On a Diophantine equation involving powers of Fibonacci numbers

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Abstract: This paper deals with the diophantine equation $F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q$, an equation on the weighted power terms of Fibonacci sequence. For the exponents $p, q \in \{1, 2\}$ the problem has already been solved in ad hoc ways using the properties of the summatory identities appear on the left-hand side of the equation. Here we suggest a uniform treatment for arbitrary positive integers p and q which works, in practice, for small values. We obtained all the solutions for $p, q \leq 10$ by testing the new approach.

Key words: Fibonacci number; diophantine equation; weighted sum.

1. Introduction. As usual, $\{F_m\}_{m\geq 0}$ denotes the sequence of Fibonacci numbers $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. Its companion sequence $\{L_m\}_{m\geq 0}$ is the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{m+2} = L_{m+1} + L_m$ for all $m \geq 0$. We assume that the reader is familiar with their Binet formula.

In this paper, we determine the solutions to the Diophantine equation

$$F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q \tag{1}$$

in positive integers (p, q, k, n) where p and q are small. This equation was first investigated by Németh *et al.* in [5] for the four possibilities $\{p, q\} \subseteq \{1, 2\}$, and all the solutions in these particular cases were obtained in elementary ad hoc ways. The purpose of this paper is to provide a uniform treatment independently from the values of p and q. As a particular case, we solve the above equation for all values of p, q which do not exceed the upper bound 10.

We consider

$$F_1^p = 1 = F_1^q = F_2^q, \quad \text{and} \quad F_1^p + 2F_2^p = 3 = F_4$$

as trivial solutions to (1). The authors in [5] have made the following

Conjecture 1. Equation (1) has only the three non-trivial solutions

$$(p, q, k, n) = (1, 1, 4, 8),\ (1, 2, 3, 4),\ (3, 3, 3, 4).$$

The above conjecture says, in particular, that there exist only finitely many solutions. Since the equation is not a standard equation, the finiteness of its number of solutions does not seem to follow in an easy way. Note that the first two quadruples were obtained in [5], while the last one is justified here in the sense that our present work confirms the conjecture by solving the equation for $\max\{p, q\} \leq 10$. The result is recorded in

Theorem 2. *Conjecture 1 is true whenever* $\max\{p, q\} \leq 10$.

Problems having similar flavour appear in the extensive literature of Fibonacci sequence. For instance, the sum $F_n^s + F_{n+1}^s$ ($n \geq 0$) gives Fibonacci numbers when $s \in \{1, 2\}$. For larger exponents s, Marques and Togbé [4] proved that if $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n, then $s = 1$ or 2. Afterwards, Luca and Oyono [2] completed the solution of the question by showing that apart from $F_1^s + F_2^s = F_3$ there is no solution $s \geq 3$ to the equation $F_n^s + F_{n+1}^s = F_m$.

A naturally arising question is what would happen if we replace the Fibonacci numbers by other linear recurrence? In the case of non degenerate binary recurrences with real roots it is likely our approach works. On the other hand, we do not think that the method extends to Tribonacci numbers or to other recurrences of order higher

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than 2 although we have made no efforts in this direction.

Now we collect some preliminary results we will use in the proof of Theorem 2. In what follows, $\log_b$ denotes the logarithm to base b, where $b > 1$ is any real number, while $\alpha := (1+\sqrt{5})/2$ is the dominant root of the Fibonacci sequence. Since the following three lemmata are widely known, we present them without proof.

Lemma 3. *For $n \geq 1$, we have $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$, and $\alpha^{n-1} \leq L_n$.*

Lemma 4. *The inequality $F_{n+1}/F_n \geq 3/2$ holds for $n \geq 2$.*

Lemma 5. *Assume that n is divisible by 4. Then $F_n - F_4 = F_{(n-4)/2}L_{(n+4)/2}$.*

At some stage of the proof of the theorem we will use the following estimates.

Lemma 6. *Equation (1) implies*

- $(k-2)p < (n-1)q$ *if* $k \geq 2$, *and*
- $(n-2)q < (k-1)p + \log_\alpha(4k)$ *if* $k \geq 3$.

Proof. Combining Lemma 3 and $F_k^p < F_n^q$ (provided by (1) and $k \geq 2$) leads immediately to the first statement.

For the second statement, Lemma 4 yields

$$\frac{F_k}{F_{k-i}} = \prod_{j=0}^{i-1}\left(\frac{F_{k-j}}{F_{k-j-1}}\right) \geq \left(\frac{3}{2}\right)^{i-1}$$

for all $i = 1, 2, \ldots, k-1$, where $k \geq 3$. Note that the lower bound could be improved to $(3/2)^i$ if $i \leq k-2$ because we avoid the quotient $F_2/F_1 = 1$. Put $\nu := 2/3$. Recalling Lemma 3, observe that

$$\begin{aligned}(2)\quad \alpha^{(n-2)q} < F_n^q &= kF_k^p \sum_{j=0}^{k-1} \frac{k-j}{k}\left(\frac{F_{k-j}}{F_k}\right)^p \\ &< kF_k^p(1 + 1 + \nu + \nu^2 + \cdots + \nu^{k-2}) \\ &< 4k\alpha^{(k-1)p}.\end{aligned}$$

Then the statement follows by taking logarithms. □

Lemma 7. *Suppose that k and n are positive integers. Then $5^k \| F_n$ if and only if $5^k \| n$.*

Proof. See Lemma 1 in [1]. □

Lemma 8. *Assume that k, p and q are positive integers, p is odd. If*

$$L_pk^2 + (L_p - 2)k - 1 = \pm 5^{p-q}L_p^2$$

holds, then $(p, q, k) = (1, 1, 1), (1, 1, 2)$.

Proof. Since L_p is never a multiple of 5, and $5^{p-q}L_p^2$ is an integer, it follows that $q \leq p$. Put $r = p - q$. If $p = 1$, then $k^2 - k - 1 = \pm 1$ gives the two solutions above. The condition $p \geq 3$ entails that the left-hand side is positive so the sign in the right-hand side must be $+$. Suppose now that $p = 3$ or $p = 5$. The left-hand side is a quadratic polynomial in k with leading coefficient $L_p \neq 0$. A verification with $q \in \{1, \ldots, p\}$ provides no more solutions.

In the sequel, we may assume $p \geq 7$. We will show that this assertion contradicts the equality in the lemma. Reducing

$$(3)\qquad L_pk^2 + (L_p - 2)k - 1 = 5^rL_p^2$$

modulo L_p, it leads to $L_p \mid 2k+1$. Thus, L_p is odd, so p is not a multiple of 3. Moreover $2k + 1 = aL_p$ holds for some odd integer a. Thus, $k = (aL_p - 1)/2$. Substituting this into (3), after some manipulations we obtain

$$(4)\qquad a^2L_p^2 - (4a+1) = 4 \cdot 5^rL_p.$$

On one hand, this gives $L_p \mid 4a + 1$, therefore $a \geq (L_p - 1)/4 \geq 7$. On the other hand, since $L_p \geq 29$, we have $4a + 1 < 5a < a^2L_p^2/2$. Consequently

$$4 \cdot 5^rL_p = a^2L_p^2 - (4a+1) > \frac{a^2L_p^2}{2},$$

therefore

$$a < \frac{2^{3/2} \cdot 5^{r/2}}{L_p^{1/2}}.$$

This implies

$$(5)\qquad (L_p - 1)L_p^{1/2} < 2^{7/2} \cdot 5^{r/2}.$$

On the other hand, rewriting (4) as

$$(L_p^2)a^2 - 4a - (4 \cdot 5^rL_p + 1) = 0,$$

and treating it as a quadratic in a, its discriminant is a perfect square. Subsequently,

$$4 + L_p^2(4 \cdot 5^rL_p + 1) = y^2$$

holds for some positive integer y. So, $4 \cdot 5^rL_p^3 + (L_p^2 + 4) = y^2$. Since $L_p^2 + 4 = 5F_p^2$ (because p is odd), we get

$$4 \cdot 5^rL_p^3 + 5F_p^2 = y^2.$$

Let c, d be such that $5^c \| F_p$ and $5^d \| y$. Then $F_p = 5^cu$, $y = 5^dv$ for some integers u, v with $\gcd(uv, 5) = 1$ and

$$4 \cdot 5^rL_p^3 + 5^{2c+1}u^2 = 5^{2d}v^2.$$

Since $5 \nmid L_p$, 5^r is the exact power of 5 in the factorisation of $4\cdot 5^r L_p^3$. From the above equation, we have that since $2c+1\neq 2d$, either $r=2c+1$, or $r=2d$, and in the last case $r\leq 2c+1$. Clearly, in both cases $r\leq 2c+1\leq 2\log_5 p+1$, where in the second inequality we used Lemma 7. Thus, $5^r\leq 5p^2$. Returning to (5), we obtain

$$L_p^{1/2}(L_p-1)\leq 2^{7/2}\cdot 5^{r/2}\leq 2^{7/2}\cdot 5^{1/2}\cdot p,$$

and since $L_p\geq \alpha^{p-1}$ (see Lemma 3), we conclude

$$\alpha^{(p-1)/2}(\alpha^{p-1}-1)\leq 2^{7/2}\cdot 5^{1/2}\cdot p,$$

an inequality false for any $p\geq 6$. □

2. Proof of the theorem. Because we already accounted for the trivial solutions, we may assume $k\geq 3$. First we handle the case $k=3$ separately. In fact we will exploit $k\geq 4$ only in (12), but without this assumption, one has more difficulties in our argument after (11). With $k=3$ we find $F_n^q=F_1^p+2F_2^p+3F_3^p=3(1+2^p)$, so $3\mid F_n$ and F_n is odd, so $4\mid n$ and $3\nmid n$. If $q=1$, then $3\cdot 2^p=F_n-3=F_n-F_4=F_{(n-4)/2}L_{(n+4)/2}$ by Lemma 5. Thus $F_{(n-4)/2}$ has its largest prime factor at most 3. The Primitive Divisor Theorem implies $(n-4)/2\leq 12$, so $n\leq 28$ and the remaining possibilities can be verified by hand. If $q=2$, we have $3(1+2^p)=\square$. Thus, $2^p+1=3\square$. Distinguishing between $p\equiv 0,1,2 \pmod 3$, we get the equation $3y^2=1+2^r x^3$, where $r\in\{0,1,2\}$, which one can solve with MAGMA [3]. If $q=3$, we handle similarly the equation $1+2^p=9y^3$. Distinguishing between p even and odd one has $9y^3=1+2^r x^2$, $r\in\{0,1\}$, and all integer solutions (x,y) to these equations can again be computed with MAGMA [3]. Assume now that $q\geq 4$. Then $3^4\mid F_n^q$. Hence, $3^3\mid 1+2^p$, so p is odd and $9\mid p$. In particular, $19\mid 2^9+1\mid 2^p+1\mid F_n^q$, so $19\mid F_n$. Thus $18\mid n$, consequently $3\mid n$, a contradiction.

So, from now on $k\geq 4$. Let the integer $p\geq 1$ be fixed (not necessarily in $\{1,2,\ldots,10\}$), and consider the term F_j^p with $j\geq 1$. Since we have $F_j=(\alpha^j-\beta^j)/\sqrt{5}$, where $\beta=(1-\sqrt{5})/2$, it follows that

$$F_j^p=\frac{(\alpha^j-\beta^j)^p}{5^{p/2}}=\frac{\alpha^{jp}}{5^{p/2}}+\zeta_{p,j},$$

where

$$|\zeta_{p,j}|<\frac{2^p\alpha^{(p-1)j}}{5^{p/2}}<\alpha^{j(p-1)}.$$

Thus,

$$\sum_{j=1}^k jF_j^p=\frac{1}{5^{p/2}}\left(\sum_{j=1}^k j\alpha^{jp}\right)+R_1,$$

where $|R_1|<\sum_{j=1}^k j\alpha^{j(p-1)}<k^2\alpha^{k(p-1)}$. The inner sum is

$$\sum_{j=1}^k jx^j=x\frac{d}{dx}\left(\sum_{j=1}^k x^j\right)=x\frac{d}{dx}\left(x\frac{x^k-1}{x-1}\right)=\frac{kx^{k+2}-(k+1)x^{k+1}+x}{(x-1)^2}$$

with $x:=\alpha^p$. Thus,

$$\sum_{j=1}^k jF_j^p=\frac{k\alpha^p-(k+1)}{5^{p/2}(\alpha^p-1)^2}\,\alpha^{p(k+1)}+\frac{\alpha^p}{5^{p/2}(\alpha^p-1)^2}+R_1.$$

Let q be a positive integer. Writing also

$$F_n^q=\frac{\alpha^{nq}}{5^{q/2}}+R_2,$$

where

$$|R_2|\leq\left(\frac{2}{5^{1/2}}\right)^q\alpha^{n(q-1)}<\alpha^{n(q-1)},$$

the above formulas lead to

$$\frac{5^{(q-p)/2}(k\alpha^p-(k+1))}{(\alpha^p-1)^2}\,\alpha^{p(k+1)}-\alpha^{qn}=5^{q/2}R_2-5^{q/2}R_1-\frac{5^{(q-p)/2}\alpha^p}{(\alpha^p-1)^2}. \tag{6}$$

Thus, in the right-hand side of (6) we see that

$$5^{q/2}|R_1|\leq 5^{q/2}k^2\alpha^{kp-k}, \tag{7}$$

and

$$\frac{5^{(q-p)/2}\alpha^p}{(\alpha^p-1)^2}\leq 5^{q/2}\alpha^3\leq 5^{q/2}\alpha^{kp-k+3}. \tag{8}$$

In the latter case, we used the facts that $\alpha^p-1\geq\alpha^{p/2}$ for all $p\geq 2$, and $\alpha/(\alpha-1)^2=\alpha^3$ (to include the case $p=1$). Bounding $5^{q/2}R_2$ takes a bit longer. Clearly, in the exponent of the upper bound on $|R_2|$ we have $n(q-1)=(n-2)q+2q-n$ and further $5^{q/2}\alpha^{2q}=(\sqrt{5}\alpha^2)^q<6^q$. Combining these and (2), we obtain

$$5^{q/2}|R_2|<5^{q/2}\alpha^{n(q-1)}<5^{q/2}\cdot 4k\alpha^{(k-1)p+2q-n}<6^{q+1}k\alpha^{(k-1)p-n}. \tag{9}$$

If $k \le n$, then the last term is not larger then $6^{q+1}k\alpha^{(k-1)p-k}$. Assume now $k > n$. Obviously, $p \ge q$ leads to $F_n^q < F_k^p$, which contradicts (1). Hence, $p < q$. Put $b = \max\{p, q\} = q$. The first statement of Lemma 6 provides

$$n > (k-2)\frac{p}{q} + 1 \ge \frac{k-2}{b} + 1,$$

which together with (9) implies

$$5^{q/2}|R_2| < 6^{q+1}k\alpha^{(k-1)p-(k-2)/b-1} < 6^{q+1}k\alpha^{kp-k/b}.$$

Comparing it with the estimate obtained for $k \le n$, we get a general upper bound on $5^{q/2}|R_2|$. Obviously, in (7) and in (8) we can replace $-k$ in the exponents by $-k/b$ to unify the two upper bounds obtained for the cases $k \le n$ and $k > n$, respectively. Let $\Delta_p := (\alpha^p - 1)^2$, and

$$z_q(k) := 5^{q/2}\alpha^3 + 6^{q+1}k + 5^{q/2}k^2.$$

Putting all the above estimates together, we get that

$$\left|\frac{5^{(q-p)/2}(k\alpha^p - k - 1)}{\Delta_p} - \alpha^{qn-p(k+1)}\right| < \frac{z_q(k)}{\alpha^{k/b+p}}. \tag{10}$$

The next goal is to analyse the exponent $\mu := qn - p(k+1)$. We distinguish two situations. Suppose first that

$$\alpha^\mu \le \frac{5^{(q-p)/2}}{3\Delta_p}. \tag{11}$$

Here observe that for $k \ge 3$ and $p \ge 1$

$$\frac{5^{(q-p)/2}}{\Delta_p}(k\alpha^p - k - 1) - \alpha^\mu \ge \frac{5^{(q-p)/2}}{\Delta_p}(k(\alpha^p - 1) - 1 - 1/3) > \frac{5^{(q-p)/2}}{\Delta_p}. \tag{12}$$

It now follows from (10) and (12), that

$$\alpha^{k/b+p} < 5^{(p-q)/2}\Delta_p z_q(k), \tag{13}$$

which gives us an upper bound on k for fixed p, q. Later, we will see that the other branch provides also a bound which is larger.

In the sequel, we suppose that the opposite of (11) is true. First consider the case when the left-hand side in (10) is zero. After rearranging the corresponding relation, we take the norms in $\mathbf{Q}[\sqrt{5}]$ and get to

$$k^2(\alpha\beta)^p - (k+1)k(\alpha^p + \beta^p) + (k+1)^2 = 5^{p-q}(\alpha\beta)^{qn-p(k+1)}N_{\mathbf{Q}\sqrt{5}}(\Delta_p).$$

By $\alpha\beta = -1$, $\alpha^p + \beta^p = L_p$, and

$$N_{\mathbf{Q}[\sqrt{5}]}(\Delta_p) = ((-1)^p - L_p + 1)^2$$

the previous equality simplifies

$$((-1)^p - L_p + 1)k^2 + (2 - L_p)k + 1 = \pm 5^{p-q}((-1)^p - L_p + 1)^2.$$

If p is even, the corresponding equation is

$$(L_p - 2)k^2 + (L_p - 2)k - 1 = \pm 5^{p-q}(L_p - 2)^2.$$

Hence, $(L_p - 2) \mid 1$, leading to $L_p = 3$, so $p = 2$. Now $k^2 + k - 1 = \pm 5^{2-q}$, where the eligible values for q is 1 or 2. Each case leads to a trivial solution.

If p is odd, then Lemma 8 handles the equation

$$L_p k^2 + (L_p - 2)k - 1 = \pm 5^{p-q}L_p^2$$

and provides only $(p, q, k) = (1, 1, 1), (1, 1, 2)$, deriving trivial solutions to (1).

Assume now that the left-hand side in (10) is nonzero. We then have

$$|k\alpha^p - (k+1) - \alpha^\mu 5^{(p-q)/2}\Delta_p| < \frac{5^{(p-q)/2}\Delta_p z_q(k)}{\alpha^{k/b+p}}. \tag{14}$$

Changing α to β in the left above we get an amount

$$|k\beta^p - (k+1) - \beta^\mu 5^{(p-q)/2}\sigma(\Delta_p)| < (2k+1) + 9 \cdot 5^{p-q}\Delta_p, \tag{15}$$

where $\sigma(\Delta_p) = (\beta^p - 1)^2 < 3$. We also used the fact that the opposite of (11) is true, therefore

$$|\beta|^\mu = \alpha^{-\mu} < 3 \cdot 5^{(p-q)/2}\Delta_p.$$

The product of the left-hand sides of (14) and (15) is the norm of a nonzero algebraic integer in $\mathbf{K}$ so it is ≥ 1. We thus get that

$$\alpha^{k/b+p} < 5^{(p-q)/2}\Delta_p z_q(k)((2k+1) + 9 \cdot 5^{p-q}\Delta_p). \tag{16}$$

Note that (13) is weaker in (16), which therefore gives a general bound for k irregardless of whether (11) holds or not.

Taking $\max\{p, q\} \le 10$, (16) gives $k \le 1104$. Now one can easily check when

$$\left(\sum_{j=1}^{k} jF_j^p\right)^{1/q}$$

is a Fibonacci number for positive integer variables $p, q \in \{1, \ldots, 10\}$ and $k \le 1104$ getting only the solutions from the statement of the theorem. The proof is complete.

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