

Space-like maximal surfaces containing entire null lines in Lorentz-Minkowski 3-space

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(Communicated by Kenji FUKAYA, M.J.A., Oct. 15, 2019)

Abstract: Consider a surface S immersed in the Lorentz-Minkowski 3-space \mathbf{R}_1^3 . A complete light-like line in \mathbf{R}_1^3 is called an *entire null line* on the surface S in \mathbf{R}_1^3 if it lies on S and consists of only null points with respect to the induced metric. In this paper, we show the existence of embedded space-like maximal graphs containing entire null lines. If such a graph is defined on a convex domain in \mathbf{R}^2 , then it must be contained in a light-like plane (cf. Remark 3.3). Our example is critical in the sense that it is defined on a certain non-convex domain.

Key words: Maximal surface; type change; zero mean curvature; Lorentz-Minkowski space.

1. Introduction. We let \mathbf{R}_1^3 be the Lorentz-Minkowski 3-space of signature $(++-)$. It is well-known that there are no complete space-like zero mean curvature immersions in \mathbf{R}_1^3 other than planes (cf. [5]). It is thus interesting to investigate singularities of zero mean curvature surfaces. In fact, maximal surfaces with singularities were investigated by several geometers, see references in [11]. Regarding these works, the second and third authors introduced the concept of *maxface* in [11], which covers a large class of space-like surfaces with zero mean curvature admitting singular points. The elliptic catenoid

$$f_0 := (\sinh v \cos u, \sinh v \sin u, v) \quad (|u| \leq \pi, v \in \mathbf{R})$$

is a typical example of maxface with a cone-like singular point (cf. Fig. 1, left).

However, if space-like maximal surfaces have analytic extensions which change their causal type, then those extensions cannot remain in the class of maxfaces. In [13], two functions A_f and B_f for immersed hypersurfaces in Lorentzian manifolds were introduced. Using them, we give here a new notion ‘ZMC-map’ as an appropriate new class for

zero mean curvature surfaces with singularities, as follows (maxfaces are all ZMC-maps but the converse is not true): Let U be a domain in the uv -plane, and let $f : U \rightarrow \mathbf{R}_1^3$ be a smooth map into the Lorentz-Minkowski 3-space \mathbf{R}_1^3 . We set

$$P := \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix}$$

and

$$B_f := \det(P),$$

where \cdot denotes the canonical Lorentzian product of \mathbf{R}_1^3 and $\det(P)$ denotes the determinant of the 2×2 matrix P . We set

$$Q := \begin{pmatrix} f_{uu} \cdot \tilde{\nu} & f_{uv} \cdot \tilde{\nu} \\ f_{vu} \cdot \tilde{\nu} & f_{vv} \cdot \tilde{\nu} \end{pmatrix},$$

where

$$\tilde{\nu} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (f_u \times_E f_v)$$

and $f_u \times_E f_v$ denotes the Euclidean vector product of \mathbf{R}^3 . Here $\tilde{\nu}$ gives a Lorentzian normal vector field of f defined on U . Consider the matrix

$$W := \tilde{P}Q$$

and set

$$A_f := \text{trace}(W),$$

where \tilde{P} is the cofactor matrix of P . We call f a *ZMC-map* (i.e. *zero mean curvature map*) if it is an immersion on an open dense subset of U and A_f

2010 Mathematics Subject Classification. Primary 53A10, 53B30; Secondary 35M10.

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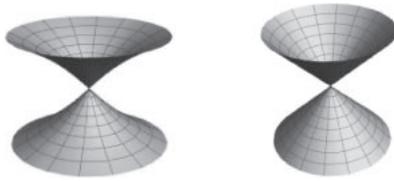


Fig. 1. The elliptic catenoid (left) and the light-cone (right).

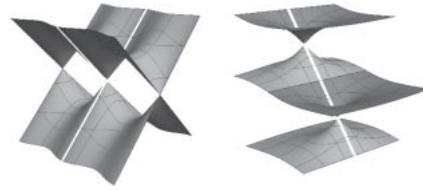


Fig. 2. Maximal surfaces with cone-like singular points lying on null lines, where the white lines indicate null points.

vanishes identically. Since the property that A_f vanishes is independent of the choice of local coordinates, ZMC-maps can be defined from an arbitrarily given 2-manifold. Moreover, one can generalize the concept of ZMC-map for a map into \mathbf{R}_1^{n+1} from an arbitrarily given n -manifold using the function A_f given in [13].

We let f be a ZMC-map. A point $p \in U$ is said to be a *space-like point* (resp. *time-like point*) of f , if $B_f(p) > 0$ (resp. $B_f(p) < 0$). A point which is neither space-like nor time-like is said to be a *null point* or a *light-like point* of f . (If p is a singular point of f , then $B_f(p) = 0$. So p is a null point.)

If $f : U \rightarrow \mathbf{R}_1^3$ is a light-like surface, that is, all points on U are null points, then A_f vanishes identically (cf. [9, Example 4]). So, by our definition, a light-like surface is also an example of a ZMC-map.

Definition 1.1. A ZMC-map is said to be *maximal type* (resp. *of mixed type*) if $B_f \geq 0$ but does not vanish identically (resp. B_f changes its sign on U). On the other hand, a ZMC-map is said to be *null* or *light-like* if B_f vanishes identically. A ZMC-map which is neither maximal, of mixed type nor light-like is said to be *time-like*.

We now fix $f : U \rightarrow \mathbf{R}_1^3$ as a ZMC-map. A null point $p \in U$ is called *degenerate* if the exterior derivative of B_f vanishes at p . The light-cone

$$f_1 := (v \cos u, v \sin u, v) \quad (|u| \leq \pi, v \in \mathbf{R}),$$

is a light-like ZMC-map consisting only of degenerate null points.

Definition 1.2. We say that a ZMC-map $f : U \rightarrow \mathbf{R}_1^3$ contains a null line segment if there exists an open interval I and a smooth curve $\gamma : I \rightarrow U$ such that $\gamma(I)$ consists only of null points and $f \circ \gamma(I)$ is a subset of a light-like line in \mathbf{R}_1^3 . In this case, $f \circ \gamma(I)$ is called a *null line*. Moreover, if $f \circ \gamma(I)$ coincides with a complete light-like line, we call it an *entire null line*.

Klyachin [9] showed the following fact:

Fact 1.3 (The line theorem for ZMC-surfaces). *Let $f : U \rightarrow \mathbf{R}_1^3$ be a ZMC-immersion such that $o \in U$ is a degenerate null point. Then, f contains a null line segment passing through $f(o)$.*

This fact was generalized to a much wider class of surfaces, including real analytic constant mean curvature surfaces in \mathbf{R}_1^3 , see [12,13]. Although there are properly embedded time-like ZMC-surfaces with an entire null line (see [3, Examples 2.2 and 2.3]), each of all examples of ZMC-surfaces with space-like points given in [1,2,6,10] containing an entire null line L has at least one cone-like singular point on L (see Fig. 2).

For example, Fig. 2, left is the hyperbolic catenoid (as a ZMC-map) given in [10]. (This surface is called the catenoid of 2nd kind in [10]) and satisfies

$$(1) \quad \sin^2 x + y^2 = t^2,$$

where (x, y, t) is the canonical coordinate system of \mathbf{R}_1^3 , and $y = \pm t, x = k\pi$ ($k \in \mathbf{Z}$) are entire null lines on the surface.

On the other hand, the maximal ZMC-surface given in [1, Theorem 5.3 (1-i)] satisfies

$$(2) \quad 2(y - t) \sin t = (x^2 + y^2 - 2yt + t^2) \cos t$$

and $y = t, x = 0$ gives the entire null line on the surface (see Fig. 2, right). By (1) and (2), these two surfaces have no self-intersections.

The purpose of this paper is to show the following

Theorem 1.4. *There exist embedded maximal ZMC-surfaces (resp. embedded ZMC-surfaces of mixed type) each of which contains an entire null line.*

Remark 1.5. This theorem gives the first example of maximal ZMC-immersion containing an entire null line. On the other hand, examples of ZMC-maps of mixed type containing an entire null

line were given in [8, Example 9.3 for $m = 1$] using a different approach.

Corollary 1.6. *There exists a family of embedded maximal ZMC-hypersurfaces (resp. embedded ZMC-hypersurfaces of mixed type) in Lorentz-Minkowski space \mathbf{R}_1^{n+1} each of which contains an $(n - 1)$ -dimensional light-like plane.*

The proof is completely same as in the proof of [7, Corollary 1.2]. The strategy to prove Theorem 1.4 is as follows: In [7], local existence of a 1-parameter family $\{f_c\}$ of ZMC-surfaces of mixed type such that each f_c contains a null line segment was shown. By improving the argument there, we will show in Section 3 that each f_c can be analytically extended so that it contains an entire null line. Also, by modifying the estimates in [7], we will also show the existence of ZMC-surfaces of maximal type each of which contains an entire null line. Unfortunately, our construction is local, and so the resulting surfaces are not proper. So the following question still remains (this is essentially the same question as in [3, Question 1]):

Question. *Are there properly embedded maximal surfaces in \mathbf{R}_1^3 , which contain at least one entire null line?*

As a partial answer of this question, a Bernstein-type theorem for entire ZMC-graphs consisting only of space-like or null points was given in the authors' previous work [3]. As we have mentioned in Remark 1.5, Hashimoto and Kato [8] recently gave a new method for constructing ZMC-maps containing null lines, using bi-complex extensions. The authors expect that this could be developed to apply to the above question.

2. ZMC-surfaces with null lines. We consider a real analytic ZMC-immersion f containing the entire null line in \mathbf{R}_1^3 . By a suitable Lorentzian transformation of the ambient space, we may assume that this entire null line is given by

$$L := \{(0, y, y) \in \mathbf{R}_1^3; y \in \mathbf{R}\}.$$

On a neighborhood of a null point, there exist a domain U in the xy -plane containing the y -axis and a real analytic function $\psi : U \rightarrow \mathbf{R}$ such that f can be expressed as the graph of a function of the form

$$(3) \quad \psi(x, y) := y + \frac{\alpha(y)}{2} x^2 + \sum_{k=3}^{\infty} \frac{\beta_k(y)}{k} x^k,$$

where α and β_k ($k = 3, 4, 5, \dots$) are certain real analytic functions on \mathbf{R} . Since

$$f(x, y) = (x, y, \psi(x, y)),$$

we have

$$\begin{aligned} A_f &= (1 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (1 - \psi_x^2)\psi_{yy}, \\ B_f &= 1 - \psi_x^2 - \psi_y^2. \end{aligned}$$

Since f is a ZMC-graph, the function $A := A_f$ vanishes identically. So we have

$$(4) \quad 0 = A_{xx}|_{x=0} = 2\alpha\alpha' + \alpha'',$$

where the prime means the derivative with respect to y . Hence there exists a constant μ such that

$$(5) \quad \alpha' + \alpha^2 + \mu = 0.$$

If $\mu > 0$, then f is space-like, and if $\mu < 0$, then f is time-like. If $\mu = 0$, the causal type of f cannot be specified, and the following four possibilities arise:

- (i) the graph of ψ is of mixed type,
- (ii) the graph of ψ consists of space-like points, except for the y -axis,
- (iii) the graph of ψ consists of time-like points except for the y -axis, and
- (iv) the graph of ψ consists only of null points.

Remark 2.1. If we weaken the condition that the image of f contains a null line segment, an example for the case (i) was shown in [7], and a general local existence theorem for such ψ satisfying (i), (ii) and (iii) was shown in [12, Prop. 6.7].

As shown in [6] and [12], by a homothetic change

$$\begin{aligned} \tilde{f}(x, y) &:= (x, y, \tilde{\psi}(x, y)) \\ \left(\tilde{\psi}(x, y) &:= \frac{1}{m} \psi(mx, my), m > 0 \right), \end{aligned}$$

one can normalize μ to be $-1, 0$ or 1 . In fact, as shown in [6],

$$\alpha^+ := -\tan(y + c) \quad \left(|c| < \frac{\pi}{2} \right)$$

is a general solution of $\alpha' + \alpha^2 + \mu = 0$ for $\mu = 1$. If $\mu = 0$, then

$$\alpha_I^0 := 0 \quad \text{and} \quad \alpha_{II}^0 := \frac{1}{y + c} \quad (c \in \mathbf{R} \setminus \{0\})$$

are the solutions, and

$$\begin{aligned} \alpha_I^- &:= \tanh(y + c) \quad (c \in \mathbf{R}), \\ \alpha_{II}^- &:= \coth(y + c) \quad (c \in \mathbf{R} \setminus \{0\}), \\ \alpha_{III}^- &:= \pm 1 \end{aligned}$$

are the solutions for $\mu = -1$. If

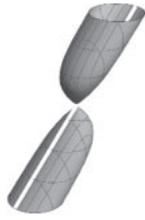


Fig. 3. A time-like ZMC-surface without self-intersection which has a cone-like singular point lying on a null line.

$$\alpha = \alpha^+, \alpha_{II}^0, \alpha_{II}^-,$$

then they are not defined on \mathbf{R} , so these three cases cannot produce any embedded ZMC-surfaces with entire null lines. In fact, the maximal surface given in (2) is of type α^+ . On the other hand, the light-cone (Fig. 1, right) and the maximal surface given in (1) is of type α_{II}^0 on each null line. Moreover, the time-like surface given by the implicit function

$$(6) \quad (y - t + \tanh t)^2 + x^2 = \tanh^2 t$$

has a null line $y = t, x = 0$ and is of type α_{II}^- (see Fig. 3). So if we seek candidates of ZMC-immersions containing entire null lines, then only the possibilities are

$$\alpha = \alpha_I^0, \alpha_I^-, \alpha_{III}^-.$$

The cases $\alpha = \alpha_I^-, \alpha_{III}^-$ give only time-like ZMC-surfaces, and properly embedded examples were known (cf. [3, Examples 2.2 and 2.3]). So only the case $\alpha := \alpha_I^0 (\equiv 0)$ is remaining.

This special case was discussed in [7], and here we would like to point out the method used in [7] is sufficient to show the following assertion:

Theorem 2.2. *There exist real analytic functions $\psi_i : U \rightarrow \mathbf{R}$ ($i = 1, 2, 3$) defined on a domain U in \mathbf{R}^2 , each of whose graphs gives a ZMC-embedding containing an entire null line of type α_I^0 , so that ψ_1, ψ_2 and ψ_3 satisfy the conditions (i), (ii) and (iii), respectively.*

The existence of ψ_1 and ψ_2 proves Theorem 1.4 in the introduction.

3. Proof of Theorem 2.2. We first consider the case (i). For each $c > 0$, a function ψ of a ZMC surface of mixed type satisfying $\alpha = 0$ was constructed in [7]. Such a surface can be characterized by the condition

$$(7) \quad \beta_3(y) = 3cy, \beta_k(0) = \beta'_k(0) = 0 \quad (k \geq 4).$$

For an arbitrary $\delta > 0$, there exists a positive constant C_δ such that ψ is defined on

$$V_\delta := (-C_\delta^{-1}, C_\delta^{-1}) \times (-\delta, \delta).$$

Remark 3.1. In [7, Page 290], the inequality

$$(8) \quad |\beta_l| \leq \frac{3c|y|^{l+2}}{(l^* + 2)^2} M^{l-3} \leq \theta_0 c^l$$

was shown, where $C_\delta := \delta M$ and $\theta_0 := 3(\delta M)^3/c$. However, there was a typographical error, and we should correct

$$(9) \quad C_\delta := \sqrt{\delta} M, \quad \theta_0 := \frac{3c}{\sqrt{\delta} M^3}.$$

Then (8) holds, correctly.

The uniqueness of the solution ψ implies that ψ can be extended on the domain $U := \bigcup_{\delta \geq 1} V_\delta$. In particular, U contains the entirety of the y -axis, and so ψ gives an example of type (i) containing the entire null line L .

We show the existence of surfaces satisfying (ii) or (iii) as a modification of the proof of [7]. We will change the initial condition and modify the estimates in [7] as follows: We set

$$(10) \quad \begin{aligned} \beta_3(y) &= 0, \quad \beta_4(y) = 4cy \quad (c \neq 0), \\ \beta_k(0) &= \beta'_k(0) = 0 \quad (k \geq 5), \end{aligned}$$

as the initial condition. Then, by [12, (6.5)],

$$B_f(x, 0) = -2cx^4 + \mathcal{O}(x^5)$$

holds. Hence, if the power series in (3) with the condition (10) converges, the graph of ψ satisfies (ii) (resp. (iii)) if $c < 0$ (resp. $c > 0$). The convergence of (3) can be proved by imitating the argument in [7]. In [7], functions b_i ($i \geq 0$) were used, where

$$b_0 := y \quad b_1 := 0, \quad b_2 := \alpha (= 0)$$

and $b_k = \beta_k$ holds for $k \geq 3$. Since $b_2 = b_3 = 0$ in [7], the series $\{P_k, Q_k, R_k\}$ were produced by the following recursive formula: We set $P_j = Q_k = R_l = 0$ for $3 \leq j \leq 5, 3 \leq k \leq 9, 3 \leq l \leq 10$, and

$$P_k := \sum_{m=4}^{k-2} \frac{2(k-2m+3)}{k-m+2} \beta_m \beta'_{k-m+2}, \quad k \geq 6,$$

$$Q_k := \sum_{m=4}^{k-6} \sum_{n=4}^{k-m-2} \frac{3n-k+m-1}{mn} \beta'_m \beta'_n \beta_{k-m-n+2},$$

$$k \geq 10,$$

$$R_k := \sum_{m=4}^{k-7} \sum_{n=4}^{k-m-3} \frac{\beta_m \beta_n \beta''_{k-m-n+2}}{k-m-n+2}, \quad k \geq 11.$$

Here P_k, Q_k and R_k are determined by β_j ($4 \leq j \leq k-2$). Thus, the functions β_k ($k \geq 5$) are inductively determined by the ordinary differential equation

$$(11) \quad \beta''_k = -k(P_k + Q_k - R_k), \quad \beta_k(0) = \beta'_k(0) = 0,$$

where $k \geq 5$. The following proposition holds, which corresponds to [7, Proposition 1.3]:

Proposition 3.2. *For each $c \neq 0$ and $\delta > 0$, we set*

$$(12) \quad M_\delta := 3 \max \{144\tau |c| |\delta|^{3/2}, \sqrt[4]{192c^2\tau}\},$$

where τ is a positive constant such that

$$t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \leq \tau \quad \left(0 < t < \frac{1}{2}\right).$$

Then the functions $\beta_l(y)$ ($l \geq 5$) satisfy the inequalities

$$(13) \quad |\beta''_l(y)| \leq |c| |y|^{l^*} M^{l-3},$$

$$(14) \quad |\beta'_l(y)| \leq \frac{3|c| |y|^{l^*+1}}{l^*+2} M^{l-3},$$

$$(15) \quad |\beta_l(y)| \leq \frac{3|c| |y|^{l^*+2}}{(l^*+2)^2} M^{l-3}$$

for any $y \in [-\delta, \delta]$, where

$$l^* := \frac{1}{2}(l-1) - 2 \quad (l = 5, 6, \dots).$$

Proof. We can prove this by induction on $l \geq 5$. The functions β_k of small indices are determined by the recursive formula (11) as follows:

$$(16) \quad \beta_4 = 4cy, \quad \beta_5 = 0, \quad \beta_6 = -8c^2y^3, \\ \beta_7 = 0, \quad \beta_8 = -32c^3y^5.$$

The estimates (13)–(15) for $l = 5$ follow from (16). In the case of [7], we set $b_3(= \beta_3) \neq 0$, but in our present case $b_3 = 0$ and the first non-trivial term begins from $b_4(= \beta_4)$. In particular, sub-terms appearing in P_k, Q_k, R_k are fewer than those in [7]. So, (13)–(15) for $l \geq 6$ follow using the same induction argument as in [7]. Therefore, the same estimates as in the proof of Proposition 1.9 in [7] are entirely valid also in this case. \square

By Proposition 3.2, for an arbitrary $\delta \geq 1$, ψ is well-defined on

$$V_\delta := (-C_\delta^{-1}, C_\delta^{-1}) \times (-\delta, \delta),$$

where $C_\delta := \sqrt{\delta}M$ (cf. (9)). In fact, like as in the case of [7], the inequality

$$(17) \quad |\beta_l| \leq \frac{3|c| |y|^{l^*+2}}{(l^*+2)^2} M^{l-3} \leq \theta_0 C_\delta^l \quad (l \geq 5)$$

can be shown, where $\theta_0 := 3|c|/(\sqrt{\delta}M^3)$. The uniqueness of the solution ψ implies that ψ can be extended on the domain

$$(18) \quad U := \bigcup_{\delta \geq 1} V_\delta.$$

In particular, U contains the entirety of the y -axis, and so ψ gives an example of type (ii) (resp. (iii)) if $c < 0$ (resp. $c > 0$) containing the entire null line L .

Remark 3.3. The domain U defined in (18) is not convex in \mathbf{R}^2 . However, this fact is crucial to construct a maximal graph containing an entire line. In fact, as pointed out in [4, Lemma 1], if there exists a maximal graph defined on a closed convex domain in \mathbf{R}^2 whose image contains an entire null line, then it must be a light-like plane. So if there exists an entire graph of mixed type containing an entire null line L , then either

- the both side of the entire null line are time-like, or
- there exists a sequence of time-like points (on the side containing space-like points) which is asymptotic to the line L ,

as a consequence.

Acknowledgement. The authors would like to express their gratitude to Dr. Atsufumi Honda and Dr. Wayne Rossman for helpful comments.

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