Contact loci, motivic Milnor fibers of nondegenerate singularities

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(Communicated by Masaki KASHIWARA, M.J.A., Jan. 14, 2020)

Abstract: Inspired by Denef-Loeser's identity of the Euler characteristic with compact supports of the contact loci with the Lefschetz numbers of a complex singularity, we study sheaf cohomology groups of contact loci of complex nondegenerate singularities. Moreover, also for these singularities, we obtain a motivic analogue of Lê Dũng Tráng's work on a monodromy relation of a complex singularity and its restriction to a generic hyperplane.

Key words: Arc spaces; contact loci; motivic Milnor fiber; nondegenerate singularity.

1. Introduction. Recently, the study of arc spaces and geometric motivic integration provides several new ideas and methods to singularity theorists. Indeed, the work [1] by Denef-Loeser gives a breakthrough point of view from which the motivic Milnor fiber is the motivic incarnation of the classic Milnor fiber. Not only permitting to recover known results but the motivic zeta function is also one of the powerful tools for the exploration of the monodromy conjecture.

In this note, we go back to the problem on the contact loci of a singularity established by Denef-Loeser [3] and the problem on the restriction of Lê Dũng Tráng [7]. Let f be a polynomial in $\mathbf{C}[x_1,\dots,x_d]$ such that $f(O)=0$, where O is the origin of $\mathbf{C}^d$. For $n\in\mathbf{N}^*$, we define the n-iterated contact locus of f at O to be a $\mathbf{C}$-variety $\mathcal{X}_{n,O}(f)$ consisting of $\varphi\in(t\mathbf{C}[t]/(t^{n+1}))^d$ with $f(\varphi)=t^n \bmod t^{n+1}$, which is endowed with the natural action of the group of n-roots of unity given by $\eta\cdot\varphi(t):=\varphi(\eta t)$. Denef and Loeser proved in [3] that

$$\chi_c(\mathcal{X}_{n,O}(f))=\Lambda(M^n),$$

where χ_c is the Euler characteristic with compact supports, $\Lambda(M^n)$ is the Lefschetz number of M^n, and M is the monodromy of (f,O). According to the recent work by McLean [9], one may expect that the singular cohomology groups of $\mathcal{X}_{n,O}(f)$ are equal to the Floer cohomology groups of M^n. This is a very difficult problem. However, by the hypothesis of nondegeneracy of f we can compute in the present note the cohomology groups of $\mathcal{X}_{n,O}(f)$ with certain sheaves (Theorems 3.2 and 3.4).

The classes of the contact loci $\mathcal{X}_{n,O}(f)$ in the Grothendieck ring of $\mathbf{C}$-varieties endowed with a good action of the group of roots of unity form the coefficients of a formal series, which is the motivic zeta function of f at O, whose limit is the opposite of the motivic Milnor fiber of f at O. Lê Dũng Tráng studied in [7] the relation between the monodromy of (f,O) and that of the restriction of (f,O) to a generic hyperplane. By supposing that the polynomial f is nondegenerate with respect to its Newton polyhedron we obtain here a motivic version of Lê Dũng Tráng's result concerning the motivic Milnor fibers (Theorem 4.1).

The detailed version of the present note, which is equipped with full proofs for its results, is given in the manuscript [8] with the same authors.

2. Preliminaries.

2.1. Monodromic Grothendieck ring of varieties. Let us consider the groups $\mu_n=\mu_n(\mathbf{C})$ of nth roots of unity, the maps $\eta\mapsto\eta^k$, for any $n,k\in\mathbf{N}^*$, and let $\hat\mu:=\varprojlim\mu_n$. Let $\mathrm{Var}_{\mathbf{C},\hat\mu}$ be the category of algebraic $\mathbf{C}$-varieties endowed with good $\hat\mu$-action. The Grothendieck group $K_0(\mathrm{Var}_{\mathbf{C},\hat\mu})$ is an abelian group generated by symbols $[X]$ for X in $\mathrm{Var}_{\mathbf{C},\hat\mu}$ such that $[X]=[Y]$ whenever X is $\hat\mu$-equivariant isomorphic to Y,

$$[X]=[Y]+[X\setminus Y]$$

for Y $\hat\mu$-action Zariski closed in X and

$$[X\times V]=[X\times\mathbf{C}^e]$$

if V is an e-dimensional complex affine space with any linear $\hat\mu$-action and the action on $\mathbf{C}^e$ is trivial. With the cartesian product $K_0(\mathrm{Var}_{\mathbf{C},\hat\mu})$ is a com-

2010 Mathematics Subject Classification. Primary 14B05, 14E18, 14M25, 32S05, 32S55.
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doi: 10.3792/pjaa.96.003

mutative ring with unity. We denote by $\mathbf{L}$ the class $[\mathbf{C}]$ and by $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$ the localization $K_0(\mathrm{Var}_{\mathbf{C},\hat{\mu}})[\mathbf{L}^{-1}]$.

Let $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}[[T]]$ be the ring of formal series over $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$, and let $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ be the subset of $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}[[T]]$ consisting of polynomials in variables $\frac{\mathbf{L}^p T^q}{(1-\mathbf{L}^p T^q)}$, with (p,q) in $\mathbf{Z}\times\mathbf{N}^*$. There exists by [1] a unique $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$-linear morphism

$$\lim_{T\to\infty}:\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}[[T]]_{\mathrm{sr}}\to\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$$

so that $\lim_{T\to\infty}\frac{\mathbf{L}^p T^q}{(1-\mathbf{L}^p T^q)}=-1$, for (p,q) in $\mathbf{Z}\times\mathbf{N}^*$.

2.2. Contact loci, motivic Milnor fibers. Let f and g be polynomials in $\mathbf{C}[x_1,\dots,x_d]$ with $f(O)=g(O)=0$, where O is the origin of $\mathbf{C}^d$. For $n\in\mathbf{N}^*$, the *n-iterated contact locus* $\mathcal{X}_{n,O}(f)$ of f at O is the set of $\varphi\in(t\mathbf{C}[t]/(t^{n+1}))^d$ with $f(\varphi)=t^n \bmod t^{n+1}$. For $n\geq m$, the *(n,m)-iterated contact locus* $\mathcal{X}_{n,m,O}(f,g)$ of the ordered pair (f,g) at O is the set of $\varphi\in(t\mathbf{C}[t]/(t^{n+1}))^d$ such that $f(\varphi)=t^n \bmod t^{n+1}$ and $\mathrm{ord}_t g(\varphi)=m$. These are $\mathbf{C}$-varieties, which are endowed with the natural μ_n-action given by $\eta\cdot\varphi(t):=\varphi(\eta t)$. As proved in [1], the formal series

$$Z_{f,O}(T):=\sum_{n\geq 1}[\mathcal{X}_{n,O}(f)]\mathbf{L}^{-nd}T^n$$

is in $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}[[T]]_{\mathrm{sr}}$, and we call

$$\mathcal{S}_{f,O}:=-\lim_{T\to\infty}Z_{f,O}(T)$$

in $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$ the *motivic Milnor fiber* of f at O. The rationality of the series

$$Z_{f,g,O}^{\Delta}(T):=\sum_{n\geq m\geq 1}[\mathcal{X}_{n,m,O}(f,g)]\mathbf{L}^{-nd}T^n$$

follows from [4, Théorème 4.1.2] and [6, Section 2.9], up to the isomorphism of rings $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}\cong\mathcal{M}_{\mathbf{C}^*}^{\mathbf{C}^*}$ (see [5, Proposition 2.6]). Its rationality can be also proved directly using [2, Lemma 3.4] with a log-resolution. We call the limit $\mathcal{S}_{f,g,O}^{\Delta}=-\lim_{T\to\infty}Z_{f,g}^{\Delta}(T)$ in $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$ *the motivic Milnor fiber of the pair* (f,g) at O.

3. Cohomology groups of contact loci of nondegenerate singularities.

3.1. Nondegeneracy. Let $f(x)=\sum_\alpha c_\alpha x^\alpha$ be in $\mathbf{C}[x]=\mathbf{C}[x_1,\dots,x_d]$ with $f(O)=0$. Denote by Γ be the Newton polyhedron of f, and by Γ_c the set of all the compact faces of Γ. For $\gamma\in\Gamma_c$, the γ-face function of f is $f_\gamma(x)=\sum_{\alpha\in\gamma}c_\alpha x^\alpha$. We call f *nondegenerate* with respect to Γ if for all $\gamma\in\Gamma_c$, f_γ are smooth on $(\mathbf{C}^*)^d$. Let $\ell_\Gamma:\mathbf{R}_{\geq 0}^d\to\mathbf{R}$ be the function defined by $\ell_\Gamma(a)=\inf_{b\in\Gamma}\langle a,b\rangle$, using the standard inner product in $\mathbf{R}^d$. For $a\in\mathbf{R}_{\geq 0}^d$, denote by γ_a the face of Γ on which the restriction of the function $\langle a,\bullet\rangle$ to Γ gets its minimum. In other words, $b\in\Gamma$ is in γ_a if and only if $\langle a,b\rangle=\ell_\Gamma(a)$. Note that γ_a is in Γ_c if and only if a is in $\mathbf{R}_{>0}^d$. In fact, when γ runs over Γ_c, the relatively open sets

$$\sigma_\gamma:=\{a\in\mathbf{R}_{>0}^d\mid\gamma=\gamma_a\}\tag{1}$$

form a partition of $\mathbf{R}_{>0}^d$ and the restriction of ℓ_Γ to each σ_γ is linear.

For $d\in\mathbf{N}^*$ we write $[d]$ for $\{1,\dots,d\}$, and for $J\subseteq[d]$, $A\subseteq\mathbf{C}$, write A^J for the set

$$\{(x_1,\dots,x_d)\mid x_j\in A\ \forall j\in J, x_i=0\ \forall i\notin J\}.$$

The cardinal of a finite set S is denoted by $|S|$.

Let f be in $\mathbf{C}[x_1,\dots,x_d]$ which is nondegenerate with respect to Γ with $f(O)=0$. Then for $J\subseteq[d]$, $f^J:=f|_{\mathbf{C}^J}$ is also nondegenerate with respect to its Newton polyhedron $\Gamma(f^J)$. If γ is a compact face of $\Gamma(f^J)$, we can define $\sigma_\gamma^J\subseteq\mathbf{R}_{>0}^J$ similarly as in (1). For $n\in\mathbf{N}^*$ and $k\in\mathbf{N}$, we denote by $\Delta_J^{(n,k)}$ the set of $a\in[n]^J$ such that $\ell_J(a)+k=n$, where ℓ_J stands for $\ell_{\Gamma(f^J)}$. For $J\subseteq[d]$ and $a\in\Delta_J^{(n,k)}$, denote by $\mathcal{X}_{J,a}$ the subvariety of $\mathcal{X}_{n,O}(f)$ consisting of φ such that $\mathrm{ord}_t x_j(\varphi)=a_j$ for all $j\in J$ and that $x_i(\varphi)\equiv 0$ for all $i\notin J$. It is obviously invariant by the μ_n-action on $\mathcal{X}_{n,O}(f)$. Let $\mathcal{P}_n$ be the index set consisting of all such pairs (J,a) such that

$$\mathcal{X}_{n,O}(f)=\bigsqcup_{(J,a)\in\mathcal{P}_n}\mathcal{X}_{J,a}.\tag{2}$$

Note that for every $\gamma\in\Gamma_c$, there exists a $J\subseteq[d]$ such that γ is contained in the hyperplanes $x_j=0$ for all $j\notin J$ and not contained in other coordinate hyperplanes. This set J is unique for each $\gamma\in\Gamma_c$, so we shall denote it by J_γ. It is a fact that the index set $\mathcal{P}_n$ in (2) is the set of (J_γ,a) such that $\gamma\in\Gamma_c$, $\ell_{J_\gamma}(a)\leq n$ and $a\in\Delta_{J_\gamma}^{(n,n-\ell_{J_\gamma}(a))}$. For $\gamma\in\Gamma_c$, we consider the $\mathbf{C}$-varieties

$$X_{\gamma,J_\gamma}(1):=\{x\in(\mathbf{C}^*)^{J_\gamma}\mid f_\gamma(x)=1\}$$

and

$$X_{\gamma,J_\gamma}(0):=\{x\in(\mathbf{C}^*)^{J_\gamma}\mid f_\gamma(x)=0\}.$$

Note that the variety $X_{\gamma_a,J_{\gamma_a}}(1)$ admits a natural $\mu_{\ell_{J_{\gamma_a}}(a)}$-action as follows:

$$e^{2\pi ir/\ell_{J_{\gamma_a}}(a)}\cdot(x_j)_{j\in J_{\gamma_a}}:=(e^{2\pi ira_j/\ell_{J_{\gamma_a}}(a)}x_j)_{j\in J_{\gamma_a}},$$

for $r \in [\ell_{J_{\gamma_a}}(a)]$. We consider the trivial action of $\hat{\mu}$ on the variety $X_{\gamma_a, J_{\gamma_a}}(0)$. Let s denote the sum function: $s(a) = \sum_{j \in J} a_j$ for $a = (a_j)_{j \in J} \in \mathbf{R}^J$.

Theorem 3.1. *Let $\gamma \in \Gamma_c$. If $a \in \sigma_\gamma^{J_\gamma} \cap \Delta_{J_\gamma}^{(n,0)}$, then there is naturally a $\hat{\mu}$-equivariant isomorphism of $\mathbf{C}$-varieties*

$$\tau : \mathcal{X}_{J_\gamma, a} \to X_{\gamma, J_\gamma}(1) \times \mathbf{C}^{|J_\gamma| \ell_{J_\gamma}(a) - s(a)}.$$

If $k \in \mathbf{N}^*$, $a \in \sigma_\gamma^{J_\gamma} \cap \Delta_{J_\gamma}^{(n,k)}$, there is a Zariski locally trivial fibration with fiber $\mathbf{C}^{|J_\gamma|(\ell_{J_\gamma}(a)+k) - s(a) - k}$:

$$\pi : \mathcal{X}_{J_\gamma, a} \to X_{\gamma, J_\gamma}(0).$$

3.2. Sheaf cohomology groups of contact loci. Let f be in $\mathbf{C}[x_1, \ldots, x_d]$ nondegenerate with respect to Γ such that $f(O) = 0$, and let $n \in \mathbf{N}^*$. As in (2) we have $\mathcal{X}_{n,O}(f) = \bigsqcup_{(J,a) \in \mathcal{P}_n} \mathcal{X}_{J,a}$. We now consider the function

$$\sigma : \mathcal{P}_n \to \mathbf{Z}$$

defined by $\sigma(J, a) = \dim_{\mathbf{C}} \mathcal{X}_{J,a}$. Then we have the following result on sheaf cohomology of contact locus $\mathcal{X}_{n,O}(f)$. Let $\mathcal{F}$ be an arbitrary sheaf of abelian groups on $\mathcal{X}_{n,O}(f)$.

Theorem 3.2. *With the previous hypothesis and notation, there is a spectral sequence*

$$E_1^{p,q} = \bigoplus_{\substack{(J,a) \in \mathcal{P}_n \\ \sigma(J,a) = p}} H_c^{p+q}(\mathcal{X}_{J,a}, \mathcal{F}) \Rightarrow H_c^{p+q}(\mathcal{X}_{n,O}(f), \mathcal{F}).$$

To obtain this spectral sequence, we define an ordering in $\mathcal{P}_n$ and we consider a filtration of $\mathcal{X}_{n,O}(f)$ by closed subsets in the usual topology

$$S_p := \bigsqcup_{\sigma(J,a) \le p} \mathcal{X}_{J,a},$$

for $p \in \mathbf{N}$. We also find a filtration of $\mathcal{F}$ as

$$\mathcal{F} = F^0(\mathcal{F}) \supseteq F^1(\mathcal{F}) \supseteq \cdots$$

in such a way that

$$F^p(\mathcal{F}) / F^{p+1}(\mathcal{F}) \cong i_{p*} \mathcal{F}_p,$$

where $\mathcal{F}_p$ is the direct image with compact support along the inclusion

$$j_p : S_p^\circ := S_p \setminus S_{p-1} \hookrightarrow S_p$$

of $\mathcal{F}|_{S_p^\circ}$, and i_p is the inclusion of S_p in $\mathcal{X}_{n,O}(f)$. This gives rise to the spectral sequence

$$E_1^{p,q} = H_c^{p+q}(\mathcal{X}_{n,O}(f), i_{p*}\mathcal{F}_p) \Rightarrow H_c^{p+q}(\mathcal{X}_{n,O}(f), \mathcal{F}),$$

and the theorem then follows thanks to the below isomorphisms, for any m in $\mathbf{Z}$,

$$\begin{aligned} H_c^m(\mathcal{X}_{n,O}(f), i_{p*}\mathcal{F}_p) &= H_c^m(S_p, \mathcal{F}_p) \\ &= H_c^m(S_p, j_{p!}\mathcal{F}|_{S_p^\circ}) \\ &= H_c^m(S_p^\circ, \mathcal{F}|_{S_p^\circ}). \end{aligned}$$

Further, we can construct a class of special sheaves on the contact loci such that when $\mathcal{F}$ belongs to the class, the previous spectral sequence is degenerate at the first page E_1. Indeed, we first rewrite $X_\gamma(0) := X_{\gamma,[d]}(0)$ and $X_\gamma(1) := X_{\gamma,[d]}(1)$ for short. Putting $r = \dim \gamma$, we can find out a toric change of coordinates

$$\nu : (\mathbf{C}^*)^r \times (\mathbf{C}^*)^{d-r} \to (\mathbf{C}^*)^d$$

so that the Laurent polynomial $f_\gamma(\nu(z))$ depends only on the first r variables. Then we get

$$X_\gamma(0) \cong \{z \in (\mathbf{C}^*)^r \mid f_\gamma(\nu(z)) = 0\} \times (\mathbf{C}^*)^{d-r}.$$

By [10, Proposition 5.1], $f_\gamma(\nu(z))$ is a nondegenerate Laurent polynomial since f_γ is. Let $\mathcal{L}_{\gamma,0}$ be the pullback via the above isomorphism of the local system

$$E_\gamma \boxtimes T^{\boxtimes(d-r)},$$

where $\boxtimes$ is the external tensor product, E_γ is an arbitrary nonconstant rank one local system on $(\mathbf{C}^*)^r$ and T is an arbitrary nonconstant rank one local system on $\mathbf{C}^*$. Similarly, if r_1 is the dimension of the convex hull of $\gamma \cup \{0\}$, we can find out a toric change of coordinates so that the polynomial $f_\gamma - 1$ involves only first r_1 variables. We also define a sheaf $\mathcal{L}_{\gamma,1}$ in the same way as previous.

Let $\mathcal{F}_a$ be the pullback of the sheaf $\mathcal{L}_{\gamma,1}^\vee \boxtimes C_a$ via the isomorphism τ in Theorem 3.1 (for $J = [d]$), where C_a is the $\mathbf{C}$-constant sheaf on $\mathbf{C}^{d\ell_\Gamma(a) - s(a)}$. For $k \in \mathbf{N}^*$ and $a \in \Delta_{[d]}^{(n,k)}$, we denote by $\mathcal{F}_{a,k}$ the sheaf $(\pi^{-1}\mathcal{L}_{\gamma,0})^\vee$ on $\mathcal{X}_{[d],a}$ which is the dual of the pullback of $\mathcal{L}_{\gamma,0}$ via π in Theorem 3.1 (for $J = [d]$). For $k \in \mathbf{N}$ and $a \in \Delta_{[d]}^{(n,k)}$, denote by $i_{a,k}$ the inclusion of $\mathcal{X}_{[d],a}$ into $\mathcal{X}_{n,O}(f)$. Then the sheaf

$$\begin{aligned} \mathcal{F} := &\bigoplus_{\gamma \in \Gamma_c} \bigoplus_{a \in \sigma_\gamma \cap \Delta_{[d]}^{(n,0)}} (i_{a,0})_! \mathcal{F}_a \\ &\oplus \bigoplus_{\gamma \in \Gamma_c, k \ge 1} \bigoplus_{a \in \sigma_\gamma \cap \Delta_{[d]}^{(n,k)}} (i_{a,k})_! \mathcal{F}_{a,k} \end{aligned} \tag{3}$$

on $\mathcal{X}_{n,O}(f)$ is exactly a sheaf that we expect, namely, it yields that the spectral sequence in Theorem 3.2 is degenerate at E_1. For $\gamma \in \Gamma_c$, $k \in \mathbf{N}$ and $p \in \mathbf{Z}$, we denote by $D_{\gamma,k,p}$ be the set of all $a \in$

$\sigma_\gamma \cap \Delta_{[d]}^{(n,k)}$ such that $d-1+dn-s(a)-k=p$. By the above construction, we have the following

Lemma 3.3. *Let f be as above, $\mathcal{F}$ as in* (3)*. For $m \in \mathbf{Z}$ with $m+d-1$ odd,*

$$H_c^m(\mathcal{X}_{n,O}(f), \mathcal{F}) = 0;$$

otherwise, putting $p := \frac{1}{2}(m+d-1) \in \mathbf{Z}$, we have

$$H_c^m(\mathcal{X}_{n,O}(f), \mathcal{F}) = \bigoplus_{\gamma\in\Gamma_c} \bigoplus_{a\in D_{\gamma,0,p}} H_c^m(\mathcal{X}_{[d],a}, \mathcal{F}_a) \oplus \bigoplus_{\gamma\in\Gamma_c, k\geq 1} \bigoplus_{a\in D_{\gamma,k,p}} H_c^m(\mathcal{X}_{[d],a}, \mathcal{F}_{a,k}).$$

Consider the case $p := \frac{1}{2}(m+d-1) \in \mathbf{Z}$. Using the morphisms τ and π in Theorem 3.1, the Kunnëth formula, the Poincare duality and computations on spectral sequences we get, for $a \in D_{\gamma,0,p}$,

$$\begin{aligned} H_c^m(\mathcal{X}_{[d],a}, \mathcal{F}_a) &\cong H^{d-1}(X_\gamma(1), \mathcal{L}_{\gamma,1})^\vee \\ &\cong H^{d-1}(X_\gamma(1), \mathcal{L}_{\gamma,1}) \end{aligned}$$

and, for $k \in \mathbf{N}^*$ and $a \in D_{\gamma,k,p}$,

$$\begin{aligned} H_c^m(\mathcal{X}_{[d],a}, \mathcal{F}_{a,k}) &\cong H^{d-1}(X_\gamma(0), \mathcal{L}_{\gamma,0})^\vee \\ &\cong H^{d-1}(X_\gamma(0), \mathcal{L}_{\gamma,0}). \end{aligned}$$

This together with the previous lemma gives us the following theorem. Remark that for every $k \geq n$, the k-summand in the direct sum in the theorem vanishes, hence the direct sum on the right hand side is finite.

Theorem 3.4. *Let f be as above, $\mathcal{F}$ as in* (3)*. For $m \in \mathbf{Z}$ with $m+d-1$ odd,*

$$H_c^m(\mathcal{X}_{n,O}(f), \mathcal{F}) = 0;$$

otherwise, with $p = \frac{1}{2}(m+d-1)$, we have

$$H_c^m(\mathcal{X}_{n,O}(f), \mathcal{F}) \cong \bigoplus_{\gamma\in\Gamma_c} H^{d-1}(X_\gamma(1), \mathcal{L}_{\gamma,1})^{|D_{\gamma,0,p}|} \oplus \bigoplus_{\gamma\in\Gamma_c, k\geq 1} H^{d-1}(X_\gamma(0), \mathcal{L}_{\gamma,0})^{|D_{\gamma,k,p}|}.$$

4. A motivic analogue of a monodromy relation of Lê Dũng Tráng. We again use the notation in Section 3.1. Let f be in $\mathbf{C}[x]$ vanishing at O and nondegenerate with respect to Γ. Since the function ℓ_{J_γ} is linear and strictly positive on $\sigma_\gamma^{J_\gamma}$ and $\dim \sigma_\gamma^{J_\gamma} = |J_\gamma| - \dim(\gamma)$, thus using Theorem 3.1 and [4, Lemme 2.1.5] we get

$$\mathcal{S}_{f,O} = \sum_{\gamma\in\Gamma_c} (-1)^{|J_\gamma|+1-\dim(\gamma)} \mathbf{X}_{\gamma,J_\gamma}, \tag{4}$$

where

$$\mathbf{X}_{\gamma,J_\gamma} := [X_{\gamma,J_\gamma}(1)] - [X_{\gamma,J_\gamma}(0)].$$

This formula covers Guibert's formula in [4],

$$\mathcal{S}_{f,O} = \sum_{\gamma\in\Gamma_c} (-1)^{d+1-\dim(\gamma)} ([X_\gamma(1)] - [X_\gamma(0)]),$$

where Guibert requires f to contain whole d variables at each of its monomials (in this case, $J_\gamma = [d]$ for all $\gamma \in \Gamma_c$), while we do not. Note that in Guibert's formula in [4, Proposition 2.1.6], the factor $\mathbf{L}-1$ after $X_\gamma(0)$ should be deleted. Furthermore, the formula (4) can be also interchanged to that of Saito (see [11, Theorem 3.3]), in which, for each $\gamma \in \Gamma_c$, Saito uses the minimal affine subspace of $\mathbf{R}^d$ containing γ while we use the minimal coordinate subspace of $\mathbf{R}^d$ containing γ.

Denote by $\widetilde{O}$ the origin of $\mathbf{C}^{d-1}$, and by $\widetilde{f}$ the polynomial $f(x_1, \ldots, x_{d-1}, 0)$. A main result of this note is stated as follows, which may be considered as a motivic analogue (in the context of nondegenerate singularities) of Lê Dũng Tráng's work on a monodromy relation of a complex singularity and its restriction to a hyperplane general to f (see [7]).

Theorem 4.1. *With the previous hypothesis and notation, the identity $\mathcal{S}_{f,O} = \mathcal{S}_{\widetilde{f},\widetilde{O}} + \mathcal{S}_{f,x_d,O}^\Delta$ holds in $\mathcal{M}_{\mathbf{C}}^{\hat{\mu}}$.*

Here is the sketch of proof of this theorem. By definition of (n,m)-iterated contact loci, we have

$$\mathcal{X}_{n,m,O}(f, x_d) = \bigsqcup_{(J_\gamma,a)\in\mathcal{P}_n, a_d=m} \mathcal{X}_{J_\gamma,a}.$$

Thus $Z_{f,x_d,O}^\Delta(T) = \sum_{\gamma\in\Gamma_c} (Z_\gamma^0 + Z_\gamma^+)$, where Z_γ^0 is

$$\sum_{\substack{a\in\sigma_\gamma^{J_\gamma}\cap\Delta_{J_\gamma}^{(\ell_{J_\gamma}(a),0)} \\ \ell_{J_\gamma}(a)\geq a_d}} [\mathcal{X}_{J_\gamma,a}] \mathbf{L}^{-d\ell_{J_\gamma}(a)} T^{\ell_{J_\gamma}(a)}$$

and Z_γ^+ is

$$\sum_{\substack{a\in\sigma_\gamma^{J_\gamma}\cap\Delta_{J_\gamma}^{(\ell_{J_\gamma}(a)+k,k)} \\ \ell_{J_\gamma}(a)+k\geq a_d, k\geq 1}} [\mathcal{X}_{J_\gamma,a}] \mathbf{L}^{-d(\ell_{J_\gamma}(a)+k)} T^{\ell_{J_\gamma}(a)+k}.$$

We are now in position to apply Theorem 3.1. Note that if $d \in J_\gamma$, then $\ell_{J_\gamma}(a) + k \geq a_d$ automatically for any $k \in \mathbf{N}$. If $d \notin J_\gamma$, the inequality $\ell_{J_\gamma}(a) + k \geq a_d$ is in the situation of [5, Lemma 2.10], in which the corresponding series has the limit zero. Thus we get

$$\mathcal{S}_{f,x_d,O}^\Delta = \sum_{\gamma\in\Gamma_c, d\in J_\gamma} (-1)^{|J_\gamma|+1-\dim(\gamma)} \mathbf{X}_{\gamma,J_\gamma}.$$

It follows from this and from (4) that

$$\mathcal{S}_{f,O} = \sum_{\gamma\in\Gamma_c, d\notin J_\gamma} (-1)^{|J_\gamma|+1-\dim(\gamma)} \mathbf{X}_{\gamma,J_\gamma} + \mathcal{S}^{\Delta}_{f,x_d,O}.$$

The condition $d \notin J_\gamma$ means that $J_\gamma \subseteq [d-1]$, hence the first sum $\sum_{\gamma\in\Gamma_c, d\notin J_\gamma}$ in the above decomposition of $\mathcal{S}_{f,O}$ is nothing else than $\mathcal{S}_{\widetilde{f},\widetilde{O}}$.

Remark 4.2. In fact, we also obtain similar results on the motivic nearby cycles in the relative version of (4) and Theorem 4.1 for nondegenerate polynomials. An important corollary of the relative version of Theorem 4.1 is a proof of the integral identity conjecture for nondegenerate polynomials.

Acknowledgments. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number FWO.101.2015.02.

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