New results on slowly varying functions in the Zygmund sense

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Abstract: Very recently Seneta [15] has provided a characterization of slowly varying functions L in the Zygmund sense by using the condition, for each $y > 0$,

$$x\left(\frac{L(x+y)}{L(x)} - 1\right) \to 0 \text{ as } x \to \infty. \tag{1}$$

We extend this result by considering a wider class of functions and a more general condition than (1). Further, a representation theorem for this wider class is provided.

Key words: Slowly varying; monotony in the Zygmund sense; class $\Gamma_a(g)$; self-neglecting function; extreme value theory; convergence rates.

1. Introduction. The notion of ultimately monotony introduced by Zygmund dates back to works like [16, p. 237] and [17, p. 186]. It says that a function $U \geq 0$ is slowly varying if for each $\epsilon > 0$ the function $x^\epsilon U(x)$ is ultimately increasing and $x^{-\epsilon}U(x)$ is ultimately decreasing. The class of this type of functions is called the Zygmund class (ZSV). A different kind of slowly varying functions was defined by Karamata [6,7] known as simply the class of slowly varying functions (KSV). It is known that any ZSV function is a KSV function, see [17, p. 186] and e.g. [14, p. 49].

A number of authors have analyzed generalizations of the functions ZSV, by considering $U(x)/x^a$ is increasing and $U(x)/x^b$ is decreasing, with $-\infty < a \leq b < \infty$ [3,4,11]. This type of functions have applications in analysis, differential equations and approximation theory [4,5,9,12]. Also, they are related with the notion of quasi-convexity, leading to applications in probability [8,13].

Very recently [15] has given an elegant characterization of the ZSV functions in terms of the condition (1).

In this paper, we extend this result by considering a wider notion of ultimately monotony than that proposed by Zygmund. Consequently, a more general limit than that involved in (1) is provided. To this aim, we take into account functions related to self-neglecting functions and to functions belonging to $\Gamma_0(g)$. This type of functions have been deeply studied in [10].

In what follows, a brief review of functions that belong to the gamma class and related classes of functions is shown. Next, the new class called $Z(g,a)$ is introduced and then a result of Zygmund type. That section then presents an extension of Theorem 2 in [15] and a characterization of the members of $Z(g,a)$.

2. Main Results.

2.1. Preliminaires. In Omey [10] the author studied the following class of functions.

The positive and measurable function g is self-neglecting (notation: $g \in SN$) if it satisfies

$$\frac{g(x+yg(x))}{g(x)} \to 1, \forall y \in \mathbf{R}$$

and locally uniformy in y.

The positive and measurable function a is in the class $\Gamma_\alpha(g)$ if $g \in SN$ and if

$$\frac{a(x+yg(x))}{a(x)} \to e^{\alpha y}, \forall y \in \mathbf{R}.$$

In [10] it is proved that this relation automatically holds locally uniformly in y.

The positive and measurable function is in the class $E\Gamma_\alpha(g,a)$ if $g \in SN$, $a \in \Gamma_0(g)$ and

$$\frac{f(x+yg(x)) - f(x)}{a(x)} \to \alpha y, \forall y \in \mathbf{R}.$$

In [10] it is proved that this relation automatically holds locally uniformly in y.

Note that if $a(x)/f(x) \to 0$, we find that $f \in \Gamma_0(g)$ with a remainder term.

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In [10] the author proved that for $g \in SN$ and $a \in \Gamma_0(g)$ we have $f \in E\Gamma_\alpha(g,a)$ if and only if f is of the form $f(x) = A(G(x))$, where $A \in \Pi_\alpha(L)$ and $G \in \Gamma_1(g)$ where $L \in RV_0$ and $g \in SN$.

In [10, Theorem 2.2] the author also showed that for $g \in SN$ and $a \in \Gamma_0(g)$ we have $f \in E\Gamma_\alpha(g,a)$ if and only if f can be represented as

$$f(x) = C + W(x) + \alpha \int_{x^\circ}^{x} \frac{a(t)}{g(t)}\,dt + V(x),$$

where $V(x)/a(x) \to 0$ and $W'(x)g(x)/a(x) \to 0$.

2.2. The class $\boldsymbol{Z(g,a)}$. In view of Zygmund we consider the following class of functions. We assume that $g \in SN$ and $a \in \Gamma_0(g)$ and we also assume that $g(x) = o(a(x))$. The positive and measurable function $U(x)$ is in the class $Z(g,a)$ if it satisfies:

$$\lim_{x\to\infty} \frac{a(x)}{g(x)} \left(\frac{U(x+yg(x))}{U(x)} - 1 \right) = 0, \forall y \in \mathbf{R}. \tag{2}$$

From (2) it follows that $U \in \Gamma_0(g)$ and also that

$$\lim_{x\to\infty} \frac{a(x)}{g(x)U(x)} \left(U(x+yg(x) - U(x) \right) = 0, \forall y \in \mathbf{R}, \tag{3}$$

so that $U \in E\Gamma_0(g,b)$, where $b(x) = g(x)U(x)/a(x) \in \Gamma_0(g)$.

From (1) it also follows that

$$\lim_{x\to\infty} \frac{a(x)}{g(x)} \log \frac{U(x+yg(x))}{U(x)} = 0, \forall y \in \mathbf{R}, \tag{4}$$

so that $\log U(x) \in E\Gamma_0(g,b)$ where $b(x) = g(x)/a(x) \in \Gamma_0(g)$.

From the previous subsection we have that in each of the relations (2)–(4), the relation holds l.u. in $y \in \mathbf{R}$.

The representation result of the previous subsection (with $\alpha = 0$) shows that $\log U(x)$ can be written as:

$$\log U(x) = C + W(x) + V(x),$$

where $V(x)a(x)/g(x) \to 0$ and $W'(x)a(x) \to 0$.

2.3. Result of Zygmund type. A positive and measurable function L is in the Zygmund class of slowly varying functions if for any fixed $\epsilon > 0$ and any $y > 0$,

$$x^\epsilon L(x) \le (x+y)^\epsilon L(x+y)$$

and

$$x^{-\epsilon} L(x) \ge (x+y)^{-\epsilon} L(x+y).$$

This class of functions is a subclass of the class of slowly varying functions in the sense of Karamata.

Seneta [15] proved that L is in the Zygmund class of SV functions if and only if

$$\lim_{x\to\infty} x \left(\frac{L(x+y)}{L(x)} - 1 \right) = 0, \forall y \in \mathbf{R}.$$

In this section we generalize the result of Seneta to the class $Z(g,a)$. We prove the following result:

Theorem 1. *Assume that $g \in SN$, $a \in \Gamma_0(g)$ and $g(x) = o(a(x))$. Let $y > 0$. The inequalities, for x large and $\epsilon > 0$,*

$$U(x+yg(x)) \times \exp \epsilon \int_{x^\circ}^{x+yg(x)} a^{-1}(t)dt \ge U(x) \times \exp \epsilon \int_{x^\circ}^{x} a^{-1}(t)dt$$

and

$$U(x+yg(x)) \times \exp -\epsilon \int_{x^\circ}^{x+yg(x)} a^{-1}(t)dt \le U(x) \times \exp -\epsilon \int_{x^\circ}^{x} a^{-1}(t)dt,$$

hold if and only if $U \in Z(g,a)$.

Proof. Assume that $U \in Z(g,a)$. Let $y > 0$. We consider for $\epsilon \in \mathbf{R}$ the function $V(x) = U(x) \times \exp \epsilon \int_{x^\circ}^{x} a^{-1}(t)dt$.

We then have

$$\begin{aligned}
&\frac{V(x+yg(x)) - V(x)}{V(x)} \\
&\quad= \frac{U(x+yg(x))}{U(x)} \exp \epsilon \int_{x}^{x+yg(x)} a^{-1}(t)dt - 1 \\
&\quad= I + II,
\end{aligned}$$

where

$$I = \left(\frac{U(x+yg(x))}{U(x)} - 1 \right) \exp \epsilon \int_{x}^{x+yg(x)} a^{-1}(t)dt,$$

$$II = \exp \epsilon \int_{x}^{x+yg(x)} a^{-1}(t)dt - 1.$$

In the second term we have (put $t = x + \theta g(x)$)

$$\int_{x}^{x+yg(x)} a^{-1}(t)dt = \frac{g(x)}{a(x)} \int_0^y \frac{a(x)}{a(x+\theta g(x))}\,d\theta.$$

Since $a \in \Gamma_0(g)$, we have

$$\int_{x}^{x+yg(x)} a^{-1}(t)dt \sim \frac{g(x)}{a(x)} \int_0^y 1 d\theta = \frac{g(x)}{a(x)}\,y$$

and we obtain that

$$\frac{a(x)}{g(x)} II \to \epsilon y.$$

Now consider the first term. By assumption we have

$$\frac{a(x)}{g(x)} \left(\frac{U(x + yg(x))}{U(x)} - 1 \right) \to 0.$$

We conclude that

$$\frac{a(x)}{g(x)} (I + II) \to \epsilon y.$$

Moreover, as we argued before, this relation holds l.u. in $y > 0$. This implies that for x large enough, the sign of $I + II$ is the same as the sign of ϵ. The inequalities of the theorem then follow.

Now we prove a converse result.

Starting from the inequalities in the theorem, for $y > 0$ and $\epsilon > 0$ we have

$$\exp \epsilon \int_x^{x+yg(x)} a^{-1}(t) dt \geq \frac{U(x + yg(x))}{U(x)} \geq \exp -\epsilon \int_x^{x+yg(x)} a^{-1}(t) dt$$

with a and g as above. It follows that

$$-\epsilon y \leq \lim \binom{\sup}{\inf} \frac{a(x)}{g(x)} \left(\frac{U(x + yg(x))}{U(x)} - 1 \right) \leq \epsilon y.$$

Since ϵ was arbitrary, we find that

$$\frac{a(x)}{g(x)} \left(\frac{U(x + yg(x))}{U(x)} - 1 \right) \to 0.$$

The theorem then follows. □

Special Cases

(a) As a first case, we take $g(x) = 1$ and $a(x) = x$, $x^\circ = 1$. In this case we find back the result of Seneta as mentioned above.

(b) As a second case, we take $g(x) = 1$ and $a(x) = \sqrt{x}, x^\circ = 0$. We find that

$$\sqrt{x} \left(\frac{U(x + y)}{U(x)} - 1 \right) \to 0$$

if and only if for x large and $\epsilon > 0$ we have

$$U(x + y) \times \exp \epsilon \int_0^{x+y} t^{-1/2} dt \geq U(x) \times \exp \epsilon \int_0^x t^{-1/2} dt$$

and

$$U(x + y) \times \exp -\epsilon \int_0^{x+y} t^{-1/2} dt \leq U(x) \times \exp -\epsilon \int_0^x t^{-1/2} dt,$$

or

$$U(x + y) \times \exp 2\epsilon (x + y)^{1/2} \geq U(x) \times \exp 2\epsilon x^{1/2}$$

and

$$U(x + y) \times \exp -2\epsilon (x + y)^{1/2} \leq U(x) \times \exp -2\epsilon x^{1/2}.$$

(c) Now we assume that $g(x) = 1$ and $a(x) \in RV_\lambda$ with $0 < \lambda < 1$. In the case we find that

$$\int_{x^\circ}^x \frac{1}{a(t)} dt \sim \frac{x}{a(x)(1 - \lambda)},$$

where $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$. Using this relationship, we can alter the inequalities in the lemma to find

$$U(x + y) \times \exp \epsilon A(x + y) \geq U(x) \times \exp \epsilon A(x)$$

and

$$U(x + y) \times \exp -\epsilon A(x + y) \leq U(x) \times \exp -\epsilon A(x),$$

where $A(x) = x/a(x) \in RV_{1-\lambda}$.

(d) As a next example, we take $V(x) = U(x) \times e^{-\tau x}$. Clearly we have

$$\frac{U(x + y)}{U(x)} - e^{\tau y} = e^{\tau y} \left(\frac{V(x + y)}{V(x)} - 1 \right)$$

and then,

$$b(x) \left(\frac{U(x + y)}{U(x)} - e^{\tau y} \right) \to 0$$

if and only if

$$b(x) \left(\frac{V(x + y)}{V(x)} - 1 \right) \to 0,$$

and the previous results can be applied.

2.4. A representation theorem for $Z(g, a)$. The following result is inspired by a result of Bojanic and Karamata [2] on the Zygmund class, see e.g. Theorem 1.5.5 in [1].

Theorem 2. *Assume that $g \in SN$, $a \in \Gamma_0(g)$, $g(x) = o(a(x))$ and $g(x) > 0$. Assume that $U \in Z(g, a)$. Then, for some function f and for $x \geq A$ for some $A > 0$,*

$$U(x) = \exp c + \int_A^x f(t)dt$$

where $a(x)f(x) \to 0$.

Proof. Let $U \in Z(g,a)$. Then, by Theorem 1, the inequalities in such a theorem hold. Taking logarithms in such inequalities, denoting $V(x) = \log U(x)$, gives for $x \geq x^\circ$ for some $x^\circ > 0$ and $\epsilon > 0$

$$V(x+yg(x)) + \epsilon \int_{x^\circ}^{x+yg(x)} a^{-1}(t)dt \geq V(x) + \epsilon \int_{x^\circ}^{x} a^{-1}(t)dt$$

and

$$V(x+yg(x)) - \epsilon \int_{x^\circ}^{x+yg(x)} a^{-1}(t)dt \leq V(x) - \epsilon \int_{x^\circ}^{x} a^{-1}(t)dt.$$

Whence we have

$$V(x+yg(x)) - V(x) \geq -\epsilon \int_{x}^{x+yg(x)} a^{-1}(t)dt$$

and

$$V(x+yg(x)) - V(x) \leq \epsilon \int_{x}^{x+yg(x)} a^{-1}(t)dt,$$

and thus

$$|V(x+yg(x)) - V(x)| \leq \epsilon \int_{x}^{x+yg(x)} a^{-1}(t)dt.$$

We then have, put $t = x + \theta g(x)$,

$$|V(x+yg(x)) - V(x)| \leq \epsilon \frac{g(x)}{a(x)} \int_0^y \frac{a(x)}{a(x+\theta g(x))} d\theta.$$

Since $a \in \Gamma_0(g)$, we have for $x \geq x^*$ for some $x^* \geq x^\circ$ and for any $\theta \in [0,y]$

$$\frac{a(x)}{a(x+\theta g(x))} \leq 1 + \epsilon.$$

Thus we have, as $x \to \infty$,

$$|V(x+yg(x)) - V(x)| \leq (1+\epsilon)\epsilon y \frac{g(x)}{a(x)} \to 0. \tag{5}$$

This convergence is uniform on $y \in [0,A)$ for any $A > 0$. Moreover, V is ultimately absolutely continuous, thus then V is ultimately differentiable almost everywhere.

Therefore, we have, for $x > A$ for some $A \geq \max(x^\circ, x^*)$,

$$V(x) = V(A) + \int_A^x f(t)dt$$

where f is a measurable function satisfying $f(x) = V'(x)$ almost everywhere in (A,∞). One may redefine $f(x) = 0$ where $V'(x)$ does not exist. Also, according to (5), we have, taking $z = x + yg(x)$, for $x \geq A$,

$$a(x)|V(z) - V(x)| \leq (1+\epsilon)\epsilon|z - x|.$$

Then, when $y \to 0$, which implies that $z \to x$, we have, for x enough large such that $V'(x)$ exists,

$$-(1+\epsilon)\epsilon \leq a(x)V'(x) \leq (1+\epsilon)\epsilon$$

which implies that $a(x)V'(x) \to 0$ as $x \to \infty$. Hence, we have

$$U(x) = \exp V(x) = \exp V(A) + \int_A^x f(t)dt.$$

□

Remarks 3.

(a) It is easy to prove that the converse of Theorem 2 holds.

(b) If L is slowly varying in the sense of Zygmund, this function thus has the canonical representation given by, see e.g. [15],

$$L(x) = \exp c + \int_B^x \frac{\epsilon(t)}{t} dt,$$

for a constant c and a bounded measurable function $\epsilon(x)$ which satisfies $\epsilon(x) \to 0$ as $x \to \infty$. Taking $g(x) = 1$ and $a(x) = x$, these conditions satisfy $g \in SN$ and $g(x) = o(a(x))$. We then have $a(x)(\epsilon(x)/x) \to 0$ as $x \to \infty$. The previous remark thus implies that $L \in Z(g,a)$.

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