On some arithmetic questions of reductive groups over algebraic extensions of local and global fields

Dedicated to Dragomir Djoković and Jack Sonn

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Abstract: In this paper we extend to algebraic extensions of local and global fields and their completions some classical results due to Borel-Serre, Tits, Conrad, Douai, Kneser and Sansuc concerning the finiteness, the surjectivity of maps between Galois cohomology groups and the obstruction to weak approximation and some related results of connected linear algebraic groups.

Key words: Weak approximation; Tate-Nakayama duality; Galois cohomology.

Introduction. Let k be a field, k_s a separable closure of k in an algebraic closure $\bar{k}$ of k, and let $\Gamma := Gal(k_s/k)$ be the absolute Galois group of k. Denote by V the set of all places (i.e., equivalent classes of valuations) of k and let k_v be the completion of k at $v \in V$. For an algebraic k-group scheme G (see [17] for basic notions we use here), let $\mathrm{H}^i_{\mathrm{fppf}}(k, G) := \mathrm{H}^i_{\mathrm{fppf}}(\bar{k}/k, G(\bar{k}))$ be the flat cohomology of G in degree i (which is ≤ 1 if G is non-commutative), which is isomorphic to the Galois cohomology $\mathrm{H}^i(k, G) := \mathrm{H}^i(\Gamma, G(k_s))$ in degree i if G is smooth.

Along with finite extensions of the field $\mathbf{Q}$ and $\mathbf{Q}_p$ (resp. $\mathbf{F}_p(t)$ and $\mathbf{F}_p((t))$ (where $\mathbf{Q}_p$, $\mathbf{F}_p$ stand for p-adic field and finite field of p elements, respectively)), many important extensions of such fields which appear in algebraic number theory are *infinite algebraic extensions*, for example, the maximal abelian extensions of a given (local or global) field k, or the maximal extension of a global k, which is unramified outside a given set S, etc. and a general study of such fields was started in [16]. When the infinite algebraic extensions k of local fields and their completions k_v enter, there are two cases to consider. First is the case when the valuation v is discrete. It seems that in this case, the arithmetic of infinite algebraic extensions k of local fields and their completions k_v is close to that of local fields. However, when v is non-discrete (i.e., the value group Δ is a *dense subgroup* of $\mathbf{R}$), the structure of such fields is more complicated. In fact, many of k_v become perfectoid fields in the sense of P. Scholze [25]. Some of the fields k of fundamental importance in number theory are perfectoid, such as $\mathbf{Q}_p(p^{1/p^\infty})$, $\mathbf{Q}_p(\mu_{p^\infty})$, $\mathbf{F}_p((t))(t^{1/p^\infty})$, and the completions of algebraic closures of above fields. It is interesting to investigate further the arithmetic of algebraic varieties in general and algebraic groups in particular over such fields. The motivation for the investigation of arithmetic of algebraic groups over infinite algebraic extensions of local or global fields is quite natural, which is to see which of the main results in the case of local and global fields hold in the general case (cf. e.g. [18], [19] and reference there). In the present paper, we are interested in answering the following question: which of the classical results in Galois cohomology theory of linear algebraic groups over local or global fields still hold in the case of *infinite* algebraic extensions of local and global fields?

We investigate here some results which are related to the finiteness, the surjectivity (bijectivity) of a coboundary map in Galois cohomology which are important in arithmetic of algebraic groups over field. In particular, we extend Kneser's Theorem on the surjectivitiy of certain coboundary maps in Galois cohomology and Conrad's Theorem (thus partially also Borel-Serre's Theorem) on the finiteness of Galois cohomology of pseudo-reductive

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groups to the case of infinite algebraic extensions of local and global fields. As an application, we apply the finiteness of Galois cohomology to show the finiteness of the obstruction to weak approximation of connected reductive groups at finite set of places.

There are some obstacles to generalize the classical results mentioned above to be noted here:
(a) Infinite algebraic extensions of local fields need not be a field of type (F) in the sense of [26, Chap. III] in general, which is very important for the finiteness results in Galois cohomology;
(b) The duality theory related with local class field theory, namely Tate-Nakayama duality theory for commutative algebraic group schemes of multiplicative type (Shatz's theory [27]) in the case of infinite algebraic extensions of local fields (or the completions of such extension fields), which is highly desirable, is not yet available;
(c) The valuation on the algebraic extensions of local fields and their completions, which is extended from the given one on the base local field may not be discrete, whereas the Tits' approach to the finiteness of Galois cohomology assumes the discreteness of the valuation.

Therefore, in order to achieve our goal in the case of infinite algebraic extensions of local fields (or their completions), one needs some further investigation and a new approach. A detailed version of the paper will be published elsewhere.

1. Tate-Nakayama duality. For a k-torus T, we denote its character module by $\hat{T} := X^*(T)$ and its co-character module by $X_*(T)$.

1.1. Localization fields and their arithmetic. Recall that if $L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots \subset k$ is a chain of finite field extensions, then one can consider the generalized number, the Steinitz number $[k : L_0]$, which is the formal product $\prod_i [L_{i+1} : L_i]$ (cf. [26, Chap. I]). One denotes $[k : L_0]_\infty$ the Steinitz number $\prod_p p^\infty$, where p runs over all prime numbers which occur infinitely many times in the sequence $[L_{i+1} : L_i]$, $i = 0, 1, \dots$.

For an algebraic extension k of a local (resp. global) field L, if its degree is infinite, we just call it for short an *infinite local* (*resp. global*) *field.* If k is an infinite global field which is equipped with a place v, one may consider the localization field $k(v)$ of k (cf. [16]) Likewise, given any field L and an infinite algebraic extension k/L equipped with a place v, one may also consider the localization $k(v)$ of k (see [16], [18] and [19]). If k is an infinite global field, then $k(v)$ is an *infinite local field* and is equipped with a non-archimedean place v. Then $k(v)$ is the same as an infinite algebraic extension of a complete valued field with finite residue field. Further, we denote by k_v the completion at v of k, which is also the completion of $k(v)$ at v. Notice that the extended place (or valuation) v on k may or may not be discrete.

1.2. Tate-Nakayama duality for infinite local fields. Next we discuss the Tate–Nakayama duality theory for algebraic group schemes of multiplicative type (Shatz's theory) in the case of infinite algebraic extensions of local fields (or the completions of such extension fields). Let k be an infnite local field over a local field L, $\mathcal{C}$ the set of all prime numbers, which are co-prime with $[k : L]_\infty$, $\Gamma_k^{\mathcal{C}}$ the Galois group of the maximal abelian pro-$\mathcal{C}$-extension $\mathscr{S}_{f,k}$ of k, that is $\Gamma_k^{\mathcal{C}} = \varprojlim_{K/k} Gal(K/k)$, where K/k runs over the set $\mathscr{E}$ of all finite abelian extensions K/k having degree co-prime with $[k : L]_\infty$. Thus from the definition and from [24, Chap. VI, Sec. 11, Thm. 22], it follows that $\mathrm{H}^2(\Gamma_k^{\mathcal{C}}, \mathbf{G}_m) \simeq \mathrm{Br}(k)$. Let M be a torsion free finitely generated (over $\mathbf{Z}$) $\Gamma_k^{\mathcal{C}}$-module and let X_M be the diagonalizable group k-scheme with character group M. Since M is torsion free, it follows that X_M is a k-torus and then the flat cohomology of X_M is isomorphic to its Galois cohomology. Denotes by $\widehat{\mathrm{H}^0(\Gamma_k^{\mathcal{C}}, M)}$ (resp. $\widehat{\mathrm{H}^0(\Gamma_k^{\mathcal{C}}, X_M)}$) the completion of $\mathrm{H}^0(\Gamma_k^{\mathcal{C}}, M)$ (resp. $\mathrm{H}^0(\Gamma_k^{\mathcal{C}}, X_M)$) with respect to the topology of subgroups of finite index (resp. of open subgroups of finite index) co-prime with $[k : L_0]_\infty$, that is, the pro-$\mathcal{C}$-topology. There is a natural pairing $\theta : M \times X_M \to \mathbf{G}_m$, which induces a cup-pairing $\theta_r : \mathrm{H}^r(\Gamma_k^{\mathcal{C}}, M) \times \mathrm{H}^{2-r}(\Gamma_k^{\mathcal{C}}, X_M) \to \mathrm{H}^2_{\mathrm{fppf}}(\Gamma_k^{\mathcal{C}}, \mathbf{G}_m) \simeq \mathrm{Br}(k)$. Then the following statements (Tate–Nakayama duality) hold.

1.1. Theorem (Tate-Nakayama duality)**.** *Let k be an infinite local field.*
(*a*) *The pairing θ induces a perfect pairing θ_0 for $r = 0$ between the compact group $\widehat{\mathrm{H}^0(\Gamma_k^{\mathcal{C}}, M)}$ and the discrete group $\mathrm{H}^2(\Gamma_k^{\mathcal{C}}, X_M)$.*
(*b*) *The groups $\mathrm{H}^1(\Gamma, M)$ and $\mathrm{H}^1(\Gamma, X_M)$ are finite and the pairing θ induces a perfect pairing for $r = 1$ between the finite groups $\mathrm{H}^1(\Gamma, M)$ and $\mathrm{H}^1(\Gamma, X_M)$, where Γ is either Γ_k or $\Gamma_k^{\mathcal{C}}$. The Tate–Nakayama duality for H^1 of tori holds. If T is an anisotropic k-torus, then $\mathrm{H}^2_{\mathrm{fppf}}(\Gamma_k^{\mathcal{C}}, T(\mathscr{S}_{f,k})) = 1$.*

(*c*) *The pairing* θ *induces a perfect pairing* θ_2 *for* $r = 2$ *between the discrete group* $\mathrm{H}^2(\Gamma_k^{\mathcal{C}}, M)$ *and the compact group* $\widehat{\mathrm{H}^0(\Gamma_k^{\mathcal{C}}, X_M)}$.
(*d*) *If* $([k^{ab} : k], [k : L]_\infty) = 1$, *then* (*a*) *and* (*c*) *also holds for* Γ_k.

2. Infinite algebraic extensions of local and global fields are of Kneser type. Recall that (see [14]) if S is a base scheme, $\tilde{G}$ a semisimple simply connected S-group scheme, with its center $Z(\tilde{G})$, $\mathrm{Ad}(G) := \tilde{G}/Z(\tilde{G})$, then S is called a scheme of *Douai type* if the coboundary map of flat cohomology $\Delta : \mathrm{H}^1_{\mathrm{fppf}}(S, \mathrm{Ad}(G)) \to \mathrm{H}^2_{\mathrm{fppf}}(S, Z(\tilde{G}))$ is surjective. We say that S is of *Kneser type*, if for any subgroup S-scheme $Z \subseteq Z(\tilde{G})$, the coboundary map of flat cohomology $\Delta : \mathrm{H}^1_{\mathrm{fppf}}(S, \tilde{G}/Z) \to \mathrm{H}^2_{\mathrm{fppf}}(S, Z(\tilde{G}))$ is surjective. We have "Kneser type" $\Rightarrow$ "Douai type" and apriori, the former notion is stronger than the latter one. It is well-known that if k is a local or global field (see [15, Thm. 2, p. 60 and Thm. 2, p. 77] for char. 0 case, and [7, Prop. 2.1], [8, Corol. 5.3], [9] and [29, Thm. A] for positive characteristic case), then k is of Kneser type. Some other examples of fields of Douai and/or of Kneser type are given in [3, Sec. 2], [14, Example 5.4, pp. 250–251] and [23, Corol. 1.7, Corol. 5.4]. We will show that any (infinite) algebraic extension of a local or global field or a completion of such extensions, is also of Kneser type. For a sheaf of groups A over some site S, the unit class ϵ_A in $\mathrm{H}^2_{\mathrm{fppf}}(S, A)$ is the equivalence class of the gerbe *Tors*(A) of all A-torsors (cf. [13, Chap. IV, Sec. 3]). We denote by $\mathrm{H}^2_{\mathrm{fppf}}(S, A)'$ the set of all neutral elements of $\mathrm{H}^2_{\mathrm{fppf}}(S, A)$. First we have

2.1. Theorem. *If a field* k *is of one of the following types*:
(*a*) *an algebraic extension of either a local field or an algebraic extension of a global field*;
(*b*) *a completion of a field of type* (*a*),
and $\tilde{G}$ *is a semisimple simply connected* k*-group, then* k *is of Kneser type and any element of* $\mathrm{H}^2_{\mathrm{fppf}}(k, \tilde{G})$ *is neutral.*

Next we consider some analogues of the well-known bijectivity results (cf. [1], [10], [14]) in the case of infinite local and global fields and also in the case of henselian fields and their completions. We refer to [1] and [14] for the definition and related properties of cohomology of quasi-abelian crossed modules, and in particular, of abelianized cohomology groups $\mathrm{H}^i_{ab}(k, G)$ and abelianization maps $ab^i : \mathrm{H}^i_{\mathrm{fppf}}(k, G) \to \mathrm{H}^i_{ab,\mathrm{fppf}}(k, G)$ of connected reductive groups.

2.2. Corollary. *Let* k *be an infinite algebraic extension of either a non-archimedean local field or a global field, with* $cd_2(k) \leq 2$ *if* $char.k = 0$ (*or the same,* k *has no real places*) *and let* G *be a connected reductive* k*-group. Then the corresponding abelianization map* $ab^1_G : \mathrm{H}^1_{\mathrm{fppf}}(k, G) \to \mathrm{H}^1_{ab,\mathrm{fppf}}(k, G)$ *is bijective.*

As analogues of [10, Prop. 3.5.3], we have the following

2.3. Proposition. *If a field* k *is a henselian field with respect to a valuation* v *and* k_v *the completion of* k *at* v, G *a connected reductive* k*-group, then the natural maps* $\mathrm{H}^i_{ab}(k, G) \to \mathrm{H}^i_{ab}(k_v, G)$ *are bijections for* $i \geq 1$.

2.4. Proposition (cf. [10, Prop. 3.5.3(3)] for commutative case)**.** *Let* k *be a henselian field with respect to a valuation* v, k_v *the completion of* k *at* v, *and let* G *be an affine* k*-group of finite type. Then the natural map* $\mathrm{H}^2_{\mathrm{fppf}}(k, G) \to \mathrm{H}^2_{\mathrm{fppf}}(k_v, G)$ *is a bijection, which induces a bijection* $\mathrm{H}^2_{\mathrm{fppf}}(k, G)' \to \mathrm{H}^2_{\mathrm{fppf}}(k_v, G)'$.

For abelianized cohomology of connected reductive groups over infinite local or gobal fields, we have

2.5. Proposition. *Let* k *be a field which is a henselian field with respect to a valuation* v *and let* k_v *be the completion of* k *at* v. *Then for any quasi-abelian crossed module* $(F \to G)$ *over* k, *the natural maps* $\mathbf{H}^i_{ab}(k, F \to G) \to \mathbf{H}^i_{ab}(k_v, F \to G)$ *are bijections for* $i \geq 1$.

3. Finiteness of Galois (flat) cohomology for pseudo-reductive groups. A k-group G is called *pseudo-reductive* (cf. [5], [6]) if G is affine, smooth, connected and the unipotent k-radical (that is, maximal, smooth, normal unipotent k-subgroup) $R_{u,k}(G)$ is trivial. The general classification of pseudo-reductive k-groups is shown in [5], [6] to be reduced to the case of generalized standard pseudo-reductive groups, basic exotic pseudo-reductive k-groups, $p \in \{2, 3\}$ and basic non-reduced pseudo-simple k-groups. By [5, Prop. 7.1.3], if k is a local field, G a pseudo-reductive k-group, which is generated by its maximal tori, then the flat cohomology set $\mathrm{H}^1_{\mathrm{fppf}}(k, G)$ is finite. A natural question arises about the finiteness of this set when k is an infinite algebraic extension of a local field and when k is or a completion of such a field (or the same, the completion of an infinite algebraic extension of a global field L). We have the following

3.1. Proposition. *Let k be a field such that for any semisimple simply connected k-group $\tilde{G}$, the flat cohomology $\mathrm{H}^1_{\mathrm{fppf}}(k,\tilde{G})$ is finite. Consider the following statements*:
(1) *For any connected reductive k-group G, the set $\mathrm{H}^1_{\mathrm{fppf}}(k,G)$ is finite*;
(2) *For any finite k-group scheme M of multiplicative type, the group $\mathrm{H}^2_{\mathrm{fppf}}(k,M)$ is finite.*
Then we have (2) $\Longrightarrow$ (1)*. Also, if k is of Kneser type, then we have* (1) $\Longrightarrow$ (2).

As a consequence, we derive the following

3.2. Corollary. *If k is either an infinite algebraic extension of a local field L, or a completion of such an extension, then for any connected reductive k-group G, $\mathrm{H}^1_{\mathrm{fppf}}(k,G)$ is finite.*

Now we arrive at another main results of the paper.

3.3. Theorem (cf. [26, Chap. III] for reductive groups and [5, Prop. 7.1.3] for pseudo-reductive groups over local fields)**.** *Let k be*
(1) *Either an infinite algebraic extension F* (*equipped with a valuation v*) *of a local field L, or*
(2) *A completion of such an extension.*
Then for any pseudo-reductive k-group G, which is generated by its maximal k-tori, the set $\mathrm{H}^1_{\mathrm{fppf}}(k,G)$ is finite.

4. Some applications to the obstruction of weak approximation. Let k be a number field (resp. global function field) and let G be a connected linear algebraic (resp. connected reductive) k-group. It is known (see [22, Sec. 3] for number field case and [30, Sec. 2] for function field case), that there is a finite set of places $S_0\subset V$ such that G has weak approximation outside S_0. Also, for any S, the obstruction to weak approximation $\mathrm{A}(S,G):=(\prod_{v\in S}G(k_v))/\overline{G(k)}$ in S and the global obstruction to weak approximation $\mathrm{A}(G):=(\prod_v G(k_v))/\overline{G(k)}$ over k are finite abelian groups.

In the case of an infinite algebraic extension k of a global field L, we may consider the localization field $k(v)$ of k at v and consider the notion of weak approximation with respect to localizations instead of completions. We say that the weak approximation in *the new sense* at a subset S of places of k holds for a k-variety X, if $X(k)$ is dense in $\prod_{v\in S}X(k(v))$. If $X=G$ is a k-group, then, as usual, we may consider the factor set $\mathrm{A}'(S,G):=(\prod_{v\in S}G(k(v)))/\overline{G(k)}$, resp. $\mathrm{A}'(G):=(\prod_v G(k(v)))/\overline{G(k)}$, which is called the *obstruction to weak approximation at S in the new sense* and the *obstruction to weak approximation over k in the new sense*, respectively. If $W\subseteq V$, we set $G_W:=\prod_{v\in W}G(k_v)$ and $G'_W:=\prod_{v\in W}G(k(v))$.

We refer the readers to [2] for the notion of z-extension and of flasque resolution used below. As an application of the results obtained in previous sections, we have the following

4.1. Theorem. *Let k be an infinite global field, S a finite set of places of k and let G be a connected reductive k-group.*
(1) *If $1\to F\to H_1\to G\to 1$ is a flasque resolution of G, then we have*

$$\mathrm{A}(S,G)\simeq \mathrm{Coker}\left(\mathrm{H}^1_{\mathrm{fppf}}(k,F)\xrightarrow{\gamma_S}\prod_{v\in S}\mathrm{H}^1_{\mathrm{fppf}}(k_v,F)\right),$$

$$\mathrm{A}(G)\simeq \mathrm{Coker}\left(\mathrm{H}^1_{\mathrm{fppf}}(k,F)\xrightarrow{\gamma_V}\prod_{v}\mathrm{H}^1_{\mathrm{fppf}}(k_v,F)\right).$$

(2) *For any z-extension $1\to Z\to H\to G\to 1$ of G, $T=H^{tor}=H/[H,H]$, we have canonical isomorphisms of finite abelian groups*

$$\mathrm{A}(S,G)\simeq\mathrm{A}(S,H)\simeq\mathrm{A}(S,T),$$
$$\mathrm{A}(G)\simeq\mathrm{A}(H)\simeq\mathrm{A}(T).$$

(3) *The set $\mathrm{A}(S,G)$ is finite and has a natural structure of an abelian group. There exists a well-defined set S_0 of places of k, which may not be finite, such that for any finite set S of places of k, we have $\mathrm{A}(S,G)=\mathrm{A}(S\cap S_0,G)$. In particular, for any $S\supset S_0$, we have $\mathrm{A}(S,G)=\mathrm{A}(S_0,G)$, and G has weak approximation property with respect to any finite set S outside S_0.*
(4) *For any finite separable extension L/k, let $G_L=G\times_k l$, S_L the extension of S to L. Then we have natural norm homomorphisms of finite abelian groups $N_{S,L/k}:\mathrm{A}(S_L,G_L)\to\mathrm{A}(S,G)$, and $N_{L/k}:\mathrm{A}(G_L)\to\mathrm{A}(G)$, which are functorial in G, and for a tower of finite separable extensions $E/K/k$, the norm homomorphisms satisfy $N_{S_E,E/k}=N_{S_K,K/k}\circ N_{S_E,E/K}$.*
(5) *The statements* (1)–(4) *also hold if we replace $\mathrm{A}(\cdot,\cdot)$ by $\mathrm{A}'(\cdot,\cdot)$, that is, k_v by $k(v)$ for all v.*

The structure of $\overline{G(k)}$ is given by the folowing

4.2. Theorem. *Let k be an infinite global field, G a connected reductive k-group and let W be any non-empty subset of V. Then*
(1) *The closure $\overline{G(k)}$ of $G(k)$ in G'_W* (*resp. G_W*) *is a normal subgroup of G'_W* (*resp. G_W*)*, which contains*

the derived subgroup $[G'_W, G'_W]$ *(resp.* $[G_W, G_W]$*).*
(2) *The following statements are equivalent*:
(*a*) *The closure* $\overline{G(k)}$ *of* $G(k)$ *in* G'_W *(resp.* G_W*) is an open subgroup of* G'_W *(resp.* G_W*).*
(*b*) G *has almost weak approximation in* W *(or weak weak approximation), i.e.,* G *has weak approximation outside a finite subset* $W_0 \subset W$.
(*c*) $\mathrm{A}(W, G)$ *is finite.*

4.3. R-equivalence and weak approximation. For connected reductive algebraic groups, there is a very close relation between some expression of the defect (obstruction) to the weak approximation and that of the R-equivalence. We refer to [3], [4], [11], [12], [28] and [31] for the notion of R-equivalence after Manin and some related results used here.

Let X be a smooth algebraic variety over a field k. We say that $x, y \in X(k)$ are *R-equivalent* if there is a sequence of points $z_i \in X(k)$, $x = z_1, y = z_n$, such that for each pair z_i, z_{i+1} there is a k-rational map $f_i : \mathbf{P}^1 \to X$, regular at 0 and 1, with $f(0) = z_i, f(1) = z_{i+1}, 1 \leq i \leq n-1$. X is called *rationally connected over* k, if any two points $x, y \in X(k)$ are R-equivalent. We then write $x \sim_R y$ and denote by $X(k)/R$ the set of R-equivalent classes of $X(k)$. Then X is rationally connected over k, if $X(k)/R = (1)$. It is known (cf. [4, Sec. 4, Prop. 10]), that if $\mathrm{char}.k = 0$, then $X(k)/R$ is a *birational invariant* of smooth complete algebraic varieties X defined over k. If G is a smooth affine k-group, then $G(k)/R$ has a natural group structure, which is compatible with the group structure on $G(k)$, i.e., the projection $G(k) \to G(k)/R$ is a group homomorphism. Moreover, $RG(k) := \{g \in G(k) \mid g \sim_R 1\}$ is a normal subgroup of $G(k)$ and we have canonically $G(k)/R \simeq G(k)/RG(k)$ (cf. [11, Lem. II.1.1(a)]). We have the following (cf. [3], [28] and [31] for geometric and global fields).

4.4. Theorem. *Let* k *be a field,* S *a finite set of places of* k *and let* G *be a connected reductive* k*-group. Assume that for we have*

$$(*) \qquad [G_S, G_S] \subseteq RG_S \subseteq \overline{G(k)},$$

where the closure is taken in G_S. *Then the quotient* $\mathrm{A}(S, G)$ *has a natural abelian group structure and we have the following exact sequence of groups*

$$(**) \qquad G(k)/R \stackrel{\varphi_S}{\to} (G_S)/RG_S \to \mathrm{A}(S, G) \to 1.$$

If k *is an infinite global field, then* $(*)$ *and* $(**)$ *hold for* k_v *and* $\mathrm{A}(\cdot,\cdot)$ *(resp. for* $k(v)$ *and* $\mathrm{A}'(\cdot,\cdot)$ *).*

4.5. The Kneser–Tits conjecture for algebraic groups. Recall the following regarding the Kneser–Tits conjecture (see [20, Chap. VII, Sec. 7.2] and [12]). For a field F and a F-isotropic semisimple F-group G, we say that the *Kneser–Tits conjecture holds for* $G(F)$ if we have $G(F) = G^+(F)$, where $G^+(F)$ denotes the subgroup of $G(F)$ generated by F-points of the unipotent radicals of all parabolic F-subgroups of G. If F is a non-archimedean local field, then by [20, Sec. 7.2, Thm. 6], the Kneser–Tits conjecture holds for all isotropic almost simple simply connected F-groups. The method of the proof is based on a detailed analysis of the Tits index of G combined with a reduction to groups of F-rank 1 due to Prasad and Raghunathan [21, Sec. 1.6].

4.6. Proposition. *Let* k *be an infinite global field, and let* v *be a place of* k.
(1) *If an almost simple simply connected group* G *is defined and isotropic on* $k(v)$ *(resp.* k_v*) then we have* $G^+(k(v)) = G(k(v))$ *(resp.* $G^+(k_v) = G(k_v)$*).*
(2) *With* G *as in* (1)*, we have* $G(k(v))/R = 1$, $G(k_v)/R = 1$.

4.7. The infinitude of global obstruction to weak approximation. In contrast to the global field case, the global obstruction to the weak approximation $\mathrm{A}(G)$ in the case of infinite global field may be infinite. Here we construct some examples of connected reductive groups G defined over an infinite global field k such that the obstruction $\mathrm{A}(G)$ is infinite.

Let L be a global field, R/L a Galois extension such that $\Gamma := Gal(R/L) \simeq (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$, and there are at least two places v of L with decomposition group $\Gamma_v = \Gamma$. (For example, we may take $L = \mathbf{Q}(\sqrt{a}, \sqrt{b})$ for suitable $a, b \in \mathbf{N}$ as in [4, p. 207].)

Let $L = L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots \subset k$ be an infinite tower of finite field extensions with union equal to k such that for all $i \geq 0$:
(1) $[L_{i+1} : L_i]$ is odd,
(2) the cardinality of the set $S_{i+1}(v)$ of extensions of v to L_{i+1} is strictly greater than that of $S_i(v)$. In particular, the set $S_k(v)$ of extensions of v to k is infinite.

Then we have

4.8. Proposition. *With above notation, assume that* L, k *satisfy the conditions* (1), (2)*. Then for the* k*-torus* $T_k := R^{(1)}_{R/L}(\mathbf{G}_m) \times_L k$, *the global obstruction to weak approximation* $\mathrm{A}(T)$ *of* T *over* *k is infinite.*

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