Euler tangent numbers modulo 720 and Genocchi numbers modulo 45

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Abstract: We establish congruences for higher order Euler polynomials modulo 720. We apply this result for constructing analogues of Stern congruences for Euler secant numbers $E_{4n} \equiv 5 (\text{mod} \ 60)$, $E_{4n+2} \equiv -1 (\text{mod} \ 60)$ to Euler tangent numbers and Genocchi numbers. We prove that Euler tangent numbers satisfy the following congruences $E_{4n+1} \equiv 16 (\text{mod} \ 720)$, and $E_{4n+3} \equiv -272 (\text{mod} \ 720)$. We establish 12-periodic property of Genocchi numbers modulo 45.

Key words: Higher-order Euler numbers; secant numbers; tangent numbers; Genocchi numbers; Ramanujan congruences.

1. Introduction. Euler numbers are defined as coefficients of Taylor series of the function

$$\text{sech}(x) + \tanh(x) = \sum_{n \geq 0} E_n \frac{x^n}{n!}.$$ 

Euler numbers and higher-order Euler numbers were studied in [4], [2]. Their congruences were studied in [3], [5], [6]. Higher-order Euler numbers are defined as coefficients of secant power

$$\text{sech}^q x = \sum_{n \geq 0} (-1)^n E_n^{(q)} \frac{x^{2n}}{(2n)!}.$$ 

Here $q$ might be any number, positive or negative. One can understand $q$ as a formal parameter and consider $E_n^{(q)}$ as a polynomial of $q$. We set

$$L_{2n}(q) = (-1)^n E_n^{(q)}.$$ 

For Euler secant numbers Stern established the following congruences ([1, p. 124], [4])

$$E_{4n} \equiv 5 (\text{mod} \ 60), \quad n > 0,$$

$$E_{4n+2} \equiv -1 (\text{mod} \ 60), \quad n \geq 0.$$ 

Later these congruences were re-discovered by Ramanujan.

The main result of our paper is the following

**Theorem 1.1.** For any $n > 0$ and for any integer $q$ the following congruences are valid

$$L_{4n}(24q - 2) \equiv L_4(24q - 2) (\text{mod} \ 720),$$

$$L_{4n+2}(24q - 2) \equiv L_6(24q - 2) (\text{mod} \ 720).$$ 

Since Euler tangent numbers are particular cases of higher-order Euler numbers, $E_{2n+1} = L_{2n}(-2)$, we obtain the following consequence of Theorem 1.1.

**Theorem 1.2.** For any $n > 0$ the following congruences hold

$$E_{4n+1} \equiv 16 (\text{mod} \ 720), \quad E_{4n+3} \equiv -272 (\text{mod} \ 720).$$

If

$$E_{4n+1} \equiv E_5 (\text{mod} \ N_1), \quad E_{4n+3} \equiv E_7 (\text{mod} \ N_3),$$

for any $n > 0$ with $N_1, N_3 \geq 720$, then $N_1 = N_3 = 720$.

The Genocchi numbers $G_n$ are a sequence of integers defined by the generating function

$$\sum_{n \geq 1} G_n \frac{x^n}{n!} = \frac{2x}{e^x + 1}.$$ 

In particular, $G_1 = 1$ and $G_n = 0$, if $n > 1$ is odd. All Genocchi numbers $G_{2n}$ are odd integers.

Combinatorial meaning of Genocchi numbers: $|G_{2n}|$ counts the number of permutations $\sigma \in S_{2n-1}$ with descends after even numbers and ascends after odd numbers. For example, if $n = 3$ then \{42135, 21435, 34215\} is a list of Genocchi permutations, and $|G_3| = 3$.

Euler tangent numbers are closely related to Bernoulli numbers $B_n$ and Genocchi numbers $G_n$, 

$$B_{2n} = \frac{2n}{4^{2n} - 2^{2n}} E_{2n-1},$$

$$G_n = 2(1 - 2^n) B_n,$$

$$G_{2n} = -2^{2-2n} n E_{2n-1}.$$ 

Application of Theorem 1.2 for Genocchi numbers gives us the following result.
Theorem 1.3. For any \( n \geq 0 \), Genocchi numbers satisfy the following congruences
\[
G_{12n} \equiv 3n \pmod{45}, \quad G_{12n+2,3} \equiv -6n - 16 \pmod{45}, \quad n > 0, \\
G_{12n+4} \equiv 3n + 1 \pmod{45}, \quad G_{12n+6} \equiv -6n - 3 \pmod{45}, \\
G_{12n+8} \equiv 3n + 17 \pmod{45}, \quad G_{12n+10} \equiv -6n + 25 \pmod{45}.
\]

Proof of theorem 1.1 is based on the following property of polynomial \( L_{2n}(q) \).

Theorem 1.4. The polynomials \( L_{2n}(q) \) can be constructed by recurrence relation
\[
(1) \quad L_{2n}(q) = \sum_{i=1}^{n} \left( \frac{2n-1}{2i-1} (q+1) - \frac{2n}{2i} \right) L_{2(n-i)}(q), \quad n > 0, \\
L_0(q) = 1.
\]

Proof of Theorem 1.4. A function \( f : A \rightarrow Q \) where \( A = \{1, 2, \ldots, 2n\}, \ Q = \{1, 2, \ldots, q\} \) is called an even pre-image if for any \( j \in Q \) the pre-image \( f^{-1}(j) \subseteq A \) has an even number of elements. Call for any \( j \in Q \) pre-image set
\[
B_j = f^{-1}(j) = \{i \mid f(i) = j\}
\]
as \( j \)-th block. Then \( |B_j| = p_j \) is even. In particular, \( p_j \) might be 0.

Let us prove that \( L_{2n}(q) \) is a number of even pre-image functions. Let \( \lambda \) be partition of \( n \) with length \( k = l(\lambda) \) and \( \lambda = 1^{r_1} 2^{r_2} \cdots n^{r_n} \) is a multiplicity form of this partition, i.e., \( r_i \) is a number of components of \( \lambda \) equal to \( i \). Then
\[
r_1 + \cdots + r_n = k, \quad 1r_1 + 2r_2 + \cdots + nr_n = n.
\]
Say that even pre-image function \( f : A \rightarrow Q \) has partition type 2\( \lambda \) if \( f \) has \( k \) non-empty blocks \( B_{\lambda_1}, \ldots, B_{\lambda_k} \), and their lengths generate partition 2\( \lambda \). Let us calculate the number of even pre-image functions with partition type \( 2^{r_1} 4^{r_2} \cdots (2n)^{r_n} \). Blocks \( B_{\lambda_i} \) with \( |B_{\lambda_i}| = 2\lambda_i \) can be selected in
\[
\frac{(2n)!}{\prod_{i=1}^{k} (2\lambda_i)!} - \frac{(2n)!}{\prod_{i=1}^{k-1} (2i)!}\]
ways. Since blocks with equal size can be permuted, number of block selections is equal to
\[
\frac{1}{r_1!r_2! \cdots r_n!} \prod_{i=1}^{n} (2i)!^{r_i}\]
First block might be a pre-image of \( q \) elements, second block might be a pre-image of \( q-1 \) elements, etc. So, the number of pre-image possibilities is equal to falling factorial
\[
q_k = q(q-1) \cdots (q-k+1).
\]
Therefore, the number of even pre-image functions is
\[
L_{2n}(q) = \sum_{\lambda \vdash n} \frac{2^{r_1} 4^{r_2} \cdots (2n)^{r_n}}{r_1!r_2! \cdots r_n!} q_{\lambda}.
\]

Taylor series of hyperbolic cosine is
\[
cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.
\]

Hence the coefficient of \( x^q \) at \( \frac{1}{q^2} \) is \( L_{2n}(q) \). This means that one can interpret higher-order Euler polynomial \( L_n(q) \) for integer \( q \) as a number of even pre-image functions \( f : A \rightarrow Q \).

Now we will show how to obtain formula (1). To do that, we will calculate the number of even pre-image functions in two ways.

First way. Suppose that \( f : A \rightarrow Q \) is an even pre-image function, \( 1 \in A \) belongs to block \( B_{\lambda_1} \), and \( |B_{\lambda_1}| = 2s \). Then, \( j_1 \) can be selected in \( q \) ways, and other elements of block \( B_{\lambda_1} \), except 1, can be selected in \( (2n-1) \) ways. The number of even pre-image functions \( g : A \setminus B_{\lambda_1} \rightarrow Q \setminus \{j_1\} \) is \( L_{2n-2s}(q-1) \). Therefore,
\[
(2) \quad L_{2n}(q) = \sum_{s=1}^{n} q \binom{2n-1}{2s-1} L_{2(n-s)}(q-1).
\]

Second way. We study pre-image possibilities for \( 1 \in Q \). If \( f^{-1}(1) = 0 \), then the number of such even pre-image functions is \( L_{2n}(q-1) \). If \( |f^{-1}(1)| = 2s \neq 0 \), then the elements of block \( B_1 \) can be selected in \( (2n-s) \) ways. The number of even pre-image functions \( h : A \setminus B_1 \rightarrow Q \setminus \{1\} \) is \( L_{2(n-s)}(q-1) \). Therefore,
\[
(3) \quad L_{2n}(q) = \sum_{s=0}^{n} \binom{2n}{2s} L_{2(n-s)}(q-1).
\]

By (2) and (3) we obtain
Therefore, have

All that remains is to change $q-1$ to $q$ to obtain (1).

**Sketch of the Proof of Theorem 1.1.** We have

$$L_4(q) = 3q^2 - 2q, \quad L_6(q) = 15q^3 - 30q^2 + 16q.$$ Therefore, 

$$L_4(24q - 2) = 16 - 336q + 1728q^2,$$

$$L_6(24q - 2) = -272 + 7584q - 69120q^2 + 207360q^3.$$ Hence Theorem 1.1 can be formulated as follows:

$$L_{4n}(24q - 2) \equiv 16 + 384q + 288q^2 (\text{mod } 720),$$

$$L_{4n+2}(24q - 2) \equiv 448 + 384q (\text{mod } 720).$$ Since $720 = 2^4 \cdot 3^2 \cdot 5$, by the Chinese remainder theorem these congruences are equivalent to the following congruences

(4) $L_{2n}(24q - 2) \equiv 0(\text{mod } 16), \quad n > 1,$

(5) $L_{2n}(24q - 2) \equiv 6q - 2(\text{mod } 9), \quad n > 0,$

(6) $L_{4n}(24q - 2) \equiv 1 + 4q + 3q^2 (\text{mod } 5), \quad n > 0,$

(7) $L_{4n+2}(24q - 2) \equiv 4q - 2 (\text{mod } 5), \quad n \geq 0.$

**Proof of congruence (4).** We proceed by induction on $n \geq 0$. If $n = 2$, then

$$L_4(24q - 2) = 3(24q - 2)^2 - 2(24q - 2) = 16(-1 + 9q)(-1 + 12q).$$ Therefore, relation (4) is true for $n = 2$. Suppose that this relation is valid for $n - 1 \geq 2$. Then by Theorem 1.4 we have

$$L_{2n}(24q - 2) = X_1 + X_2 + X_3,$$

where

$$X_1 = ((24q - 1) - 1) L_0(24q - 2) = 24q - 2,$$ $$X_2 = \left( \frac{2n - 1}{2(n - 1) - 1} \right) (24q - 1) - \left( \frac{2n}{2(n - 1)} \right) L_{2(n-1)}(24q - 2),$$ $$X_3 = \sum_{i=1}^{n-2} \left( \frac{2n - 1}{2i - 1} \right) (24q - 1) - \left( \frac{2n}{2i} \right) L_{2(n-i)}(24q - 2).$$ By the inductive hypothesis we have

$$L_{2(n-i)}(24q - 2) \equiv 0(\text{mod } 16), \quad 1 \leq i \leq n - 2.$$ Hence

$$X_3 = \sum_{i=1}^{n-2} \left( \frac{2n - 1}{2i - 1} \right) (24q - 1) - \left( \frac{2n}{2i} \right).$$

Therefore,

$$L_{2(n-i)}(24q - 2) = 0(\text{mod } 16).$$

Note that $X$ is even for any $n$ : if $n$ is odd, then $(n - 1)$ is even, and if $n$ is even, $(-n - 6q + 12nq)$ is even. Hence, we have

$$L_{2n}(24q - 2) \equiv 0(\text{mod } 16),$$

and the congruence (4) is proved.

**Proof of the congruence (5).** For $n = 0$ our statement is evident. Suppose that it is true for $n - 1$. Then by Theorem 1.4

$$L_{2n}(24q - 2) = ((24q - 1) - 1) L_0(24q - 2) + Y,$$

where

$$Y = \sum_{i=1}^{n-1} \left( \frac{2n - 1}{2i - 1} \right) (24q - 1) - \left( \frac{2n}{2i} \right).$$

By induction hypothesis

$$L_{2(n-i)}(24q - 2) \equiv 6q - 2(\text{mod } 9), \quad 1 \leq i \leq n - 1.$$ Further,

$$\sum_{i=1}^{n-1} \left( \frac{2n - 1}{2i - 1} \right) = 4^{n-1} - 1, \quad \text{if } n > 1,$$

$$\sum_{i=1}^{n-1} \left( \frac{2n}{2i - 1} \right) = 2^{2n-1} - 2, \quad \text{if } n > 0.$$ Therefore, we have the following modulo 9

$$Y \equiv \sum_{i=1}^{n-1} \left( \frac{2n - 1}{2i - 1} \right) (24q - 1) - \left( \frac{2n}{2i} \right) (6q - 2)$$

$$\equiv (6q - 2)(24q - 1) \sum_{i=1}^{n-1} \left( \frac{2n - 1}{2i - 1} \right)$$

$$- (6q - 2) \sum_{i=1}^{n-1} \left( \frac{2n}{2i} \right).$$
\[\equiv (6q - 2)((24q - 1)(4q - 1) - (2^{2n-1} - 2))\]
\[\equiv 3(4q - 1)(6q - 2)(8q - 1).\]

Since \(3(4q - 1) \equiv 0 \pmod{9}\), we obtain
\[Y \equiv 0 \pmod{9}.
\]

Therefore,
\[L_{2n}(24q - 2) = ((24q - 1) - 1)L_0(24q - 2) + Y \equiv 24q - 2 \equiv 6q - 2 \pmod{9},\]
and the congruence (5) is proved completely.

Similar arguments show that (6) and (7) are valid as well.

All that remains is to use Chinese remainder theorem to get
\[E_{4n+1} \equiv E_5 \pmod{720}, \quad E_{4n+3} \equiv E_7 \pmod{720},\]
for any \(n > 0\). Suppose that for some integers \(N_1 \geq 720\) and \(N_3 \geq 720\) the following congruences are valid
\[E_{4n+1} \equiv E_5 \pmod{N_1}, \quad E_{4n+3} \equiv E_7 \pmod{N_3},\]
for any \(n > 0\). In particular, they are valid for \(n = 2, 3\). We have
\[E_5 = 16, \quad E_7 = -272, \quad E_9 = 7936, \]
\[E_{11} = -353792, \quad E_{13} = 22368256, \]
\[E_{15} = -1903757312,\]
and,
\[\text{GCD}(E_9 - E_5, E_{13} - E_9) = 720,\]
\[\text{GCD}(E_7 - E_{11}, E_7 - E_{15}) = 720.\]
Therefore,
\[N_1 = 720, \quad N_3 = 720.\]

So, the number 720 as a base of congruence in Theorem 1.1 is optimal.

**Sketch of the proof of Theorem 1.3.** The proof is based on the following four facts. First, by Theorem 1.2 \(E_{4n+1} \equiv 16 \pmod{45}\), \(E_{4n+3} \equiv -272 \pmod{45}\). Second, we use the following connection between Genocchi numbers and tangent numbers \(G_{2n} = nE_{2n-1} - 2^{2n-1} n\). Third, Genocchi numbers are odd. Fourth, \(16^n \pmod{45} \equiv 1, 16, 31\), if \(n \equiv 0, 1, 2 \pmod{3}\) respectively.

Let us give the proof of congruences for \(G_{12n}, G_{12n+2}, G_{12n+4}, G_{12n+6}\). Other congruences mentioned in Theorem 1.3 can be established by similar arguments.

We have
\[G_{12n} = -2^{2-12n} \cdot 6n \cdot E_{12n-1} = -16^{-3n} \cdot 3n \cdot 2^3 E_{12n-1},\]
\[2^3 E_{12n-1} \equiv -2^3 \cdot 272 \pmod{45} \equiv -16 \pmod{45}.
\]
Therefore,
\[G_{12n} \equiv 1 \cdot 48n \pmod{45} \equiv 3n \pmod{45}.
\]

Further,
\[G_{12n+2} = -2^{-12n} \cdot (6n + 1) \cdot E_{12n+1},\]
\[E_{12n+1} \equiv 16 \pmod{45},\]
and,
\[G_{12n+2} \equiv -16^{-3n} \cdot 16(6n + 1) \pmod{45} \equiv -6n - 16 \pmod{45}.
\]

We have
\[G_{12n+4} = -2^{-12n} \cdot (6n + 2) E_{12n+3} \]
\[= -16^{-3n} \cdot 2^{-1}(3n + 1) E_{12n+3},\]
\[2^{-1} E_{12n+3} \equiv -1 \pmod{45}.
\]
Hence,
\[G_{12n+4} \equiv 3n + 1 \pmod{45}.\]

Further,
\[G_{12n+6} = -2^{-12n} \cdot (6n + 3) E_{12n+5} \]
\[= -16^{-3n} \cdot (6n + 3) 16^{-1} E_{12n+5},\]
\[16^{-1} E_{12n+5} \equiv 1 \pmod{45}.\]
Therefore,
\[G_{12n+6} \equiv -16^{-3n} \cdot (6n + 3) \equiv -6n - 3 \pmod{45}.
\]

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**References**


