partial\beta_1 \cup (\partial\beta_2 - x)$, and whose vertices $V(P_0)$ consists of $\partial\vec{D}_1 \cap \partial\vec{D}_2 - \partial\lambda_0$. The diagram of Figure 10 is a DS-diagram of a derived spine $P_0 = \tilde{P} - \lambda_0$. We denote this DS-diagram by Δ_0. But the DS-diagram Δ_0 have no E-cycle in general. So we will deform Δ_0 into one with E-cycle by applying the algorithm of [7].

4.2. Remodeling a DS-diagram into one with E-cycle. Applying the remodeling algorithm to the DS-diagram Δ_0 of the derived spine P_0 in the previous subsection, we construct a DS-diagram Δ_1 with E-cycle. The algorithm is carried out along the mark-lines γ_1 and γ_2, and this operation produces some new vertices. These new vertices of the spine P_1 represented by Δ_1 will be denoted by

$$a_i\ (1 \le i \le 12)\,, \quad b_i\ (1 \le i \le 9)\,, \quad c_i\ (1 \le i \le p)\,, \quad d_i\ (1 \le i \le p)\,,$$

where the number p depends on the initial Heegaard diagram. The set $V(P_1)$ of vertices of P_1 consists of these new vertices and the original vertices $V(P_0)$. We can see (cf. §4.5) that the codes of the new vertices are

$$\phi(a_1) = \ell\,,\quad \phi(a_2) = r\,,\quad \phi(a_3) = r\,,\quad \phi(a_4) = \ell\,,\quad \phi(a_5) = \ell\,,\quad \phi(a_6) = r\,,$$
$$\phi(a_7) = r\,,\quad \phi(a_8) = \ell\,,\quad \phi(a_9) = \ell\,,\quad \phi(a_{10}) = r\,,\quad \phi(a_{11}) = \ell\,,\quad \phi(a_{12}) = r\,,$$

$$\phi(b_1) = r\,,\quad \phi(b_2) = r\,,\quad \phi(b_3) = r\,,\quad \phi(b_4) = \ell\,,\quad \phi(b_5) = r\,,\quad \phi(b_6) = r\,,$$
$$\phi(b_7) = \ell\,,\quad \phi(b_8) = \ell\,,\quad \phi(b_9) = r\,,$$

$$\phi(c_1) = r\,,\quad \phi(c_p) = \ell\,,$$

$$\phi(d_1) = \ell\,,\quad \phi(d_p) = r.$$

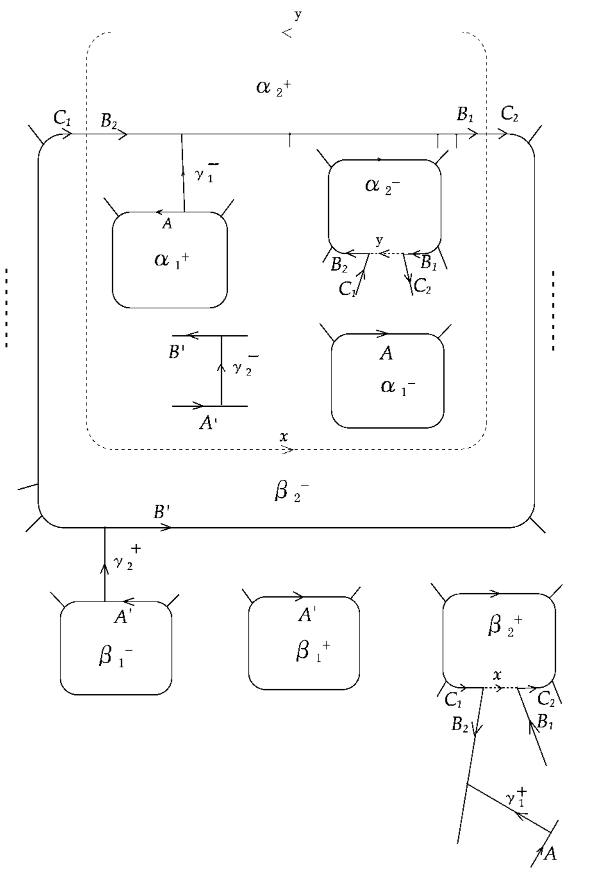

FIGURE 10. The DS-diagram Δ_0 of the derived spine $P_0 = \tilde{P} - \lambda_0$.

The codes $\phi(c_i)$, $\phi(d_i)$ $(2 \le i \le p-1)$ depend on the initial Heegaard diagram. As a consequence of the remodeling operation together with Lemma 4.1.1, we get the following lemma, whose proof is given in §4.5.

LEMMA 4.2.1. *Let M be an oriented closed* 3*-manifold with* $HG(M) = 2$*. Then there exists a DS-diagram with E-cycle* Δ_1 *which satisfies* $M = M(\Delta_1)$ *and has a representation of the E-data*:

$$\begin{aligned}\mathcal{E}(\Delta_1) = \{& a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} a_3^{+r} a_4^{+\ell} a_5^{-\ell} a_6^{-r} a_7^{+r} a_8^{+\ell} a_9^{-\ell} a_{10}^{-r} a_2^{+r} a_5^{+\ell} a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^- \\ & a_6^{+r} a_9^{+\ell} a_8^{-\ell} a_3^{-r} a_{10}^{+r} c_p^{+\ell} \Upsilon_1^+ c_1^{+r} b_1^{+r} b_2^{+r} b_3^{+r} d_1^{+\ell} \Upsilon_2^+ d_p^{+r} a_{11}^{+\ell} a_4^{-\ell} a_7^{-r} a_{12}^{+r} a_1^{+\ell} \Lambda_2^- \\ & \Omega_{21}^+ b_4^{-\ell} b_5^{-r} b_6^{+r} b_2^{-r} b_7^{+\ell} b_8^{-\ell} b_9^{-r} \Omega_{11}^+ c_1^{-r} d_1^{-\ell} \Omega_{12}^+ \\ & b_5^{+r} b_3^{-r} b_8^{+\ell} b_7^{-\ell} b_6^{-r} b_9^{+r} b_1^{-r} b_4^{+\ell} \Omega_{22}^+ \}\,,\end{aligned} \tag{1}$$

where

(1) $\Upsilon_1^+ = c_{p-1}^+ c_{p-2}^+ \cdots c_2^+$ *and* $\Upsilon_2^+ = d_2^+ d_3^+ \cdots d_{p-1}^+$,

(2) Λ_1^-, Λ_2^- *are negative subwords and* Ω_{11}^+, Ω_{12}^+, Ω_{21}^+ *and* Ω_{22}^+ *are positive subwords,*

(3) $\Lambda_1^- \cup \Lambda_2^- - (\bigcup c_i^- \cup \bigcup d_i^-)$, Ω_{11}^+, Ω_{12}^+, Ω_{21}^+ *and* Ω_{22}^+ *consist of the vertices inherited from the original Heegaard-diagram,*

(4) $f(\Lambda_1^-) \cup f(\Lambda_2^-) = f(\Omega_{11}^+) \cup f(\Omega_{12}^+) \cup f(\Omega_{21}^+) \cup f(\Omega_{22}^+) \cup f(\Upsilon_1^+) \cup f(\Upsilon_2^+)$.

4.3. Some more moves. The E-data of Lemma 4.2.1 has $bl(\Delta_1) = 14$. In order to prove the main theorem, we have to reduce the block number of Δ_1. So we prepare several moves which are some compositions of regular moves and their inverses.

4.3.1. Exchange of vertices. Composing the regular moves $R_1^{\pm 1}$ and $R_2^{\pm 1}$, we can obtain eight moves $T_{\sigma_2,\sigma_3}^{\sigma_1}$ ($\sigma_j = \ell$ or r) (cf. [9]) which causes the following replacement of subwords in the coded sequence $\mathcal{E}(\Delta)$:

$$\begin{array}{lllll}
T_{\ell,\ell}^{\ell} & : & \text{(i) } b^{+\ell}a^{-\ell} \to a^{-\ell}b^{+\ell}, & \text{(ii) } a^{+\ell} \to a^{+\ell}x^{+\ell}, & \text{(iii) } b^{-\ell} \to x^{-\ell}b^{-\ell} \\
T_{\ell,r}^{\ell} & : & \text{(i) } b^{+r}a^{-\ell} \to a^{-\ell}b^{+r}, & \text{(ii) } a^{+\ell} \to a^{+\ell}x^{+r}, & \text{(iii) } b^{-r} \to b^{-r}x^{-r} \\
T_{r,\ell}^{\ell} & : & \text{(i) } b^{+\ell}a^{-r} \to a^{-r}b^{+\ell}, & \text{(ii) } a^{+r} \to x^{+r}a^{+r}, & \text{(iii) } b^{-\ell} \to x^{-r}b^{-\ell} \\
T_{r,r}^{\ell} & : & \text{(i) } b^{+r}a^{-r} \to a^{-r}b^{+r}, & \text{(ii) } a^{+r} \to x^{+\ell}a^{+r}, & \text{(iii) } b^{-r} \to b^{-r}x^{-\ell} \\
T_{\ell,\ell}^{r} & : & \text{(i) } a^{-\ell}b^{+\ell} \to b^{+\ell}a^{-\ell}, & \text{(ii) } a^{+\ell} \to a^{+\ell}x^{+r}, & \text{(iii) } b^{-\ell} \to x^{-r}b^{-\ell} \\
T_{\ell,r}^{r} & : & \text{(i) } a^{-\ell}b^{+r} \to b^{+r}a^{-\ell}, & \text{(ii) } a^{+\ell} \to a^{+\ell}x^{+\ell}, & \text{(iii) } b^{-r} \to b^{-r}x^{-\ell} \\
T_{r,\ell}^{r} & : & \text{(i) } a^{-r}b^{+\ell} \to b^{+\ell}a^{-r}, & \text{(ii) } a^{+r} \to x^{+\ell}a^{+r}, & \text{(iii) } b^{-\ell} \to x^{-\ell}b^{-\ell} \\
T_{r,r}^{r} & : & \text{(i) } a^{-r}b^{+r} \to b^{+r}a^{-r}, & \text{(ii) } a^{+r} \to x^{+r}a^{+r}, & \text{(iii) } b^{-r} \to b^{-r}x^{-r}\ .
\end{array}$$

where $x^{\pm\phi(x)}$ are new symbols appearing by each moves.

Each of the above eight moves $T_{\sigma_2,\sigma_3}^{\sigma_1}$ exchanges the positions of $a^{-\sigma_2}$ and $b^{+\sigma_3}$ in the coded sequence $\mathcal{E}(\Delta)$, and so we can use them for decreasing the block number without altering the regular equivalence class.

4.3.2. Block reducing lemma. By adequate successive applications of the moves $T_{\sigma,\sigma'}^{\ell}$ or $T_{\sigma,\sigma'}^{r}$, we have the following two lemmas. We call each of those lemmas a *block reducing lemma*.

LEMMA 4.3.1. *Let* $\Delta = (G, f, P; e)$ *be a DS-diagram with E-cycle having an arrangement of E-data of the form*

$$\mathcal{A}(\Delta) = W W_0^- U_1^+ U_2^- \cdots U_{2m-1}^+ U_{2m}^-$$

where W *is a subword,* W_0^- *and* U_{2k}^- *are negative blocks,* U_{2k+1}^+ *are positive blocks. If*

$$\left(\bigcup_{i=1}^{k} f(U_{2i-1}^+)\right) \cap f(U_{2k}^-) = \emptyset \tag{2}$$

for any $k = 1 \cdots m$, then we can obtain a DS-diagram Δ' with E-cycle such that $\Delta' \cong \Delta$ and $bl(\Delta') = bl(\Delta) - m$.

LEMMA 4.3.2. *Let $\Delta = (G, f, P; e)$ be a DS-diagram with E-cycle having an arrangement of E-data of the form*

$$\mathcal{A}(\Delta) = W W_0^+ U_1^- U_2^+ \cdots U_{2m-1}^- U_{2m}^+$$

where W is a subword, W_0^+ and U_{2k}^+ are positive blocks, U_{2k+1}^- are negative blocks. If

$$\left(\bigcup_{i=1}^{k} f(U_{2i-1}^-) \right) \cap f(U_{2k}^+) = \emptyset \tag{3}$$

for any $k = 1 \cdots m$, then we can obtain a DS-diagram Δ' with E-cycle such that $\Delta' \cong \Delta$ and $rbl(\Delta') = bl(\Delta) - m$.

These block reducing lemmas can be proved by applying the following lemmas 4.3.3 and 4.3.4 which were shown in [2] and give the moves exchanging the position of a positive subword and a negative subword in the arrangement $\mathcal{A}(\Delta)$.

LEMMA 4.3.3. *Let $\Delta = (G, f, P; e)$ be a DS-diagram having an arrangement of E-data of the form*

$$\mathcal{A}(\Delta) = U_1^+ U_2^- W_0^- W_1^+ W_2^- \cdots\cdots W_{2n-1}^+ W_{2n}^- W_{2n+1}^+ \tag{4}$$

where U_1^+, W_{2k-1}^+ are positive subwords and U_2^-, W_{2k}^- are negative subwords.

If $f(U_1^+) \cap f(U_2^-) = \emptyset$, $U_1^+ = b_\nu^+ b_{\nu-1}^+ \cdots b_2^+ b_1^+$ and $U_2^- = a_1^- a_2^- \cdots a_{\mu-1}^- a_\mu^-$, then, by adequate successive applications of the moves $T_{\sigma,\sigma'}^{\ell}$, we can obtain a DS-diagram Δ' such that $\Delta' \cong \Delta$ and the E-data $\mathcal{E}(\Delta')$ has an arrangement

$$\mathcal{A}(\Delta') = U_2^- U_1^+ \tilde{W}_0^- \tilde{W}_1^+ \tilde{W}_2^- \cdots\cdots \tilde{W}_{2n}^- \tilde{W}_{2n+1}^+ , \tag{5}$$

satisfying the following conditions (i)–(iii):

(i) *$\tilde{W}_{2k}^+$ (or $\tilde{W}_{2k-1}^-$) is a positive (respectively negative) subword which differs from W_{2k}^+ (respectively W_{2k-1}^-) only by the new symbols created by the moves,*

(ii) *$\tilde{W}_{2k}^+$ includes some new symbols if and only if the symbol a_i^+ appears in the original subword W_{2k}^+ for some $i = 1, 2, \cdots, \mu$,*

(iii) *$\tilde{W}_{2k-1}^-$ includes some new symbols if and only if the original subword W_{2k-1}^- contains the symbol b_j^- for some $j = 1, 2, \cdots, \nu$.*

By similar successive applications of $T_{\sigma,\sigma'}^{r}$ instead of $T_{\sigma,\sigma'}^{\ell}$, we have that

LEMMA 4.3.4. *Let $\Delta = (G, f, P; e)$ be a DS-diagram having an arrangement of E-data of the form*

$$\mathcal{A}(\Delta) = U_1^- U_2^+ W_0^+ W_1^- W_2^+ \cdots\cdots W_{2n-1}^- W_{2n}^+ W_{2n+1}^- .$$

If $f(U_1^-) \cap f(U_2^+) = \emptyset$, $U_1^- = b_\nu^- b_{\nu-1}^- \cdots b_2^- b_1^-$ *and* $U_2^+ = a_1^+ a_2^+ \cdots a_{\mu-1}^+ a_\mu^+$, *then, by adequate successive applications of the moves* $T^r_{\sigma,\sigma'}$, *we can obtain a DS-diagram* Δ' *such that* $\Delta' \cong \Delta$ *and the E-data* $\mathcal{E}(\Delta')$ *has an arrangement*

$$\mathcal{A}(\Delta') = U_2^+ U_1^- \tilde{W}_0^+ \tilde{W}_1^- \tilde{W}_2^+ \cdots\cdots \tilde{W}_{2n}^+ \tilde{W}_{2n+1}^- ,$$

satisfying the following conditions (i)–(iii):

(i) $\tilde{W}_{2k}^-$ (*or* $\tilde{W}_{2k-1}^+$) *is a negative* (*respectively positive*) *subword which differs from* W_{2k}^- (*respectively* W_{2k-1}^+) *only by the new symbols created by the moves,*

(ii) $\tilde{W}_{2k}^-$ *includes some new symbols if and only if the symbol* a_i^- *appears in the original subword* W_{2k}^- *for some* $i = 1, 2, \cdots, \mu$,

(iii) $\tilde{W}_{2k-1}^+$ *includes some new symbols if and only if the original subword* W_{2k-1}^+ *contains the symbol* b_j^+ *for some* $j = 1, 2, \cdots, \nu$.

In the case where the formula (4) is a blockwise representation, namely $W_0^- = W_{2n+1}^+ = \emptyset$, the resulting arrangement (5) gives a blockwise representation

$$\mathcal{A}(\Delta') = (U_1^+ \tilde{W}_1^+) \tilde{W}_2^- \tilde{W}_3^+ \cdots\cdots \tilde{W}_{2n-1}^+ (\tilde{W}_{2n}^- U_2^-) ,$$

and we have $bl(\Delta') = bl(\Delta) - 1$.

PROOF OF LEMMA 4.3.1. Let $\Delta = (G, f, P; e)$ be a DS-diagram with E-cycle having an arrangement of E-data of the form

$$\mathcal{A}(\Delta) = W W_0^- U_1^+ U_2^- \cdots U_{2m-1}^+ U_{2m}^-$$

having the condition (2). Since $f(U_1^+) \cap f(U_2^-) = \emptyset$, we can apply Lemma 4.3.3 and get a regularly equivalent DS-diagram $\Delta_1 = (G_1, f_1, P_1; e_1)$ having an arrangement of E-data of the form

$$\mathcal{A}(\Delta_1) = W_1 (W_{1,0}^- U_{1,2}^-)(U_{1,1}^+ U_{1,3}^+) U_{1,4}^- \cdots U_{1,2m-1}^+ U_{1,2m}^-$$

where each subword satisfies that

(1-i) W_1 differs from W only by new symbols $x_{1,j}^\pm$ produced by the moves,

(1-ii) $W_{1,0}^-$ differs from W_0^+ only by new symbols $x_{1,j}^-$ produced by the moves,

(1-iii) $U_{1,1}^+ = U_1^+$ and $U_{1,2}^- = U_2^-$

(1-iv) $U_{1,2k-1}^+$ ($k = 2, 3, \cdots m$) differs from U_{2k-1}^+ only by new symbols $x_{1,j}^+$ produced by the moves,

(1-v) $U_{1,2k}^-$ ($k = 2, 3, \cdots m$) differs from U_{2k}^- only by new symbols $x_{1,j}^-$ produced by the moves.

This implies that $bl(\Delta_1) = bl(\Delta) - 1$. By the fact $f(U_{2k}^-) \cap f(U_2^-) = \emptyset$ $(k = 2, \cdots, m)$ and the conditions (2), (1-iii)-(1-v), we can see the conditions

$$\left(\bigcup_{i=1}^{k} f_1(U_{1,2i-1}^+)\right) \cap f_1(U_{1,2k}^-) = \emptyset \tag{6}$$

for any $k = 2, 3, \cdots m$. Since $(f_1(U_{1,1}^+) \cup f_1(U_{1,3}^+)) \cap f_1(U_{1,4}^-) = \emptyset$, we can apply Lemma 4.3.3 and get a regularly equivalent DS-diagram $\Delta_2 = (G_2, f_2, P_2; e_2)$ having an arrangement of the form

$$\mathcal{A}(\Delta_2) = W_2(W_{2,0}^- U_{2,2}^- U_{2,4}^-)(U_{2,1}^+ U_{2,3}^+ U_{2,5}^+) U_{2,6}^- \cdots U_{2,2m-1}^+ U_{2,2m}^-$$

where each subword satisfies that

(2-i) W_2 differs from W_1 only by new symbols $x_{2,j}^{\pm}$ produced by the moves,

(2-ii) $W_{2,0}^-$ differs from W_1^+ only by new symbols $x_{2,j}^-$ produced by the moves,

(2-iii) $U_{2,2k-1}^+ = U_{1,2k-1}^+$ $(k = 1, 2)$ and $U_{2,2k}^- = U_{1,2k}^-$ $(k = 2)$

(2-iv) $U_{2,2k-1}^+$ $(k = 3, 4, \cdots m)$ differs from $U_{1,2k-1}^+$ only by new symbols $x_{2,j}^+$ produced by the moves,

(2-v) $U_{2,2k}^-$ $(k = 1, 3, 4, \cdots m)$ differs from $U_{1,2k}^-$ only by new symbols $x_{2,j}^-$ produced by the moves.

This implies that $bl(\Delta_2) = bl(\Delta) - 2$. By the fact $f(U_{2k}^-) \cap f(U_4^-) = \emptyset$ $(k = 3, \cdots m)$ and the conditions (6), (2-iii)–(2-v), we can see the conditions that

$$\left(\bigcup_{i=1}^{k} f(U_{2,2i-1}^+)\right) \cap f(U_{2,2k}^-) = \emptyset \tag{7}$$

for any $k = 3, 4, \cdots m$.

Applying Lemma 4.3.3 m-times similarly, we get a sequence $\Delta_k = (G_k, f_k, P_k; e_k)$ $(k = 1, 2, \cdots m)$ of regularly equivalent DS-diagrams, such that Δ_k is obtained from Δ_{k-1} by exchanging the positive subword $(U_{k-1,1}^+ U_{k-1,3}^+ \cdots U_{k-1,2k-1}^+)$ and the negative subwords $U_{k-1,2k}^-$. By $x_{k,j}^{\pm}$ $(1 \leq j \leq s_k)$, we denote the new symbols created by the k-th application of Lemma 4.3.3 which deforms Δ_{k-1} into Δ_k, where the number s_k is given by

$$s_k = (\#U_{2k}^-) \times \left(\prod_{j=1}^{k} (\#U_{2j-1}^+)\right).$$

The conditions (2) imply that the arrangement of m-th DS-diagram Δ_m can be represented as

$$\mathcal{A}(\Delta_m) = W_m \Theta_1^- \Theta_2^+$$
$$(\Theta_1^- = W_{m,0}^- U_{m,2}^- U_{m,4}^- U_{m,6}^- \cdots U_{m,2m}^-,\ \Theta_2^+ = U_{m,1}^+ U_{m,3}^+ U_{m,5}^+ \cdots U_{m,2m-1}^+),$$

and satisfies that

(m-i) W_m differs from W by some of new symbols $x^{\pm}_{k',j}$ for $k \le k' \le m$,

(m-ii) $W^-_{m,0}$ differs from W^-_0 by some of new symbols $x^-_{k',j}$ for $k \le k' \le m$, and

(m-iii) $U^+_{m,2k-1}$ differs from U^+_{2k-1} by some of new symbols $x^+_{k',j}$ for $k \le k' \le m$, and

(m-iv) $U^-_{m,2k}$ differs from U^-_{2k} by new symbols $x^-_{k,j}$.

This implies that $bl(\Delta_m) = bl(\Delta) - m$. So we complete the proof by $\Delta_m = \Delta'$. □

Lemma 4.3.2 can be proved by using the moves in Lemma 4.3.4 instead of those in Lemma 4.3.3.

4.3.3. Pair eliminating. There are some cases where, even if the block reducing lemma is not directly applicable, we can reduce the block number after preparatory moves. Here we will introduce such preparatory moves, called *pair eliminatings*.

Consider the case that the given E-data includes subwords $W_0 = a^{+\ell} W^- b^{+r}$ and $V_0 = a^{-\ell} b^{-r}$, where $W^- = w_1^{-\phi(w_1)} w_2^{-\phi(w_2)} \cdots w_\zeta^{-\phi(w_\zeta)}$ is a negative subword disjoint from V_0. Then, making successive ζ times applications of the moves $T^{\ell}_{\phi(w_i),\ell}$ ($\sigma = \ell$ or r), we obtain an E-data in which the subwords W_0, V_0 and each of $w_i^{+\phi(w_i)}$ ($i = 1, 2, \cdots \zeta$) are replaced as following

$$\text{(i)} \quad a^{+\ell} W^- b^{+r} \to W^- a^{+\ell} b^{+r}\,, \quad \text{(ii)} \quad a^{-\ell} b^{-r} \to V^- a^{-\ell} b^{-r}\,,$$
$$\text{(iii)} \quad w_i^{+\phi(w_i)} \to W_i^+ (i = 1, 2, \cdots \zeta)$$

where $V^- = v_1^{-\phi(v_1)} v_2^{-\phi(v_2)} \cdots v_\zeta^{-\phi(v_\zeta)}$ is a negative subword, and

$$W_i^+ = \begin{cases} w_i^{+\ell} v_i^{+\ell} & \text{if } \phi(w_i) = \ell\,, \\ v_i^{+r} w_i^{+r} & \text{if } \phi(w_i) = r \end{cases}$$

is a positive subword. The symbols above $v_i^{\pm\phi(v_i)}$ ($i = 1, 2, \cdots \zeta$) are new symbols created by the moves $T^{\ell}_{\sigma,\ell}$.

Furthermore, making an application of the move R_2^+, we obtain an E-data in which the subwords $W^- a^{+\ell} b^{+r}$, $V^- a^{-\ell} b^{-r}$ are replaced by W^-, V^- respectively. Then, we obtain the move which causes the following replacement of subwords on the coded sequence

$$\mathfrak{S}_+^{\ell,r} \;:\; \text{(i)} \quad a^{+\ell} W^- b^{+r} \;\to\; W^-\,, \quad \text{(ii)} \quad a^{-\ell} b^{-r} \;\to\; V^-\,,$$
$$\text{(iii)} \quad w_i^{+\phi(w_i)} \;\to\; W_i^+ \,(i = 1, 2, \cdots \zeta)\,.$$

This move eliminates the vertices $a^{\pm}$ and $b^{\pm}$ in the arrangement $\mathcal{A}(\Delta)$. So we call $\mathfrak{S}_+^{\ell,r}$ a *pair eliminating* and a pair of vertices a and b an *eliminating pair*.

Composing $T^{\sigma_1}_{\sigma_2,\sigma_3}$ and R_2^+ ($\sigma_j = \ell$ *or* r) similarly, we can obtain three other pair eliminatings $\mathfrak{S}_+^{r,\ell}$, $\mathfrak{S}_-^{\ell,r}$ and $\mathfrak{S}_-^{r,\ell}$, in a coded sequence:

$\mathfrak{S}_+^{r,\ell}$: (i) $a^{+r}W^-b^{+\ell} \to W^-$, (ii) $b^{-\ell}a^{-r} \to \bar{V}^-$, (iii) $w_i^{+\phi(w_i)} \to \overline{W_i}^+$ $(i = 1, 2, \cdots \zeta)$

$\mathfrak{S}_-^{\ell,r}$: (i) $a^{-\ell}W^+b^{-r} \to W^+$, (ii) $b^{+r}a^{+\ell} \to \bar{V}^+$, (iii) $w_i^{-\phi(w_i)} \to \overline{W_i}^-$ $(i = 1, 2, \cdots \zeta)$

$\mathfrak{S}_-^{r,\ell}$: (i) $a^{-r}W^+b^{-\ell} \to W^+$, (ii) $a^{+r}b^{+\ell} \to V^+$, (iii) $w_i^{-\phi(w_i)} \to W_i^-$ $(i = 1, 2, \cdots \zeta)$

where

$$W^- = w_1^{-\phi(w_1)}w_2^{-\phi(w_2)}\cdots w_\zeta^{-\phi(w_\zeta)}, \qquad W^+ = w_1^{+\phi(w_1)}w_2^{+\phi(w_2)}\cdots w_\zeta^{+\phi(w_\zeta)},$$

$$\bar{V}^- = v_\zeta^{-\phi(v_\zeta)}\cdots v_2^{-\phi(y_2)}v_1^{-\phi(v_1)}, \qquad \overline{W_i}^+ = \begin{cases} w_i^{+\ell}v_i^{+r} & \text{if } \phi(w_i) = \ell\,, \\ v_i^{+\ell}w_i^{+r} & \text{if } \phi(w_i) = r\,, \end{cases}$$

$$\bar{V}^+ = v_\zeta^{+\phi(v_\zeta)}\cdots v_2^{+\phi(y_2)}v_1^{+\phi(v_1)}, \qquad \overline{W_i}^- = \begin{cases} w_i^{-r}v_i^{-\ell} & \text{if } \phi(w_i) = r\,, \\ v_i^{-r}w_i^{-\ell} & \text{if } \phi(w_i) = \ell\,, \end{cases}$$

$$V^+ = v_1^{+\phi(v_1)}v_2^{+\phi(v_2)}\cdots v_\zeta^{+\phi(v_\zeta)} \quad \text{and} \quad W_i^- = \begin{cases} w_i^{-r}v_i^{-r} & \text{if } \phi(w_i) = r\,, \\ v_i^{-\ell}w_i^{-\ell} & \text{if } \phi(w_i) = \ell\,. \end{cases}$$

The pair eliminating is useful for removing vertices which prevent us from applying the block reducing lemma or other moves. A new DS-diagram Δ' obtained by applying the pair eliminating operation to a DS-diagram Δ satisfies that $\Delta' \cong \Delta$ and $bl(\Delta') = bl(\Delta)$ or $bl(\Delta) - 1$ or $bl(\Delta) - 2$.

4.4. Reducing the block number. In this subsection, we reduce a block number of Δ_1 from fourteen to two by applying three times of pair eliminatings and two times of block reducing lemmas introduced in §4.3. The coded sequences in the following moves are so complicated that we bracket the subwords where we apply a pair eliminating.

First we apply a pair eliminating $\mathfrak{S}_+^{\ell,r}$ to the E-data

$$\begin{aligned}\mathcal{E}(\Delta_1) = \{&a_1^{-\ell}d_p^{-r}c_p^{-\ell}a_2^{-r}a_3^{+r}(a_4^{+\ell}a_5^{-\ell}a_6^{-r}a_7^{+r})a_8^{+\ell}a_9^{-\ell}a_{10}^{-r}a_2^{+r}(a_5^{+\ell})a_{11}^{-\ell}a_{12}^{-r}\Lambda_1^- \\ &(a_6^{+r})a_9^{+\ell}a_8^{-\ell}a_3^{-r}a_{10}^{+r}c_p^{+\ell}\Upsilon_1^+c_1^{+r}b_1^{+r}b_2^{+r}b_3^{+r}d_1^{+\ell}\Upsilon_2^+d_p^{+r}a_{11}^{+\ell} \\ &(a_4^{-\ell}a_7^{-r})a_{12}^{+r}a_1^{+\ell}\Lambda_2^-\Omega_{21}^+b_4^{-\ell}b_5^{-r}b_6^{+r}b_2^{-r}b_7^{+\ell}b_8^{-\ell}b_9^{-r}\Omega_{11}^+c_1^{-r}d_1^{-\ell}\Omega_{12}^+ \\ &b_5^{+r}b_3^{-r}b_8^{+\ell}b_7^{-\ell}b_6^{-r}b_9^{+r}b_1^{-r}b_4^{+\ell}\Omega_{22}^+\}\,.\end{aligned} \tag{8}$$

In order to explain how the pair eliminating is applied, we rewrite $\mathcal{E}(\Delta_1)$ by

$$\mathcal{E}(\Delta_1) = U_1W_0U_2V_0U_3$$

where

$$\begin{aligned}
U_1 &= a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} a_3^{+r},\\
W_0 &= a_4^{+\ell}(a_5^{-\ell} a_6^{-r}) a_7^{+r},\\
U_2 &= a_8^{+\ell} a_9^{-\ell} a_{10}^{-r} a_2^{+r} (a_5^{+\ell}) a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^{-} (a_6^{+r}) a_9^{+\ell} a_8^{-\ell} a_3^{-r} a_{10}^{+r} c_p^{+\ell}\\
&\quad \Upsilon_1^{+} c_1^{+r} b_1^{+r} b_2^{+r} b_3^{+r} d_1^{+\ell} \Upsilon_2^{+} d_p^{+r} a_{11}^{+\ell},\\
V_0 &= a_4^{-\ell} a_7^{-r} \quad \text{and}\\
U_3 &= a_{12}^{+r} a_1^{+\ell} \Lambda_2^{-} \Omega_{21}^{+} b_4^{-\ell} b_5^{-r} b_6^{+r} b_2^{-r} b_7^{+\ell} b_8^{-\ell} b_9^{-r} \Omega_{11}^{+} c_1^{-r} d_1^{-\ell} \Omega_{12}^{+}\\
&\quad b_5^{+r} b_3^{-r} b_8^{+\ell} b_7^{-\ell} b_6^{-r} b_9^{+r} b_1^{-r} b_4^{+\ell} \Omega_{22}^{+}.
\end{aligned}$$

Applying the pair eliminating, the subwords W_0, V_0 and the vertices $a_5^{+\ell}$ and a_6^{+r} are replaced as following

$$\text{(i)}\quad a_4^{+\ell} W^{-} a_7^{+r} \to W^{-} = a_5^{-\ell} a_6^{-r}, \quad \text{(ii)}\quad a_4^{-\ell} a_7^{-r} \to V^{+} = m_1^{-\ell} m_2^{-r},$$

$$\text{(iii)}\quad \begin{array}{l} a_5^{+\ell} \to W_1^{+} = a_5^{+\ell} m_1^{+\ell} \\ a_6^{+r} \to W_2^{+} = m_2^{+r} a_6^{+r}. \end{array}$$

As a consequence, we obtain a DS-diagram Δ_2 with an E-data

$$\begin{aligned}
\mathcal{E}(\Delta_2) &= U_1 W^{-} \widehat{U_2} V^{-} U_3\\
&= \{a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} (a_3^{+r} (a_5^{-\ell} a_6^{-r}) a_8^{+\ell}) a_9^{-\ell} a_{10}^{-r} a_2^{+r} (a_5^{+\ell} m_1^{+\ell}) a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^{-}\\
&\quad (m_2^{+r} a_6^{+r}) a_9^{+\ell} (a_8^{-\ell} a_3^{-r}) a_{10}^{+r} c_p^{+\ell} \Upsilon_1^{+} c_1^{+r} b_1^{+r} b_2^{+r} b_3^{+r} d_1^{+\ell} \Upsilon_2^{+} d_p^{+r} a_{11}^{+\ell}\\
&\quad (m_1^{-\ell} m_2^{-r}) a_{12}^{+r} a_1^{+\ell} \Lambda_2^{-} \Omega_{21}^{+} b_4^{-\ell} b_5^{-r} b_6^{+r} b_2^{-r} b_7^{+\ell} b_8^{-\ell} b_9^{-r} \Omega_{11}^{+} c_1^{-r} d_1^{-\ell} \Omega_{12}^{+}\\
&\quad b_5^{+r} b_3^{-r} b_8^{+\ell} b_7^{-\ell} b_6^{-r} b_9^{+r} b_1^{-r} b_4^{+\ell} \Omega_{22}^{+}\}.
\end{aligned} \tag{9}$$

The block number of Δ_2 is still fourteen. This move from Δ_1 to Δ_2 enables us to make the following move.

Applying the pair eliminating $\mathfrak{S}_{+}^{r,\ell}$ to $\mathcal{E}(\Delta_2)$, which causes the replacement

$$\text{(i)}\quad a_3^{+r} W^{-} a_8^{+\ell} \to W^{-} = a_5^{-\ell} a_6^{-r}, \quad \text{(ii)}\quad a_8^{-\ell} a_3^{-r} \to \bar{V}^{-} = m_4^{-\ell} m_3^{-r},$$

$$\text{(iii)}\quad \begin{array}{l} a_5^{+\ell} \to a_5^{+\ell} m_3^{+r} \\ a_6^{+r} \to m_4^{+\ell} a_6^{+r}, \end{array}$$

we obtain a DS-diagram Δ_3 with an E-data

$$\begin{aligned}
\mathcal{E}(\Delta_3) = \{&a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} (a_5^{-\ell} a_6^{-r}) a_9^{-\ell} a_{10}^{-r} a_2^{+r} (a_5^{+\ell} m_3^{+r} m_1^{+\ell}) a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^{-}\\
&(m_2^{+r} m_4^{+\ell} a_6^{+r}) a_9^{+\ell} (m_4^{-\ell} m_3^{-r}) a_{10}^{+r} c_p^{+\ell} \Upsilon_1^{+} c_1^{+r} b_1^{+r} b_2^{+r} b_3^{+r} d_1^{+\ell} \Upsilon_2^{+} d_p^{+r} a_{11}^{+\ell}\\
&(m_1^{-\ell} m_2^{-r}) a_{12}^{+r} a_1^{+\ell} \Lambda_2^{-} \Omega_{21}^{+} b_4^{-\ell} b_5^{-r} b_6^{+r} b_2^{-r} b_7^{+\ell} b_8^{-\ell} b_9^{-r} \Omega_{11}^{+} c_1^{-r} d_1^{-\ell} \Omega_{12}^{+}\\
&b_5^{+r} b_3^{-r} b_8^{+\ell} b_7^{-\ell} b_6^{-r} b_9^{+r} b_1^{-r} b_4^{+\ell} \Omega_{22}^{+}\}.
\end{aligned} \tag{10}$$

This move reduces the block number by two, that is, $bl(\Delta_3) = bl(\Delta_2) - 2 = 12$.

Applying the pair eliminating $\mathfrak{S}_{+}^{r,\ell}$ to

$$\begin{aligned}\mathcal{E}(\Delta_3) = \{& a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} a_5^{-\ell} a_6^{-r} a_9^{-\ell} a_{10}^{-r} a_2^{+r} a_5^{+\ell} m_3^{+r} m_1^{+\ell} a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^{-} \\ & m_2^{+r} m_4^{+\ell} a_6^{+r} a_9^{+\ell} m_4^{-\ell} m_3^{-r} a_{10}^{+r} c_p^{+\ell} \Upsilon_1^{+} c_1^{+r} b_1^{+r} (b_2^{+r}) b_3^{+r} d_1^{+\ell} \Upsilon_2^{+} d_p^{+r} a_{11}^{+\ell} \\ & m_1^{-\ell} m_2^{-r} a_{12}^{+r} a_1^{+\ell} \Lambda_2^{-} \Omega_{21}^{+} b_4^{-\ell} b_5^{-r} (b_6^{+r} b_2^{-r} b_7^{+\ell}) b_8^{-\ell} b_9^{-r} \Omega_{11}^{+} c_1^{-r} d_1^{-\ell} \Omega_{12}^{+} \\ & b_5^{+r} b_3^{-r} b_8^{+\ell} (b_7^{-\ell} b_6^{-r}) b_9^{+r} b_1^{-r} b_4^{+\ell} \Omega_{22}^{+} \} ,\end{aligned} \tag{11}$$

which causes the replacement

(i) $b_6^{+r} W^{-} b_7^{+\ell} \to W^{-} = b_2^{-r}$, (ii) $b_7^{-\ell} b_6^{-r} \to \bar{V}^{-} = m_5^{-\ell}$, (iii) $b_2^{+r} \to m_5^{+\ell} b_2^{+r}$,

we obtain Δ_4 with an E-data

$$\begin{aligned}\mathcal{E}(\Delta_4) = \{& a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} a_5^{-\ell} a_6^{-r} a_9^{-\ell} a_{10}^{-r} a_2^{+r} a_5^{+\ell} m_3^{+r} m_1^{+\ell} a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^{-} \\ & m_2^{+r} m_4^{+\ell} a_6^{+r} a_9^{+\ell} m_4{}^{-\ell} m_3{}^{-r} a_{10}^{+r} c_p^{+\ell} \Upsilon_1^{+} c_1^{+r} b_1^{+r} (m_5^{+\ell} b_2^{+r}) b_3^{+r} d_1^{+\ell} \Upsilon_2^{+} d_p^{+r} a_{11}^{+\ell} \\ & m_1{}^{-\ell} m_2{}^{-r} a_{12}^{+r} a_1^{+\ell} \Lambda_2^{-} \Omega_{21}^{+} b_4^{-\ell} b_5^{-r} (b_2^{-r}) b_8^{-\ell} b_9^{-r} \Omega_{11}^{+} c_1^{-r} d_1^{-\ell} \Omega_{12}^{+} \\ & b_5^{+r} b_3^{-r} b_8^{+\ell} (m_5^{-\ell}) b_9^{+r} b_1^{-r} b_4^{+\ell} \Omega_{22}^{+} \} .\end{aligned} \tag{12}$$

This move causes $bl(\Delta_4) = bl(\Delta_3) - 2 = 10$.

Now we apply the block reducing lemma. In order to confirm the conditions for the block reducing lemma, we rewrite the E-data $\mathcal{E}(\Delta_4)$ by a blockwise representation:

$$\begin{aligned}\mathcal{E}(\Delta_4) = & W_{4,1}^{-} W_{4,2}^{+} W_{4,3}^{-} W_{4,4}^{+} W_{4,5}^{-} W_{4,6}^{+} W_{4,7}^{-} W_{4,8}^{+} W_{4,9}^{-} W_{4,10}^{+} \\ & W_{4,11}^{-} W_{4,12}^{+} W_{4,13}^{-} W_{4,14}^{+} W_{4,15}^{-} W_{4,16}^{+} W_{4,17}^{-} W_{4,18}^{+} W_{4,19}^{-} W_{4,20}^{+}\end{aligned} \tag{13}$$

where

$$\begin{aligned}
W_{4,1}^{-} &= a_1^{-\ell} d_p^{-r} c_p^{-\ell} a_2^{-r} a_5^{-\ell} a_6^{-r} a_9^{-\ell} a_{10}^{-r} & W_{4,2}^{+} &= a_2^{+r} a_5^{+\ell} m_3^{+r} m_1^{+\ell} \\
W_{4,3}^{-} &= a_{11}^{-\ell} a_{12}^{-r} \Lambda_1^{-} & W_{4,4}^{+} &= m_2^{+r} m_4^{+\ell} a_6^{+r} a_9^{+\ell} \\
W_{4,5}^{-} &= m_4{}^{-\ell} m_3{}^{-r} & W_{4,6}^{+} &= a_{10}^{+r} c_p^{+\ell} \Upsilon_1^{+} c_1^{+r} b_1^{+r} m_5^{+\ell} b_2^{+r} b_3^{+r} d_1^{+\ell} \Upsilon_2^{+} \\
& & & \quad d_p^{+r} a_{11}^{+\ell} \\
W_{4,7}^{-} &= m_1{}^{-\ell} m_2{}^{-r} & W_{4,8}^{+} &= a_{12}^{+r} a_1^{+\ell} \\
W_{4,9}^{-} &= \Lambda_2^{-} & W_{4,10}^{+} &= \Omega_{21}^{+} \\
W_{4,11}^{-} &= b_4^{-\ell} b_5^{-r} b_2^{-r} b_8^{-\ell} b_9^{-r} & W_{4,12}^{+} &= \Omega_{11}^{+} \\
W_{4,13}^{-} &= c_1^{-r} d_1^{-\ell} & W_{4,14}^{+} &= \Omega_{12}^{+} b_5^{+r} \\
W_{4,15}^{-} &= b_3^{-r} & W_{4,16}^{+} &= b_8^{+\ell} \\
W_{4,17}^{-} &= (m_5^{-\ell}) & W_{4,18}^{+} &= b_9^{+r} \\
W_{4,19}^{-} &= b_1^{-r} & W_{4,20}^{+} &= b_4^{+\ell} \Omega_{22}^{+} .
\end{aligned}$$

Obviously, $f(W_{4,5}^{-}) \cap f(W_{4,6}^{+}) = \emptyset$ and $\{f(W_{4,5}^{-}) \cup f(W_{4,7}^{-}\} \cap f(W_{4,8}^{+}) = \emptyset$. We can apply Lemma 4.3.2 to the blocks $W_{4,5}^{-} W_{4,6}^{+} W_{4,7}^{-} W_{4,8}^{+}$ in $\mathcal{E}(\Delta_4)$. Then we obtain a DS-diagram Δ_5

which satisfies $bl(\Delta_5) = bl(\Delta_4) - 2$. The E-data of Δ_5 can be represented as

$$\begin{aligned}\mathcal{E}(\Delta_5) = {} & W^-_{5,1}W^+_{5,2}W^-_{5,3}W^+_{5,4}W^-_{5,9}W^+_{5,10}W^-_{5,11}W^+_{5,12} \\ & W^-_{5,13}W^+_{5,14}W^-_{5,15}W^+_{5,16}W^-_{5,17}W^+_{5,18}W^-_{5,19}W^+_{5,20}\end{aligned} \tag{14}$$

where

$$\begin{array}{ll} W^-_{5,1} = \tilde{W}^-_{4,1} & W^+_{5,2} = \tilde{W}^+_{4,2} \\ W^-_{5,3} = \tilde{W}^-_{4,3} & W^+_{5,4} = \tilde{W}^+_{4,4}W^+_{4,6}W^+_{4,8} \\ W^-_{5,9} = W^-_{4,5}W^-_{4,7}\tilde{W}^-_{4,9} & W^+_{5,10} = W^+_{4,10} \\ W^-_{5,11} = \tilde{W}^-_{4,11} & W^+_{5,12} = W^+_{4,12} \\ W^-_{5,13} = \tilde{W}^-_{4,13} & W^+_{5,14} = W^+_{4,14} \\ W^-_{5,15} = \tilde{W}^-_{4,15} & W^+_{5,16} = W^+_{4,16} \\ W^-_{5,17} = \tilde{W}^-_{4,17} & W^+_{5,18} = W^+_{4,18} \\ W^-_{5,19} = \tilde{W}^-_{4,19} & W^+_{5,20} = W^+_{4,20}. \end{array}$$

A subword $\tilde{W}^-_{4,i}$ (or $\tilde{W}^+_{4,j}$) differs from $W^-_{4,i}$ (or $W^+_{4,j}$) only by the new vertices created by the moves from Δ_4 to Δ_5. The fact $W^+_{5,j} = W^+_{4,j}$ ($j = 10, 12, 14, 16, 18$ or 20) implies that

$$\begin{aligned}
& f(W^+_{5,10}) \cap f(W^-_{5,11}) = \emptyset\,, \\
& \{f(W^+_{5,10}) \cup f(W^+_{5,12})\} \cap f(W^-_{5,13}) = \emptyset\,, \\
& \{f(W^+_{5,10}) \cup f(W^+_{5,12}) \cup f(W^+_{5,14})\} \cap f(W^-_{5,15}) = \emptyset\,, \\
& \{f(W^+_{5,10}) \cup f(W^+_{5,12}) \cup f(W^+_{5,14}) \cup f(W^+_{5,16})\} \cap f(W^-_{5,17}) = \emptyset\,, \\
& \{f(W^+_{5,10}) \cup f(W^+_{5,12}) \cup f(W^+_{5,14}) \cup f(W^+_{5,16}) \cup f(W^+_{5,18})\} \cap f(W^-_{5,19}) = \emptyset \quad \text{and} \\
& \{f(W^+_{5,10}) \cup f(W^+_{5,12}) \cup f(W^+_{5,14}) \cup f(W^+_{5,16}) \cup f(W^+_{5,18}) \cup f(W^+_{5,20})\} \cap f(W^-_{5,1}) = \emptyset\,.
\end{aligned}$$

Therefore applying Lemma 4.3.1, we obtain a DS-diagram Δ which has $bl(\Delta) = 2$. This shows that it is sufficient for the proof of our main theorem to prove Lemma 4.2.1. □

4.5. Proofs of Lemma 4.1.1 and Lemma 4.2.1

4.5.1. Proof of Lemma 4.1.1. According to [11], for a fake surface $\tilde{P}$ induced from a Heegaard diagram $(M_1, M_2; \vec{D}_1, \vec{D}_2)$ with genus 2, we can find two faces λ_1 and λ'_1 on $\tilde{P}$ having the following conditions

(1) two faces λ_1 and λ'_1 are included in $F \subset \tilde{P}$,

(2) there are five edges A, B, D, A' and B' such that

$$\begin{array}{lll} A \subset \tilde{h}(\partial\lambda_1^- \cap \partial\alpha_1^+)\,, & B \subset \tilde{h}(\partial\lambda_1^- \cap \partial\alpha_2^+)\,, & D \subset \tilde{h}(\partial\lambda_1^+ \cap \partial\beta_2^+)\,, \\ A' \subset \tilde{h}(\partial\lambda_1'^+ \cap \partial\beta_1^-) \quad \text{and} & B' \subset \tilde{h}(\partial\lambda_1'^+ \cap \partial\beta_2^-)\,, & \end{array}$$

(3) these edges are arranged on $\partial\lambda_1$ and $\partial\lambda'_1$ in the order

$$\partial\lambda_1 = \cdots A \cdots DB \cdots \quad \text{and} \quad \partial\lambda'_1 = \cdots A' \cdots B' \cdots\,.$$

Each symbols $\lambda_1^{\pm}$, ${\lambda'_1}^{\pm}$, $\alpha_i^{\pm}$ and $\beta_i^{\pm}$ means each component of the inverse images $\tilde{h}^{-1}(\lambda_1)$, $\tilde{h}^{-1}(\lambda'_1)$, $\tilde{h}^{-1}(\alpha_i)$ and $\tilde{h}^{-1}(\beta_i)$ respectively, such that λ_1^{-}, ${\lambda'_1}^{-}$ and $\alpha_i^{\pm}$ are on $\partial\mathbf{B}_1^3$, and λ_1^{+}, ${\lambda'_1}^{+}$ and $\beta_i^{\pm}$ are on $\partial\mathbf{B}_2^3$.

Let $p \in A$, $q \in B$, $p' \in A'$ and $q' \in B'$ be four points. Then there are two simple arcs γ_1 on λ_1 and γ_2 on λ'_1 which satisfies that

(1) γ_1 is a directed arc on λ_1 going from p to q and γ_2 is a directed arc on λ'_1 going from p' to q',

(2) $\gamma_1 \cap \gamma_2 = \emptyset$ on F,

(3) $\tilde{h}^{-}(\gamma_1)$ has two components $\gamma_1^{+} \subset \lambda_1^{+}$ and $\gamma_1^{-} \subset \lambda_1^{-}$,

(4) $\tilde{h}^{-}(\gamma_2)$ has two components $\gamma_2^{+} \subset {\lambda'_1}^{+}$ and $\gamma_2^{-} \subset {\lambda'_1}^{-}$.

The arcs $\gamma_1^{\pm}$ and $\gamma_2^{\pm}$ on $\partial\mathbf{B}_1^3 \cup \partial\mathbf{B}_2^3$ are as described in Figure 11.

For a Heegaard diagram $(M_1, M_2; \vec{D}_1, \vec{D}_2)$ with the arcs γ_1 and γ_2 on $\tilde{P}$, we apply a disk-slide to the meridian disks α_2 and β_2 so that the edges B and C cross each other and a face λ_0 is produced, as described in Figure 12. This Heegaard diagram gives a fake surface admitting a 2-gon λ_0 and pre mark-lines γ_1^{-} and γ_2^{+}. This complete the proof. □

4.5.2. Proof of Lemma 4.2.1. The lemma will be proved by carrying out the algorithm in [7]. First we briefly explain some terminologies, see [7] for detail. For each edge (or face) σ of Δ, we call σ a 1-*cell* (or a 2-*cell*) of Δ, we may call $f(\sigma)$ in P a 1-*label* (or a 2-*label*, respectively) of Δ. Let $\Delta = (G, f, P)$ be a general DS-diagram. Consider a pair of 2-cells in Δ with the same label α. We denote one of them by α^{+} and the other by α^{-}. In this way, we can separate whole 2-cells in Δ into two classes $\{\alpha_1^{+}, \alpha_2^{+}, \cdots, \alpha_{n+1}^{+}\}$ and $\{\alpha_1^{-}, \alpha_2^{-}, \cdots, \alpha_{n+1}^{-}\}$. The closure $\mathcal{Z}^{+}$ of $\alpha_1^{+} \cup \alpha_2^{+} \cup \cdots \cup \alpha_{n+1}^{+}$ (or $\mathcal{Z}^{-}$ of $\alpha_1^{-} \cup \alpha_2^{-} \cup \cdots \cup \alpha_{n+1}^{-}$) is called the *positive zone* (or the *negative zone*, respectively). We will call such a pair $(\mathcal{Z}^{+}, \mathcal{Z}^{-})$ a *bicoloring* of the DS-diagram Δ. If both of $\mathcal{Z}^{+}$ and $\mathcal{Z}^{-}$ are connected, we will call $(\mathcal{Z}^{+}, \mathcal{Z}^{-})$ a *split bicoloring* of Δ. A DS-diagram Δ is *splittable* if Δ has a split bicoloring.

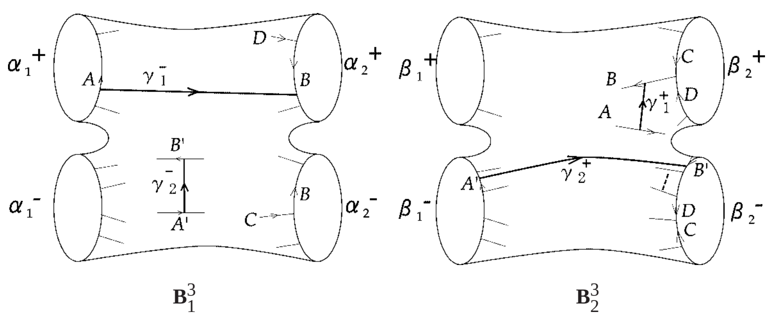

FIGURE 11. The arcs $\gamma_1^{\pm}$ and $\gamma_2^{\pm}$ on $\partial\mathbf{B}_1^3 \cup \partial\mathbf{B}_2^3$.

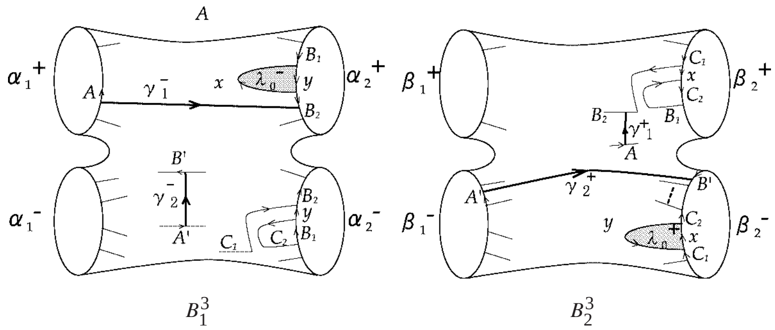

FIGURE 12. The disk-slide producing λ_0.

Let Δ be a splittable DS-diagram with a split bicoloring $(\mathcal{Z}^+, \mathcal{Z}^-)$. Let $a_1, a_2, \cdots, a_m$ be a sequence of successive 1-cells on a simple loop $\mathcal{Z}^+ \cap \mathcal{Z}^-$ such that $cl(a_1 \cup a_2 \cup \cdots \cup a_m) = \mathcal{Z}^+ \cap \mathcal{Z}^-$. Let A_i be the label of a_i, $1 \leq i \leq m$. Then we say that $\Gamma = A_1 A_2 \cdots A_m$ is a *splitting cycle* of Δ associated with $(\mathcal{Z}^+, \mathcal{Z}^-)$. If a DS-diagram Δ has a splitting cycle $\Gamma = A_1 A_2 \cdots A_m$ satisfying $A_i \neq A_j$ for each $i \neq j$, then $\mathcal{Z}^+ \cap \mathcal{Z}^-$ is an E-cycle of Δ.

For remodeling a DS-diagram into one with E-cycle, we use *elementary deformations* $I^\pm$, $II^\pm$ and *digging operations* as deformations of DS-diagrams which preserve the represented manifold, see [7].

Now we apply the remodeling algorithm to the DS-diagram Δ_0 of the derived spine P_0 obtained by Lemma 4.1.1. We decide a bicoloring $(\mathcal{Z}^+, \mathcal{Z}^-)$ of Δ_0 by $Z^+ = \{\alpha_i^+, \beta_i^+, \lambda_j^+\}$ and $Z^- = \{\alpha_i^-, \beta_i^-, \lambda_j^-\}$. The choice of λ_0 makes it possible to define this bicoloring. The intersection $\mathcal{Z}^+ \cap \mathcal{Z}^-$ consists of three cycles, and so this bicoloring $(\mathcal{Z}^+, \mathcal{Z}^-)$ is not yet a split one. The three cycles of $\mathcal{Z}^+ \cap \mathcal{Z}^-$ can be written as

$$\begin{aligned} \partial\alpha_1^+ &= \{A\Gamma(\alpha_1^+)\}\,, \\ \partial\beta_1^- &= \{(A')^{-1}\Gamma(\beta_1^-)\} \quad \text{and} \\ (\partial\alpha_2^+ - y) \cup (\partial\beta_2^- - x) &= \{B\Gamma(\alpha_2^+)C\Gamma(\beta_{21}^-)(B')^{-1}\Gamma(\beta_{22}^-)\}\,, \end{aligned}$$

where $\Gamma(\alpha_1^+)$, $\Gamma(\alpha_2^+)$, $\Gamma(\beta_1^-)$, $\Gamma(\beta_{21}^-)$ and $\Gamma(\beta_{22}^-)$ are consecutive sequences of 1-labels, see Figure 13.

We apply DS-deformations in Theorem 2.3 of [7] and digging operations to make $Z^+ \cap Z^-$ a simple loop. The DS-deformaions in Theorem 2.3 of [7] are applied to the 1-labels B and B'. As a consequence of these deformations, the three cycles of $\mathcal{Z}^+ \cap \mathcal{Z}^-$ are deformed into

$$\begin{aligned} \partial\alpha_1^+ &= \{A\Gamma(\alpha_1^+)\}\,, \\ \partial\beta_1^- &= \{(A')^{-1}\Gamma(\beta_1^-)\} \quad \text{and} \\ (\partial\alpha_2^+ - y) \cup (\partial\beta_2^- - x) &= \{\Gamma(B)\Gamma(\alpha_2^+)C\Gamma(\beta_{21}^-)\Gamma(B')\Gamma(\beta_{22}^-)\}\,, \end{aligned}$$

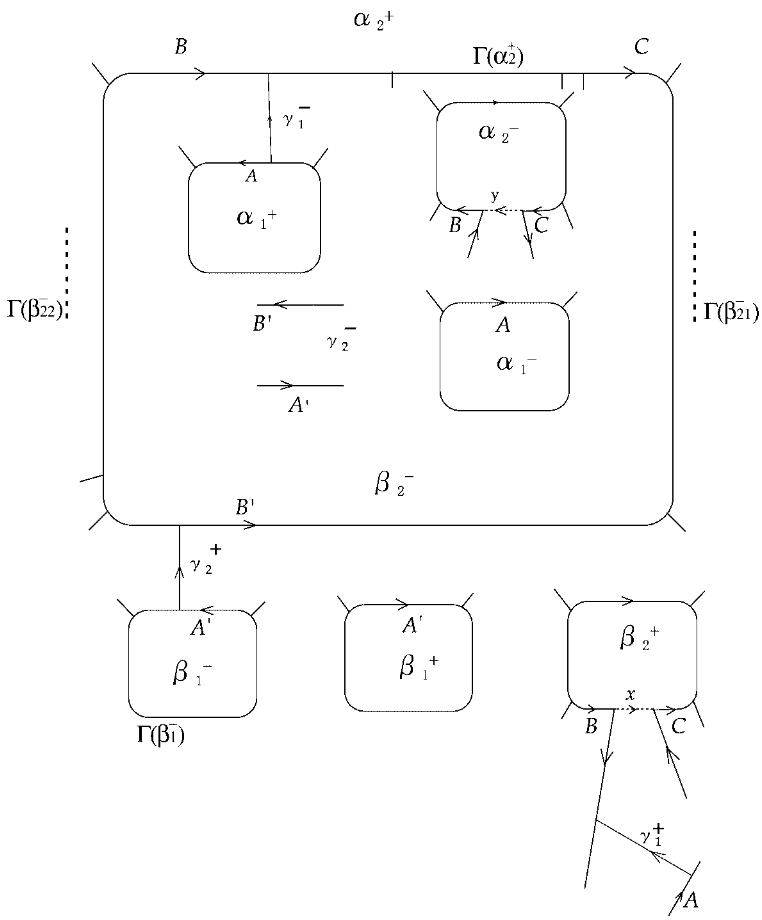

FIGURE 13. The DS-diagram Δ_0.

where

$$\Gamma(B) = B_1 V B_* Q_1 S P_2 B_* P_1 T Q_2 B_* U B_2$$

and

$$\Gamma(B') = (B_2')^{-1}(U')^{-1}(B_*')^{-1}(Q_2')^{-1}(T')^{-1}(P_1')^{-1}(B_*')^{-1}(P_2')^{-1} (S')^{-1}(Q_1')^{-1}(B_*')^{-1}(V')^{-1}(B_1')^{-1}.$$

Defining a mark-line γ_1^- from A to B_* (or γ_2^+ from A' to B_*') in parallel with the pre mark-line γ_1^- (or respectively γ_2^+), we apply digging operations along mark-lines γ_1^- and γ_2^+. By Lemma 2.1 of [7], these digging operations connect the three cycles into only one cycle. Now we obtain a splittable DS-diagram Δ_s having a splitting cycle

$$\Gamma_1 = \{\Gamma_1(B, A)\Gamma(\alpha_2^+)C\Gamma(\beta_{21}^-)\Gamma_1(B', A')\Gamma(\beta_{22}^-)\},$$

where

$$\Gamma_1(B, A) = B_1 V B_\flat E^{-1} B_\sharp Q_1 S P_2 B_\flat D_\sharp (A_\sharp \Gamma(\alpha_1^+) A_\flat) D_\flat B_\sharp P_1 T Q_2 B_\flat B B_\sharp U_\flat U U_\sharp B_2$$

and

$$\begin{aligned}\Gamma_1(B', A') = {} & (B_2')^{-1}(U_\sharp')^{-1}(U')^{-1}(U_\flat')^{-1}(B_\sharp')^{-1}(B')^{-1}(B_\flat')^{-1}(Q_2')^{-1}(T')^{-1}(P_1')^{-1}(B_\sharp')^{-1}\\ & (D_\flat')^{-1}((A_\flat')^{-1}\Gamma(\beta_1^-)(A_\sharp')^{-1})(D_\sharp')^{-1}\\ & (B_\flat')^{-1}(P_2')^{-1}(S')^{-1}(Q_1')^{-1}(B_\sharp')^{-1}E'(B_\flat')^{-1}(V')^{-1}(B_1')^{-1}.\end{aligned}$$

We apply Theorem 3.1 of [7] to the 1-labels $B_\sharp$ and $B_\sharp'$.

$$\Gamma_2 = \{\Gamma_2(B, A)\Gamma(\alpha_2^+)C\Gamma(\beta_{21}^-)\Gamma_2(B', A')\Gamma(\beta_{22}^-)\},$$

where

$$\Gamma_2(B, A) = B_1 V B_\flat E^{-1} L Q_1 S P_2 B_\flat D_\sharp (A_\sharp \Gamma(\alpha_1^+) A_\flat) D_\flat N P_1 T Q_2 B_\flat B M U_\flat U U_\sharp B_2$$

and

$$\begin{aligned}\Gamma_2(B', A') = {} & (B_2')^{-1}(U_\sharp')^{-1}(U')^{-1}(U_\flat')^{-1}(M')^{-1}(B')^{-1}(B_\flat')^{-1}(Q_2')^{-1}(T')^{-1}(P_1')^{-1}(N')^{-1}\\ & (D_\flat')^{-1}((A_\flat')^{-1}\Gamma(\beta_1^-)(A_\sharp')^{-1})(D_\sharp')^{-1}\\ & (B_\flat')^{-1}(P_2')^{-1}(S')^{-1}(Q_1')^{-1}(L')^{-1}E'(B_\flat')^{-1}(V')^{-1}(B_1')^{-1}.\end{aligned}$$

Remove two 2-gons with each of boundary circles $U_\flat D_\flat$ and $U_\flat'^{-1} D_\flat'^{-1}$ by applying elementary deformation I^- for each of 2-gons. The splitting cycle Γ_3 is

$$\Gamma_3 = \{\Gamma_3(B, A)\Gamma(\alpha_2^+)C\Gamma(\beta_{21}^-)\Gamma_3(B', A')\Gamma(\beta_{22}^-)\},$$

where

$$\Gamma_3(B, A) = B_1 V B_\flat E^{-1} L Q_1 S P_2 B_\flat D_\sharp (A_\sharp \Gamma(\alpha_1^+) A_\flat) P_1 T Q_2 B_\flat B U U_\sharp B_2$$

and

$$\begin{aligned}\Gamma_3(B', A') = {} & (B_2')^{-1}(U_\sharp')^{-1}(U')^{-1}(B')^{-1}(B_\flat')^{-1}(Q_2')^{-1}(T')^{-1}(P_1')^{-1}\\ & ((A_\flat')^{-1}\Gamma(\beta_1^-)(A_\sharp')^{-1})(D_\sharp')^{-1}\\ & (B_\flat')^{-1}(P_2')^{-1}(S')^{-1}(Q_1')^{-1}(L')^{-1}E'(B_\flat')^{-1}(V')^{-1}(B_1')^{-1}.\end{aligned}$$

For a pair of two 1-labels $B_\flat$ and $B_\flat{}'$, we apply DS-deformations according to Theorem 4.2 of [7]. Firstly, we apply an elementary deformation II^+ to $B_\flat{}'$. Then, $\mathcal{Z}^+ \cap \mathcal{Z}^-$ consists two cycles as

$$\mathcal{Z}^+ \cap \mathcal{Z}^- = \{\Gamma_4(B, A)\Gamma(\alpha_2^+)C\Gamma(\beta_{21}^-)\Gamma_4(B', A')\Gamma(\beta_{22}^-)\} \cup \{GHI\}$$

where

$$\Gamma_4(B, A) = B_1 V B_\flat E^{-1} L Q_1 S P_2 B_\flat D_\sharp (A_\sharp \Gamma(\alpha_1^+) A_\flat) P_1 T Q_2 B_\flat B U U_\sharp B_2$$

and

$$\Gamma_4(B', A') = (B_2')^{-1}(U_\sharp')^{-1}(U')^{-1}(B')^{-1}(Q_2')^{-1}(T')^{-1}(P_1')^{-1}(A_\flat')^{-1}\Gamma(\beta_1^-)(A_\sharp')^{-1} (D_\sharp')^{-1}(P_2')^{-1}(S')^{-1}(Q_1')^{-1}(L')^{-1}E'(V')^{-1}(B_1')^{-1}.$$

Secondly, we take a simple arc γ^+ from $B_\flat$ to H in $\mathcal{Z}^+$. γ^+ may intersects with p 1-cells with 1-labels $W_1, W_2, \cdots W_{p-1}$ and V where each W_i is an edge inherited from the initial Heegaard diagram and V is a new edge obtained by the deformations of remodeling, see Figure 14. The 1-label W_1 is an edge included in $\partial\beta_1$ and each W_i $(i = 2, 3 \cdots, p-1)$ is an edge included in $\partial\alpha_1 \cup \partial\alpha_2$. Along γ^+ as a mark-line, we apply a digging operation.

Lastly, applying an elementary deformation II^+ to X_1 and an elementary deformation II^+ to X_2 (edges X_1 and X_2, see Figure 12-c in [7]), we obtain a DS-diagram $\Delta_1 = (G, f, P; e)$ with an E-cycle e. The splitting cycle Γ_e which is the sequence of successive 1-labels on e is

$$\Gamma_e = \{\Gamma_5(B, A)\tilde{\Gamma}(\alpha_2^+)\Gamma(C)\Gamma(\beta_{21}^-)\Gamma_5(B', A')\Gamma(\beta_{22}^-)\},$$

where

$$\begin{aligned}\Gamma_5(B, A) &= B_1(V_\flat V V_\sharp J_1 X_* J_2 E^{-1} L Q_1 S P_2 L_1 Z L_2 D_\sharp) A_\sharp \tilde{\Gamma}(\alpha_1^+) A_\flat \\ &\quad (P_1 T Q_2 K_1 T_p T_{p-1} \Gamma(T) T_1 H_1 I G H_2 S_1 \Gamma(S) S_{p-1} K_2 B U U_\sharp) B_2 \\ &= B_1 \tilde{\Gamma}_5(B, A) B_2\end{aligned}$$

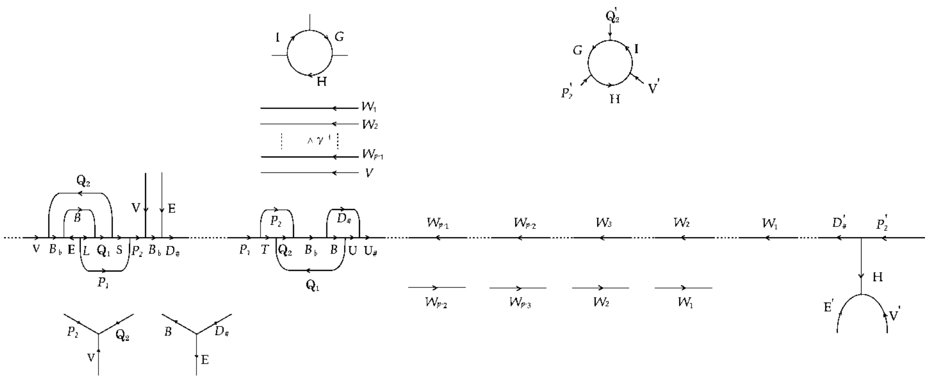

FIGURE 14. Mark line γ^+ from $B_\flat$ to H.

and

$$\begin{aligned}\Gamma_5(B', A') &= (B_2')^{-1}(U_\sharp')^{-1}(U')^{-1}(B')^{-1}(Q_2')^{-1}(T')^{-1}(P_1')^{-1}(A_\flat')^{-1}\tilde{\Gamma}(\beta_1^-)(A_\sharp')^{-1}\\ &\qquad ((D_\sharp')^{-1}(P_2')^{-1})(S')^{-1}(Q_1')^{-1}(L')^{-1}E'(V')^{-1}(B_1')^{-1}\\ &= (B_2')^{-1}\tilde{\Gamma}_5(B', A')(B_1')^{-1}\,.\end{aligned}$$

The consecutive sequence of 1-labels $\Gamma(T) = T_{p-2}\cdots T_2$ and $\Gamma(S) = S_2\cdots S_{p-2}$ included in $\Gamma_5(B, A)$, consist the new 1-labels which appear by the digging operation along γ^+, see Figure 15. If $\Gamma(\alpha_i^+)$ contains the 1-label W_j for some $j = 2, 3\cdots p-1$, $\tilde{\Gamma}(\alpha_i^+)$ includes the 1-labels $W_{j\sharp}W_jW_{j\flat}$. If $C = W_k$, for some $k = 2, 3\cdots p-1$, then $\Gamma(C) = W_{k\sharp}W_kW_{k\flat}$. For $\Gamma(\beta_1^-) = \Gamma(\beta_{11}^-)W_1\Gamma(\beta_{12}^-)$, $\tilde{\Gamma}(\beta_1^-) = \Gamma(\beta_{11}^-)W_{1\sharp}W_1W_{1\flat}\Gamma(\beta_{12}^-)$.

From now on, we will see the vertices on the E-cycle of the DS-diagram Δ_1 in order to examine the E-data and to prove Lemma 4.2.1. The splitting cycle Γ_e leads us to the E-data $\mathcal{E}(\Delta_1)$. We can write it as

$$\mathcal{E}(\Delta_1) = \{\Psi_B\Lambda_2^-\Omega_{21}^+\Psi_{B'}\Omega_{22}^+\}\,, \tag{15}$$

where

$$\begin{aligned}\Psi_B = {}& a_1^{-\ell}d_p^{-r}c_p^{-\ell}a_2^{-r}a_3^{+r}a_4^{+\ell}a_5^{-\ell}a_6^{-r}a_7^{+r}a_8^{+\ell}a_9^{-\ell}a_{10}^{-r}a_2^{+r}a_5^{+\ell}a_{11}^{-\ell}a_{12}^{-r}\Lambda_1^-\\ & a_6^{+r}a_9^{+\ell}a_8^{-\ell}a_3^{-r}a_{10}^{+r}c_p^{+\ell}\Upsilon_1^+c_1^{+r}b_1^{+r}b_2^{+r}b_3^{+r}d_1^{+\ell}\Upsilon_2^+d_p^{+r}a_{11}^{+\ell}a_4^{-\ell}a_7^{-r}a_{12}^{+r}a_1^{+\ell}\,,\end{aligned}$$

and

$$\Psi_{B'} = b_4^{-\ell}b_5^{-r}b_6^{+r}b_2^{-r}b_7^{+\ell}b_8^{-\ell}b_9^{-r}\Omega_{11}^+c_1^{-r}d_1^{-\ell}\Omega_{12}^+b_5^{+r}b_3^{-r}b_8^{+\ell}b_7^{-\ell}b_6^{-r}b_9^{+r}b_1^{-r}b_4^{+\ell}\,.$$

The subword Ψ_B (or $\Psi_{B'}$) consists of the vertices which are adjacent to the 1-labels of $\tilde{\Gamma}_5(B, A)$ (or $\tilde{\Gamma}_5(B', A')$, respectively). The subwords Λ_i^-, Ω_{ij}^+, Υ_1^+ and Υ_2^+ consist of

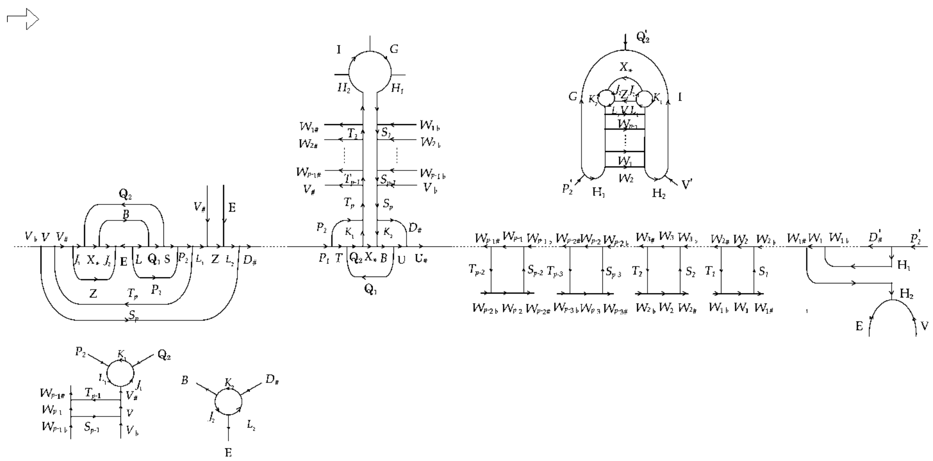

FIGURE 15. The DS-diagram Δ_1 with E-cycle.

the vertices which are adjacent to the 1-labels of $\tilde{\Gamma}(\alpha_i^+)$ (or $\tilde{\Gamma}(\alpha_2^+)W_{1\sharp}W_1$, if $C = W_k$.), $\Gamma(\beta_{ij}^-)$, $\Gamma(T)$ and $\Gamma(S)$, respectively. For the vertices included in $\Upsilon_1^+ = c_{p-1}^+ c_{p-2}^+ \cdots c_2^+$ and $\Upsilon_2^+ = d_2^+ d_3^+ \cdots d_{p-1}^+$, the vertices c_i^- and d_i^- $(i = 2, 3, \cdots p-1)$ are included in the sequence of 1-labels $W_{i\sharp}W_i W_{i\flat}$ on the E-cycle. This implies that $f(\Lambda_1^-) \cup f(\Lambda_2^-) = f(\Omega_{11}^+) \cup f(\Omega_{12}^+) \cup f(\Omega_{21}^+) \cup f(\Omega_{22}^+) \cup f(\Upsilon_1^+) \cup f(\Upsilon_2^+)$. The vertices $a_i^\pm$ $(1 \le i \le 12)$, $b_i^\pm$ $(1 \le i \le 9)$, $c_i^\pm$ $(1 \le i \le p)$ and $d_i^\pm$ $(1 \le i \le p)$ are new vertices which appear by deformations of remodeling. This shows that the E-data (15) is one required in Lemma 4.2.1 $\square$

REMARK 1. By the moves from the coded sequence (15) to the coded sequence (14) in §4.4, each subword in (15) is inherited to the subwords in (14) as follows,

$$\begin{aligned}
&\Psi_B \to W_{5,1}^- W_{5,2}^+ W_{5,3}^- W_{5,4}^+ W_{4,5}^- W_{4,7}^- ,\\
&\Lambda_2^- \to \tilde{W}_{4,9}^- ,\\
&\Omega_{21}^+ \to \Omega_{21}^+ = W_{5,10}^+ ,\\
&\Psi_{B'} \to W_{5,11}^- W_{5,12}^+ W_{5,13}^- W_{5,14}^+ W_{5,15}^- W_{5,16}^+ W_{5,17}^- W_{5,18}^+ W_{5,19}^- b_4^{+\ell} \quad \text{and}\\
&\Omega_{22}^+ \to \Omega_{22}^+ .
\end{aligned}$$

5. Difficulty for the case of $HG(M(\Delta)) = 3$

In the main theorem, we have considered the case of $HG(M(\Delta)) = 2$. Here, we will mention briefly that the same method as in the case of $HG(M(\Delta)) = 2$ seems not to be applicable in the case of $HG(M(\Delta)) = 3$.

Let $(M_1, M_2; \vec{D}_1, \vec{D}_2)$ be a Heegaard diagram with genus 3, and let $\vec{D}_1 = \{\alpha_1, \alpha_2, \alpha_1'\}$ and $\vec{D}_2 = \{\beta_1, \beta_2, \beta_1'\}$ be complete meridian disk systems of M_1 and M_2 respectively. In order to apply the similar way of moves of the case $HG(M(\Delta)) = 2$, we have to prepare the following four pre mark-lines

γ_1^- with the initial point $p_1 \in A_1 \subset \partial\alpha_1{}^+$ and the terminal point $q_1 \in B_1 \subset \partial\alpha_2{}^+$,
$\gamma_1'^-$ with the initial point $p_1' \in A_1' \subset \partial\alpha_1'{}^+$ and the terminal point $q_1' \in B_1' \subset \partial\alpha_2{}^+$,
γ_2^+ with the initial point $p_2 \in A_2 \subset \partial\beta_1{}^-$ and the terminal point $q_2 \in B_2 \subset \partial\beta_2{}^-$ and
$\gamma_2'^+$ with the initial point $p_2' \in A_2' \subset \partial\beta_1'{}^-$ and the terminal point $q_2' \in B_2' \subset \partial\beta_2{}^-$,

where

(1) these arcs do not intersect with each other,
(2) both of γ_2^+ and γ'^+_2 intersect with neither edges B_1 nor B_1'.

We also obtain a DS-diagram Δ_1 in a similar method to the case of genus 2 by making 2-gon near the mark-line $\gamma_1'^{-}$. The splitting cycle of Δ_1 can be written as

$$\begin{aligned}\Gamma(\Delta_1) = \Gamma_5(B_1, A_1)\tilde{\Gamma}(\alpha_2^+)\Gamma(C)\Gamma(\beta_{21}^-)\Gamma_5(B_2, A_2)\Gamma(\beta_{22}^-)\\ \Gamma(\beta_{21}'^{-})\Gamma_5(B_2', A_2')\Gamma(\beta_{22}'^{-})\Gamma_5(B_1', A_1')\tilde{\Gamma}(\alpha_2'^{+})\end{aligned}$$

and it leads us to the E-data

$$\mathcal{E}(\Delta_1) = \Psi_{B_1}\Lambda_2^-\Omega_{21}^+\Psi_{B_2}\Omega_{22}^+\Omega'^{+}_{21}\Psi_{B_2'}\Omega'^{+}_{22}\Psi_{B_1'}\Lambda'^{-}_2 .$$

Applying the moves for reducing the block number to Δ_1, we obtain a DS-diagram Δ' with the coded sequence

$$\begin{aligned}\mathcal{E}(\Delta') = {} & (W_{5,1}^-W_{5,2}^+W_{5,3}^-W_{5,4}^+W_{4,5}^-W_{4,7}^-)\tilde{W}_{4,9}^-W_{5,10}^+\\ & (W_{5,11}^-W_{5,12}^+W_{5,13}^-W_{5,14}^+W_{5,15}^-W_{5,16}^+W_{5,17}^-W_{5,18}^+W_{5,19}^-b_4^{+\ell})\Omega_{22}^+\\ & W'^{+}_{5,10}(W'^{-}_{5,11}W'^{+}_{5,12}W'^{-}_{5,13}W'^{+}_{5,14}W'^{-}_{5,15}W'^{+}_{5,16}W'^{-}_{5,17}W'^{+}_{5,18}W'^{-}_{5,19}b_4'^{+\ell})\Omega_{22}'^{+}\\ & (W'^{-}_{5,1}W'^{+}_{5,2}W'^{-}_{5,3}W'^{+}_{5,4}W'^{-}_{4,5}W'^{-}_{4,7})\tilde{W}'^{-}_{4,9}\\ = {} & \left(\prod_{k=1}^{2} W_{5,2k-1}^-W_{5,2k}^+\right)\left(\prod_{k=5}^{10} W_{5,2k-1}^-W_{5,2k}^+\right)\\ & W'^{+}_{5,10}\left(\prod_{k=6}^{10} W'^{-}_{5,2k-1}W'^{+}_{5,2k}\right)\left(\prod_{k=1}^{2} W'^{-}_{5,2k-1}W'^{+}_{5,2k}\right)W'^{-}_{5,9}\end{aligned}$$

where $W'^{-}_{5,9} = W'^{-}_{4,5}W'^{-}_{4,7}\tilde{W}'^{-}_{4,9}$ and $W'^{+}_{5,20} = b_4'^{+\ell}\Omega_{22}'^{+}$ and each subwords of $\mathcal{E}(\Delta_1)$ are inherited to the subwords of $\mathcal{E}(\Delta')$ as follows,

$$\begin{aligned}\Psi_{B_1} & \to W_{5,1}^-W_{5,2}^+W_{5,3}^-W_{5,4}^+W_{4,5}^-W_{4,7}^- ,\\ \Lambda_2^- & \to \tilde{W}_{4,9}^- ,\\ \Omega_{21}^+ & \to \Omega_{21}^+ = W_{5,10}^+ ,\\ \Psi_{B_2} & \to W_{5,11}^-W_{5,12}^+W_{5,13}^-W_{5,14}^+W_{5,15}^-W_{5,16}^+W_{5,17}^-W_{5,18}^+W_{5,19}^-b_4^{+\ell} ,\\ \Omega_{22}^+ & \to \Omega_{22}^+ ,\\ \Omega_{21}'^{+} & \to \Omega_{21}'^{+} = W'^{+}_{5,10} ,\\ \Psi_{B_2'} & \to W'^{-}_{5,11}W'^{+}_{5,12}W'^{-}_{5,13}W'^{+}_{5,14}W'^{-}_{5,15}W'^{+}_{5,16}W'^{-}_{5,17}W'^{+}_{5,18}W'^{-}_{5,19}b_4'^{+\ell} ,\\ \Omega_{22}'^{+} & \to \Omega_{22}'^{+} ,\\ \Psi_{B_1'} & \to W'^{-}_{5,1}W'^{+}_{5,2}W'^{-}_{5,3}W'^{+}_{5,4}W'^{-}_{4,5}W'^{-}_{4,7} ,\\ \Lambda_2'^{-} & \to \tilde{W}'^{-}_{4,9} .\end{aligned}$$

The subwords $W_{i,j}^{\varepsilon}$ and $W'^{\varepsilon}_{i,j}$ ($\varepsilon = +$ or $-$) consists of similar symbols and order to the subwords $W_{i,j}^{\varepsilon}$ of (14).

The subword $(\prod_{k=5}^{10} W^-_{5,2k-1} W^+_{5,2k}) W'^+_{5,10} (\prod_{k=6}^{10} W'^-_{5,2k-1} W'^+_{5,2k}) W'^-_{5,1} W'^+_{5,2}$ can be deformed to the blocks Θ^-_1 and Θ^+_2. But the other subword $W'^-_{5,3}$ $W'^+_{5,4}$ $W'^-_{5,9}$ $W^-_{5,1}$ $W^+_{5,2}$ $W^-_{5,3} W^+_{5,4}$ can not be deformed to less blocks any more by the similar confirmation to the proof of Theorem 1. So we obtain a DS-diagram with block number 4. If we apply the different way of moves, we may obtain a DS-diagram with block number 3. But it is difficult to find a method to do it now.

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