On minimality of the invariant Hilbert scheme associated to Popov's $SL(2)$-variety

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Abstract: This article gives a necessary and sufficient condition for the invariant Hilbert scheme studied in [Kub] to be the minimal resolution of a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety.

Key words: Invariant Hilbert scheme; spherical variety; minimal resolution.

1. Introduction. A variety with an action of a reductive algebraic group is called *quasihomogeneous* if it contains a dense open orbit. This article considers quasihomogeneous $SL(2)$-varieties. Three dimensional affine normal quasihomogeneous $SL(2)$-varieties containing more than one orbit have been actively studied since their complete classification was obtained by Popov [Pop73]. The classification requires discrete parameters $l = p/q \in \{\mathbf{Q} \cap (0,1]\}$ and $m \in \mathbf{N}$, and the variety $E_{l,m}$ corresponding to a pair (l, m) is smooth if and only if $l = 1$ and m is arbitrary; otherwise it has a unique singular point at the origin, which turns out to be $SL(2)$-invariant. After the work of Popov, Panyushev [Pan88] constructed the minimal resolution of singularities $\mathcal{W}$ of $E_{l,m}$. Here a resolution of singularities $f: Y \to X$ is said to be *minimal* if the canonical divisor K_Y of Y is f-nef, i.e., if $K_Y \cdot C \geq 0$ holds for any curve $C \subset Y$ that is contracted to a point under f. Batyrev and Haddad [BH08] described $E_{l,m}$ as a categorical quotient of a hypersurface H_{q-p} in $\mathbf{C}^5$ modulo an action of a diagonalizable group $G_0 \times G_m \cong \mathbf{C}^* \times \mu_m$. In the same article, they defined an additional $\mathbf{C}^*$-action on $E_{l,m}$ and showed that $E_{l,m}$ becomes a spherical $SL(2) \times \mathbf{C}^*$-variety (see §2.3). In [Kub], we used the quotient description to study $E_{l,m}$ through the invariant Hilbert scheme $\mathcal{H} := \mathrm{Hilb}^{G_0 \times G_m}_{h_{H_{q-p}}}(H_{q-p})$ that comes together with the Hilbert–Chow morphism

$$\gamma : \mathcal{H} \to H_{q-p} /\!/ (G_0 \times G_m) \cong E_{l,m},$$

and we obtained the following result (see §2.1 for the definition of the invariant Hilbert scheme).

Theorem 1 ([Kub, Corollaries 4.3 and 10.3 and Theorem 5.4]). *For any pair (l, m), $\mathcal{H}$ is irreducible and reduced, and γ is an equivariant resolution of singularities. Moreover, $\mathcal{H}$ is described as follows*:

(i) *If $l = 1$ and m is arbitrary, then $\mathcal{H}$ is isomorphic to $E_{1,m}$.*

(ii) *If $l < 1$ and if $E_{l,m}$ is toric (i.e., if $q - p$ divides m, see Theorem 12), then $\mathcal{H}$ is isomorphic to the blow-up $Bl_O(E_{l,m})$ of $E_{l,m}$ at the origin.*

(iii) *If $l < 1$ and if $E_{l,m}$ is non-toric, then $\mathcal{H}$ is isomorphic to the minimal resolution of a weighted blow-up $Bl^{\omega}_O(E_{l,m})$ of weight ω, where ω depends on the parameters l and m (see §2.3 for the definition of ω).*

It is then natural to ask if $\mathcal{H}$ is minimal over $E_{l,m}$. In this article, we give a necessary and sufficient condition for $\mathcal{H}$ to coincide with the minimal resolution $\mathcal{W}$ of $E_{l,m}$ constructed by Panyushev. Set

$$k := g.c.d.(m, q-p), \quad a := \frac{m}{k}, \quad b := \frac{q-p}{k}.$$

Then the main result can be formulated as follows:

Theorem 2. *γ is the minimal resolution of $E_{l,m}$ if and only if $1 + b \leq ap$.*

Remark 3. If $E_{l,m}$ is toric, then γ is the minimal resolution if and only if $p > 1$ or $m \neq q - 1$.

2. Preliminaries. In §2.1, we review the definition of the invariant Hilbert scheme introduced by Alexeev and Brion ([AB05, Bri13]). In §2.2

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and §2.3, we collect some known results on spherical varieties and quasihomogeneous $SL(2)$-varieties, respectively.

2.1. The invariant Hilbert scheme. Let G be a reductive algebraic group, and let V be a G-module. Consider the isotypic decomposition $V \cong \bigoplus_{M\in\mathrm{Irr}(G)} \mathrm{Hom}^G(M,V)\otimes M$ of V, where $\mathrm{Irr}(G)$ denotes the set of isomorphism classes of irreducible representations of G. If the dimension of $\mathrm{Hom}^G(M,V)$ is finite for every $M\in\mathrm{Irr}(G)$, it defines a function $h_V:\mathrm{Irr}(G)\to\mathbf{Z}_{\geq 0}$ that sends an irreducible representation M to its multiplicity $\dim\mathrm{Hom}^G(M,V)$ in V. This function h_V is called the *Hilbert function* of V.

Given an affine G-variety X and a Hilbert function $h:\mathrm{Irr}(G)\to\mathbf{Z}_{\geq 0}$, the *invariant Hilbert scheme* $\mathrm{Hilb}_h^G(X)$ associated with the triple (G,X,h) is a moduli space that parametrizes closed G-subschemes Z of X such that $\mathbf{C}[Z]\cong\bigoplus_{M\in\mathrm{Irr}(G)} M^{\oplus h(M)}$ as G-modules. Let $\pi:X\to X /\!/ G := \mathrm{Spec}(\mathbf{C}[X]^G)$ be the quotient morphism, and let $U\subset X/\!/G$ be the flat locus of π. Then, the coordinate ring of every scheme-theoretic fiber of $\pi:\pi^{-1}(U)\to U$ has the same Hilbert function, which is called the *Hilbert function of a general fiber* of π, and we denote it by h_X. The associated invariant Hilbert scheme $\mathrm{Hilb}_{h_X}^G(X)$ is known to become a candidate for a resolution of singularities of $X/\!/G$ via the *Hilbert–Chow morphism*

$$\gamma:\mathrm{Hilb}_{h_X}^G(X)\to X/\!/G$$

that sends a closed G-subscheme Z to a point $Z/\!/G$: the morphism γ is projective and induces an isomorphism over the flat locus $U\subset X/\!/G$. For details, refer to [Bri13].

Remark 4. If G is finite, then the Hilbert function of a general fiber of $\pi:X\to X/G$ is the Hilbert function of the regular representation $\mathbf{C}[G]$, and the associated invariant Hilbert scheme $\mathrm{Hilb}_{h_{\mathbf{C}[G]}}^G(X)$ coincides with the G-Hilbert scheme G-$\mathrm{Hilb}(X)$ of Ito and Nakamura [IN96]. The G-Hilbert scheme G-$\mathrm{Hilb}(X)$ is known to give a crepant resolution of singularities of the quotient variety X/G if X is a smooth variety of dimension less than four and if the G-action is Gorenstein ([IN96, Nak01, BKR01]).

2.2. Canonical divisor of spherical varieties. Spherical varieties are classified by combinatorial objects called *colored fans*, which are generalization of fans for toric varieties (see e.g. [Kno91, Per14] for details). Let G be a connected reductive algebraic group, and let H be an algebraic subgroup of G. A normal G-variety X is called *spherical* if it contains a dense open orbit under a Borel subgroup B of G. By a *spherical embedding*, we mean a normal G-variety X together with an equivariant open embedding $G/H\hookrightarrow X$ of a homogeneous spherical variety G/H. Below we gather known results that we use in the next section.

Definition 5. Keep the notation above.

(i) We denote by $\mathcal{M}$ the set of rational B-eigenfunctions on G/H, i.e., $\mathcal{M}=\{f\in\mathbf{C}(G/H)^*:\exists\chi_f\in\mathfrak{X}(B)\ \forall b\in B\ b\cdot f=\chi_f(b)f\}$, where $\mathfrak{X}(B)$ stands for the group of characters of B. The image of a homomorphism $\tau:\mathcal{M}\to\mathfrak{X}(B)$ defined by $f\mapsto\chi_f$ is a finitely generated free abelian group, which we denote by Γ. Since G/H contains a dense open B-orbit, the kernel of τ consists of constant functions.

(ii) We denote by $\mathcal{D}(X)$ the set of B-stable prime divisors on X. We simply write $\mathcal{D}$ for $\mathcal{D}(G/H)$. A *color* of X is a B-stable but not G-stable prime divisor.

(iii) Any $D\in\mathcal{D}$ defines a point ρ_D in $Q:=\mathrm{Hom}(\Gamma,\mathbf{Q})$ such that $\rho_D(\chi_f)=v_D(f)$ for any $\chi_f\in\Gamma$, where v_D stands for the valuation defined by the divisor D.

(iv) Let $\mathcal{V}$ denote the set of G-invariant valuations on $\mathbf{C}(G/H)^*$. Any $v\in\mathcal{V}$ defines a point $\rho_v\in Q$ in a similar way as above, and the map $\mathcal{V}\to Q$, $v\mapsto\rho_v$ is injective. The image of $\mathcal{V}$, which we denote by the same symbol, is a cone in Q, call the *valuation cone*.

Definition 6 ([Pas17, Definition 2.8])**.** A primitive element of a ray of the opposite $-\mathcal{V}^\vee$ of the dual in $\Gamma\otimes_{\mathbf{Z}}\mathbf{Q}$ is called a *spherical root* of X.

Theorem 7 ([Pas17, Theorem 2.15])**.** *Let $D\in\mathcal{D}$, and choose a simple root α with respect to B such that $P_\alpha\cdot D\neq D$, where P_α denotes the minimal parabolic subgroup corresponding to α. Then, one and only one of the following cases occurs*:

(i) *α is a spherical root of G/H;*

(ii) *2α is a spherical root of G/H;*

(iii) *neither α nor 2α is a spherical root of G/H.*

Remark 8 ([Pas17, §2])**.** The anticanonical divisor of a spherical embedding $G/H\hookrightarrow X$ can be written in the form

$$-K_X = \sum_{D \in \mathcal{D}(X) \setminus \mathcal{D}} D + \sum_{D \in \mathcal{D}} a_D D,$$

where a_D is determined according to the type of D classified in Theorem 7, which does not depend on the choice of the simple root. Denote by $P \subset G$ the stabilizer of the open B-orbit of G/H, and by S_P the set of simple roots α such that $-\alpha$ is not a weight of the Lie algebra of P. Then the integer a_D is given as follows: if D is of type (i) or (ii), then $a_D = 1$; if D is of type (iii), then $a_D = \sum_{\alpha \in \mathcal{R}_P^+} \langle \alpha, \alpha^\vee \rangle$, where $\mathcal{R}_P^+$ stands for the set of positive roots with at least one non-zero coefficient for a simple root of S_P.

Remark 9. Keep the notation of Remark 8. According to [Pas17], $-K_X$ is associated to a piecewise linear function h_{-K_X} on the colored fan $\mathfrak{F}(X)$ of X, which is linear on each colored cone $(\mathcal{C}, \mathcal{F})$ in $\mathfrak{F}(X)$, such that the restriction $h_{\mathcal{C}} := h_{-K_X}|_{\mathcal{C}}$ to $(\mathcal{C}, \mathcal{F})$ is given as $h_{\mathcal{C}}(\rho_D) = a_D$ for any $D \in \mathcal{F}$ (with the notation of Definition 5 (iii)), and $h_{\mathcal{C}}(v) = 1$ for any primitive element v of a ray of $\mathcal{C}$ that is not generated by some ρ_D with $D \in \mathcal{F}$.

Remark 10. Assume that X is a $\mathbf{Q}$-Gorenstein spherical G/H-embedding. Given a G-equivariant resolution of singularities $f : Y \to X$, one has $K_Y = f^* K_X + \sum_{i \in I} a_i F_i$ for some $a_i \in \mathbf{Q}$, where $\{F_i : i \in I\}$ is the set of exceptional divisors of f. Let $(\mathcal{C}, \mathcal{F})$ be a colored cone of $\mathfrak{F}(X)$ such that $\rho_{F_i} \in \mathcal{C}$ under the notation of Definition 5 (iii). Then, according to the proof of [Pas17, Proposition 5.2], a_i can be calculated as $h_{\mathcal{C}}(\rho_{F_i}) - 1$.

2.3. Classification of quasihomogeneous $SL(2)$-varieties and related works. Popov's classification is as follows:

Theorem 11 ([Pop73, Corollary of Proposition 9])**.** *Every 3-dimensional affine normal quasihomogeneous $SL(2)$-variety containing more than one orbit is uniquely determined by a pair of numbers $(l, m) \in \{\mathbf{Q} \cap (0, 1]\} \times \mathbf{N}$.*

We denote by $E_{l,m}$ the variety corresponding to a pair (l, m). It is known that a necessary and sufficient condition for $E_{l,m}$ to be a toric variety can be given in terms of the parameters:

Theorem 12 ([Gaï08], see also [BH08, Corollary 2.7])**.** *$E_{l,m}$ is toric if and only if $q - p$ divides m.*

Below we recall theorems from [BH08], starting with the quotient construction of $E_{l,m}$. We take X_0, X_1, X_2, X_3, X_4 to be the coordinates of $\mathbf{C}^5$ and consider a hypersurface $H_{q-p} \subset \mathbf{C}^5$ defined by the equation $X_0^{q-p} = X_1 X_4 - X_2 X_3$. Then, $SL(2)$ acts trivially on X_0 and by left multiplication on $\begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}$, preserving H_{q-p}. We also consider actions of the following diagonalizable groups:

$$G_0 := \{\mathrm{diag}(t, t^{-p}, t^{-p}, t^q, t^q) : t \in \mathbf{C}^*\} \cong \mathbf{C}^*,$$
$$G_m := \{\mathrm{diag}(1, \zeta^{-1}, \zeta^{-1}, \zeta, \zeta) : \zeta^m = 1\} \cong \mu_m.$$

We see that the $SL(2)$-action on $\mathbf{C}^5$ commutes with the action of $G := G_0 \times G_m$.

Theorem 13 ([BH08, Theorem 1.6])**.** *The affine quotient $H_{q-p} /\!/ G$ is isomorphic to $E_{l,m}$.*

Remark 14. The quotient description $E_{l,m} \cong H_{q-p} /\!/ G$ essentially comes from the theory of Cox rings: according to the proof of [BH08, Theorem 1.7], G contains a subgroup isomorphic to $G'_k = \{\mathrm{diag}(\zeta, 1, 1, 1, 1) : \zeta^k = 1\}$, and the coordinate ring of the G'_k-quotient of H_{q-p} is isomorphic to the Cox ring of $E_{l,m}$ ([BH08, Corollary 2.6]).

Remark 15. The dense $SL(2)$-orbit $\mathfrak{U} \subset E_{l,m}$ is isomorphic to

$$(H_{q-p} \cap \{X_0 \neq 0\}) /\!/ G \cong \mathrm{Spec}(\mathbf{C}[X, Y, Z, W]^{G_m}),$$

where X, Y, Z, W are G_0-invariant monomials defined as follows: $X := X_0^p X_1$, $Y := X_0^{-q} X_3$, $Z := X_0^p X_2$, $W := X_0^{-q} X_4$ (see the proof of [BH08, Theorem 1.6]).

Batyrev and Haddad studied the $SL(2)$-variety $E_{l,m}$ further by using the quotient description. First, they considered an action of $\mathbf{C}^*$ on $E_{l,m}$, which is induced by that of the diagonal matrices

$$\{\mathrm{diag}(1, s^{-1}, s^{-1}, s, s) : s \in \mathbf{C}^*\}$$

on H_{q-p}, and showed that $E_{l,m}$ becomes a spherical $SL(2) \times \mathbf{C}^*$-variety ([BH08, Proposition 4.1]). Let B be the Borel subgroup of $SL(2)$ consisting of upper triangular matrices, and set $\widetilde{B} := B \times \mathbf{C}^*$. Then, $E_{l,m}$ contains exactly three $\widetilde{B}$-stable prime divisors:

$$D := (H_{q-p} \cap \{X_0 = 0\}) /\!/ G,$$
$$S^- := (H_{q-p} \cap \{X_4 = 0\}) /\!/ G,$$
$$S^+ := (H_{q-p} \cap \{X_2 = 0\}) /\!/ G.$$

Note that D is stable under the action of $SL(2) \times \mathbf{C}^*$, while S^- and S^+ are not: S^- and S^+ are colors.

Remark 16. According to [BH08, Proposition 3.6], S^+ is isomorphic to the affine normal toric variety defined by the following semigroup:

$$M_{l,m}^+ := \{(i,j) \in \mathbf{Z}^2_{\geq 0} : j \leq li,\ m|(i-j)\}.$$

Let $\mathbf{e}_1 = (\frac{1}{m}, -\frac{1}{m})$, $\mathbf{e}_2 = (0,1) \in \mathbf{R}^2$. Then, the cone σ corresponding to S^+ is spanned by the vectors $ap\mathbf{e}_1 - b\mathbf{e}_2$ and $\mathbf{e}_2$. Therefore, if $1 + b \leq ap$, S^+ is always singular and isomorphic to $\mathbf{C}^2/\mu_{ap}$, where the action is given by $(z_1, z_2) \mapsto (\zeta z_1, \zeta^b z_2)$ for $\zeta^{ap} = 1$. If $b \geq ap$, then S^+ is smooth if and only if $ap = 1$; otherwise S^+ is isomorphic to the cyclic quotient singularity of type $\frac{1}{ap}(1, y)$, where y is the remainder of b divided by ap. We note that $b = ap$ happens only if $E_{l,m}$ is toric, in which case the condition $b = ap$ is equivalent to $p = 1$ and $m = q - 1$. We will refer to this remark again in Remark 23 and Example 25.

Second, they described an equivariant flip

$$\begin{tikzcd} E_{l,m}^- \arrow[rr, dotted] \arrow[dr] & & E_{l,m}^+ \arrow[dl] \\ & E_{l,m} & \end{tikzcd}$$

by different GIT quotients $E_{l,m}^-$ and $E_{l,m}^+$ of H_{q-p} corresponding to some non-trivial characters. Third, they constructed a weighted blow-up $E'_{l,m} := Bl^{\omega}_O(E_{l,m})$ of $E_{l,m}$ with a weight ω defined by the above-mentioned $\mathbf{C}^*$-action on $E_{l,m}$. The exceptional divisor D' of the weighted blow-up $E'_{l,m} \to E_{l,m}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and we obtain surjective morphisms $\gamma^- : E'_{l,m} \to E^-_{l,m}$ and $\gamma^+ : E'_{l,m} \to E^+_{l,m}$ by contracting $\mathbf{P}^1 \times \mathbf{P}^1$ in different directions to $\mathbf{P}^1$. Moreover, $E'_{l,m}$ has cyclic quotient singularities $\mathbf{C}^2/\mu_b$ of type $\frac{1}{b}(1,t)$ along the curve C that is embedded diagonally into $\mathbf{P}^1 \times \mathbf{P}^1 \cong D'$, where $t := (s+1)b - ap$ by setting s to be the quotient of mp divided by $q - p$ (see [BH08, §3] for details, see also [Kub, §5]).

Remark 17. By Theorem 12, $E_{l,m}$ is toric if and only if $b = 1$. Therefore, $E'_{l,m}$ is smooth if and only if $E_{l,m}$ is toric. Furthermore, if $E_{l,m}$ is toric, then the weight ω is trivial, in which case $E'_{l,m}$ is the usual blow-up.

Proposition 18 ([BH08, Proposition 3.13])**.** *Let $C^{\pm}$ be the image of D' under the morphism $\gamma^{\pm}$. Then the canonical divisor $K_{E^{\pm}_{l,m}}$ of $E^{\pm}_{l,m}$ has the following intersection number with $C^{\pm}$:*

$$K_{E^-_{l,m}} \cdot C^- = -\frac{(1+b)k}{aq^2}, \quad K_{E^+_{l,m}} \cdot C^+ = \frac{(1+b)k}{ap^2}.$$

3. Proof of Theorem 2. In [Kub], we have seen that the invariant Hilbert scheme $\mathcal{H}$ is obtained by minimally resolving the locally trivial family of quotient singularities $\mathbf{C}^2/\mu_b$, so that the Hilbert–Chow morphism $\gamma : \mathcal{H} \to E_{l,m}$ factors through $E'_{l,m}$, namely $\gamma = \psi \circ \varphi$ with the notation of the equivariant commutative diagram below.

$$\begin{tikzcd} & \mathcal{H} \arrow[dl] \arrow[d, "\psi"] \arrow[dr] & \\ E^-_{l,m} \arrow[dr] & E'_{l,m} \arrow[l, "\gamma^-"'] \arrow[r, "\gamma^+"] \arrow[d, "\varphi"] & E^+_{l,m} \arrow[dl] \\ & E_{l,m} & \end{tikzcd}$$

In proving Theorem 2, it is sufficient to show that $K_{E'_{l,m}}$ is φ-nef if and only if $1 + b \leq ap$, concerning that ψ is the minimal resolution. Moreover, we have the following

Lemma 19. *$K_{E'_{l,m}}$ is φ-nef if and only if $K_{E'_{l,m}}$ is γ^--nef and γ^+-nef.*

Proof. Let $\widetilde{C^-}$ and $\widetilde{C^+}$ be generators of the Picard group $\mathrm{Pic}(D') \cong \mathbf{Z}^2$ such that $\gamma^{\pm}(\widetilde{C^{\mp}})$ is a point. Then, the classes $[\widetilde{C^-}]$ and $[\widetilde{C^+}]$ generate the Kleiman-Mori cone $\overline{\mathrm{NE}}(E'_{l,m}/E_{l,m})$ of φ, and the lemma follows by taking into account that γ^- (resp. γ^+) is the contraction of the extremal ray generated by $[\widetilde{C^+}]$ (resp. $[\widetilde{C^-}]$). $\square$

The canonical divisor $K_{E'_{l,m}}$ can be expressed in two ways with some $\alpha, \beta \in \mathbf{Q}$ as follows:

$$K_{E'_{l,m}} = (\gamma^-)^* K_{E^-_{l,m}} + \alpha D' = (\gamma^+)^* K_{E^+_{l,m}} + \beta D',$$

concerning that $E^-_{l,m}$ and $E^+_{l,m}$ are $\mathbf{Q}$-factorial.

Lemma 20. *$K_{E'_{l,m}}$ has the following intersection numbers with $\widetilde{C^-}$ and $\widetilde{C^+}$:*

$$K_{E'_{l,m}} \cdot \widetilde{C^-} = \frac{\beta(1+b)k}{(\alpha-\beta)aq^2}, \quad K_{E'_{l,m}} \cdot \widetilde{C^+} = \frac{\alpha(1+b)k}{(\alpha-\beta)ap^2}.$$

Proof. We have

$$K_{E'_{l,m}} \cdot \widetilde{C^-} = K_{E^-_{l,m}} \cdot C^- + \alpha D' \cdot \widetilde{C^-} = \beta D' \cdot \widetilde{C^-}$$

and

$$K_{E'_{l,m}} \cdot \widetilde{C^+} = \alpha D' \cdot \widetilde{C^+} = K_{E^+_{l,m}} \cdot C^+ + \beta D' \cdot \widetilde{C^+},$$

so that the lemma follows from Proposition 18. $\square$

In the following, we calculate the coefficients α and β by using combinatorial datum of the colored cones of the simple spherical varieties $E_{l,m}$, $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$. We denote by $\mathfrak{X}(\widetilde{B})$ the group of characters of $\widetilde{B}$, and by $\mathcal{M}$ the lattice of rational $\widetilde{B}$-eigenfunctions on the dense open orbit $\mathfrak{U}$. Then we have $\mathcal{M} = \{Z^i W^j \in \mathbf{C}(\mathfrak{U})^* : m|(i-j)\}$, which is

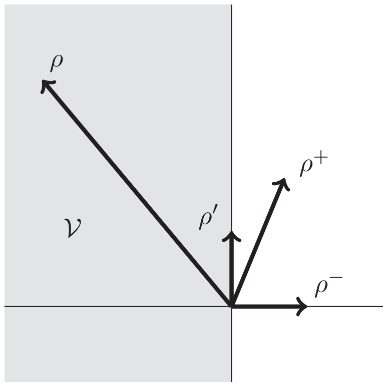

Fig. 1.

generated by ZW and Z^m. Since $T \times \mathbf{C}^* \subset \widetilde{B}$ acts on Z^iW^j via $(t,s)\cdot Z^iW^j = t^{i+j}s^{i-j}Z^iW^j$, where T is the maximal torus of $SL(2)$, the natural homomorphism $\tau : \mathcal{M} \to \mathfrak{X}(\widetilde{B}) \cong \mathbf{Z}^2$ is given by $Z^iW^j \mapsto (i+j, i-j)$. Let Γ be the image of τ, and set $\mathbf{v}_1 := \tau(ZW) = (2,0)$ and $\mathbf{v}_2 := \tau(Z^m) = (m,m)$. Note that $\mathbf{v}_1$ is a simple root of $(SL(2)\times\mathbf{C}^*, \widetilde{B})$, and that $P_{\mathbf{v}_1} = SL(2)\times\mathbf{C}^*$ with the notation of Theorem 7. If we denote the dual basis of $\{\mathbf{v}_1,\mathbf{v}_2\}$ by $\{\mathbf{u}_1,\mathbf{u}_2\}$, the lattice vectors ρ, ρ^-, ρ^+, and ρ' in $\Gamma^\vee := \mathrm{Hom}(\Gamma,\mathbf{Z}) \subset Q := \mathrm{Hom}(\Gamma,\mathbf{Q})$ defined by the $\widetilde{B}$-stable divisors D, S^-, S^+, and D' can be expressed as follows (see Fig. 1): $\rho = -b\mathbf{u}_1 + ap\mathbf{u}_2$, $\rho^- = \mathbf{u}_1$, $\rho^+ = \mathbf{u}_1 + m\mathbf{u}_2$, $\rho' = \mathbf{u}_2$. The valuation cone $\mathcal{V}\subset Q$ is given as $\mathcal{V} = \{x\mathbf{u}_1 + y\mathbf{u}_2 \in Q : x \le 0\}$ (see [BH08, §4], see also [Had10, Proposition 4.2.5]), and $-\mathcal{V}^\vee$ is the ray generated by $\mathbf{v}_1$, which turns out that $\mathbf{v}_1$ is a spherical root. Therefore, the divisors S^- and S^+ are of type (i) in Theorem 7. The colored cones of $E_{l,m}$, $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$ are described as follows:

$$\begin{cases} \mathcal{C} = \mathcal{C}(E_{l,m}) = \mathbf{Q}_{\ge 0}\rho + \mathbf{Q}_{\ge 0}\rho^- \\ \mathcal{F} = \mathcal{F}(E_{l,m}) = \{\rho^+, \rho^-\} \end{cases},$$

$$\begin{cases} \mathcal{C}^- = \mathcal{C}(E^-_{l,m}) = \mathbf{Q}_{\ge 0}\rho + \mathbf{Q}_{\ge 0}\rho^+ \\ \mathcal{F}^- = \mathcal{F}(E^-_{l,m}) = \{\rho^+\} \end{cases},$$

$$\begin{cases} \mathcal{C}^+ = \mathcal{C}(E^+_{l,m}) = \mathbf{Q}_{\ge 0}\rho + \mathbf{Q}_{\ge 0}\rho^- \\ \mathcal{F}^+ = \mathcal{F}(E^+_{l,m}) = \{\rho^-\} \end{cases},$$

$$\begin{cases} \mathcal{C}' = \mathcal{C}(E'_{l,m}) = \mathbf{Q}_{\ge 0}\rho + \mathbf{Q}_{\ge 0}\rho' \\ \mathcal{F}' = \mathcal{F}(E'_{l,m}) = \emptyset \end{cases}.$$

Remark 21. Colored cones of $E_{l,m}$, $E^-_{l,m}$, $E^+_{l,m}$, and $E'_{l,m}$ were computed by Batyrev and Haddad [BH08, §4]. However, we have included the calculation above to specify the basis of Q, which is different from the one chosen in [BH08, §4] and more convenient for our later discussion.

Let $h_{-K_{E^-_{l,m}}} = h_{\mathcal{C}^-}$ and $h_{-K_{E^+_{l,m}}} = h_{\mathcal{C}^+}$ be linear functions associated to $-K_{E^-_{l,m}}$ and $-K_{E^+_{l,m}}$, respectively, in the sense of Remark 9.

Lemma 22. *One has*

$$h_{\mathcal{C}^-} = \frac{p-k}{q}\mathbf{v}_1 + \frac{1+b}{aq}\mathbf{v}_2, \quad h_{\mathcal{C}^+} = \mathbf{v}_1 + \frac{1+b}{ap}\mathbf{v}_2.$$

Proof. By Remark 8, the anticanonical divisor of $E^-_{l,m}$ (and hence of $E^+_{l,m}$) can be written in the form $-K_{E^-_{l,m}} = D + a_{S^-}S^- + a_{S^+}S^+$, and the coefficients are $a_{S^-} = a_{S^+} = 1$. Therefore, the functions $h_{\mathcal{C}^-}$ and $h_{\mathcal{C}^+}$ satisfy $h_{\mathcal{C}^-}(\rho) = h_{\mathcal{C}^-}(\rho^+) = 1$ and $h_{\mathcal{C}^+}(\rho) = h_{\mathcal{C}^+}(\rho^-) = 1$. The lemma follows from these conditions on $h_{\mathcal{C}^-}$ and $h_{\mathcal{C}^+}$ by a direct calculation. □

Proof of Theorem 2. By Remark 10, one has

$$\alpha = h_{\mathcal{C}^-}(\rho') - 1 = \frac{1+b}{aq} - 1$$

and

$$\beta = h_{\mathcal{C}^+}(\rho') - 1 = \frac{1+b}{ap} - 1.$$

In particular, $\alpha - \beta < 0$. Therefore, in view of Lemma 20, we have $K_{E^-_{l,m}} \cdot \widetilde{C^-} \ge 0$ and $K_{E^+_{l,m}} \cdot \widetilde{C^+} \ge 0$ if and only if $1+b \le ap$. □

Remark 23. As mentioned in §1, the existence of the minimal resolution $\mathcal{W}$ of $E_{l,m}$ was proved by Panyushev [Pan88]. He constructed it as the minimal resolution of $E^+_{l,m} \cong SL(2)\times_B S^+$, which is described by the Hirzebruch–Jung continued fraction arising from the cone σ of the toric surface S^+ (see Remark 16 for the definition of σ). It follows that γ factors as

$$\mathcal{H} \to \mathcal{W} \to E^+_{l,m} \to E_{l,m}.$$

Therefore, Theorem 2 implies that $\mathcal{H}$ and $\mathcal{W}$ coincide if and only if $1+b\le ap$. Consider the subdivision of σ obtained by adding a new ray $\mathbf{R}_{\ge 0}\mathbf{e}_1$, which defines the morphism $E'_{l,m} \to E^+_{l,m}$. If $1+b\le ap$, then the subdivision coincides with the first step of that defined by the Hirzebruch–Jung continued fraction for constructing the minimal resolution $\mathcal{W}$, concerning that the cone σ is in the normal form in the sense of [CLS11, §10.1] if and only if $1+b\le ap$.

Example 24. Let $l = \frac{p}{q} = \frac{1}{3}$, and let $m = 3$. Then, $E_{\frac{1}{3},3}$ is non-toric. In this case, $E'_{\frac{1}{3},3}$ has a locally trivial family of A_1-singularities, and $\mathcal{H}$ is obtained by minimally resolving them. In terms of the colored fan of the spherical varieties $\mathcal{H}$ and $E'_{\frac{1}{3},3}$, the morphism $\mathcal{H} \to E'_{\frac{1}{3},3}$ corresponds to adding a new ray spanned by $\rho_1 = -\mathbf{u}_1 + 2\mathbf{u}_2$ to $\mathcal{C}'$ (see [Kub, §5]). Moreover, since $k = 1, a = 3, b = 2$, the Hilbert–Chow morphism $\gamma : \mathcal{H} \to E_{\frac{1}{3},3}$ is the minimal resolution, namely $\mathcal{H} \cong \mathcal{W}$.

Example 25. Assume that $E_{l,m}$ is toric. Then, $\mathcal{W}$ is described as follows: first of all, since $b = 1$, the cone σ is spanned by $ap\mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{e}_2$ (see Remark 16). Then we need to consider the following two cases.

Case 1: $p = 1$ and $m = q - 1$ ($\Leftrightarrow ap = 1$). In this case, S^+ is smooth, and $\mathcal{W}$ is isomorphic to $E^+_{l,m}$.

Case 2: $p > 1$ or $m \neq q - 1$. In this case, the singularity of S^+ is resolved by adding a single ray $\mathbf{R}_{\geq 0}\mathbf{e}_1$, and this subdivision corresponds to the morphism $\mathcal{W} \to E^+_{l,m}$. Taking Remark 23 into account, we see that $E'_{l,m} \cong \mathcal{W}$. On the other hand, we have $\mathcal{H} \cong E'_{l,m}$ by Theorem 1. Therefore, it follows that $\mathcal{W} \cong \mathcal{H}$, which is compatible with Remark 3.

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