On the mod 2 cohomology of the classifying space of the exceptional Lie group E_6

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Abstract: We determine the mod 2 cohomology ring of the classifying space of the exceptional Lie group E_6 and the action of the Steenrod algebra on it.

Key words: Cohomology; classifying space; exceptional Lie group.

1. Introduction. In 1975, Kono and Mimura [5] determined the ring structure of the mod 2 cohomology ring $H^*(BE_6)$ of the classifying space BE_6 of the compact simply-connected exceptional Lie group E_6 except for the relation of degree 68. Their result is stated as follows:

Theorem 1.1 (Kono-Mimura [5]). *The mod 2 cohomology ring of the classifying space BE_6 of the exceptional Lie group E_6 is*

$$H^*(BE_6) = \mathbf{Z}/2[y_4, y_6, y_7, y_{10}, y_{18}, y_{34}, y_{32}, y_{48}]/I,$$

such that $\deg y_i = i$ *and I is the ideal generated by*

$$y_7y_{10}, \quad y_7y_{18}, \quad y_7y_{34}, \quad r_{68},$$

where $r_{68} = y_{34}^2 + y_{18}^2 y_{32} + y_{10}^2 y_{48} + higher\ terms.$

Thus the remaining problem on the mod 2 cohomology ring of BE_6 is the determination of the higher terms in r_{68}. Indeed, Toda [10] announced the result in 1973, but the detailed account never appeared in the literature. The purpose of this paper is to complement Toda's method and to give a description of $H^*(BE_6)$ as an algebra over the mod 2 Steenrod algebra. Our strategy for determining r_{68} is stated as follows: Let y_4 be the unique generator of $H^4(BE_6)$. We define the generators y_i for $i = 6, 7, 10, 18, 34$ by using the cohomology operations (see (3.1) and (3.2)). Let ρ_6 be the representation

$$\rho_6 : E_6 \longrightarrow SU(27) \tag{1.1}$$

whose highest weight is the fundamental weight ω_1 (for the fundamental weight of E_6, see [4, Chapter VI, §4]). Then ρ_6 induces a map of classifying spaces $BE_6 \to BSU(27)$. The induced homomorphism in cohomology is denoted by

$$\rho_6^* : H^*(BSU(27)) \longrightarrow H^*(BE_6).$$

The i-th Chern class $c_i(\rho_6)$ of ρ_6 is defined by $\rho_6^*(c_i)$ where c_i is the i-th universal Chern class in $H^*(BSU(27))$ (See [2, Appendix]). We define the remaining generators y_i for $i = 32, 48$ by using the Chern classes of ρ_6 (see (3.4) and (3.5)). Then by applying a squaring operation $\mathrm{Sq}^{32}\mathrm{Sq}^{16}\mathrm{Sq}^{8}\mathrm{Sq}^{4}$ to the Chern class $c_4(\rho_6) = y_4^2$, we obtain r_{68} (see Proposition 5.1):

$$r_{68} = y_{34}^2 + y_{18}^2 y_{32} + y_{10}^2 y_{48} + y_6 y_{10} y_{18} y_{34} + y_4 y_{10} y_{18}^3 + y_4 y_{10}^3 y_{34}.$$

We remark that other relations are also obtained from the Chern class $c_8(\rho_6) = y_4^4 + y_6 y_{10}$. More precisely, the relation $y_7 y_{10} = 0$ is obtained from

$$\mathrm{Sq}^1(y_4^4 + y_6 y_{10} + c_8(\rho_6)) = 0,$$

and $y_7 y_{18} = y_7 y_{34} = 0$ are respectively obtained by applying Sq^8, $\mathrm{Sq}^{16}\mathrm{Sq}^8$ to $y_7 y_{10} = 0$. Thus all the relations of $H^*(BE_6)$ are obtained from the Chern classes of ρ_6 and the Wu formula.

In general, given a map $X \to Y$ of topological spaces, the cohomology $H^*(X)$ has the structure of an $H^*(Y)$-algebra over the mod 2 Steenrod algebra $\mathcal{A}$. In other words, it is an algebra over the Massey-Peterson algebra $H^*(Y) \odot \mathcal{A}$, or the semitensor product of algebras in [6]. Using our result, we can determine the structure of the mod 2 cohomology of BE_6 as an algebra over the Massey-Peterson algebra $H^*(BSU(27)) \odot \mathcal{A}$ whose $H^*(BSU(27))$-algebra structure is given by the homomorphism ρ_6^*. Notice that, as an algebra over the Massey-Peterson algebra $H^*(BSU(27)) \odot \mathcal{A}$, the mod 2 cohomology

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ring $H^*(BE_6)$ is generated by the single element y_4 and all the relations are obtained from $c_4(\rho_6) = y_4^2$ and $c_8(\rho_6) = y_4^4 + y_6 y_{10}$.

2. The mod 2 cohomology of BSpin(10). Recall the mod 2 cohomology of BSpin(10) which will be needed in §3. According to Adams' book [1], there exists an inclusion $j_{\mathrm{Spin}(10)} : \mathrm{Spin}(10) \hookrightarrow E_6$. The restriction of ρ_6 to Spin(10) is given as follows:

$$\rho_6 \circ j_{\mathrm{Spin}(10)} = 1 + \lambda_1 + \Delta_+,$$

where $\lambda_1 : \mathrm{Spin}(10) \to SO(10) \to SU(10)$ is the standard representation, and $\Delta_+ : \mathrm{Spin}(10) \to SU(16)$ is the spin representation.

The mod 2 cohomology of BSpin(10) is

$$H^*(B\mathrm{Spin}(10)) = \mathbf{Z}/2[w_4, w_6, w_7, w_8, w_{10}, u_{32}]/I,$$

where u_{32} is the 16-th Chern class $c_{16}(\Delta_+)$ of the spin representation Δ_+ and I is the ideal generated by $w_7 w_{10}$ (see Quillen's paper [9] for the details). The action of Steenrod squares on w_j's is given by the Wu formula:

$$\mathrm{Sq}^i w_j = \sum_{t=0}^{i} \binom{j-i-1+t}{t} w_{i-t} w_{j+t},$$

where $w_0 = 1$ and $w_i = 0$ for $i \notin \{4, 6, 7, 8, 10\}$. Let Sq be the total Steenrod square, that is, $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \mathrm{Sq}^3 + \cdots$. Then, we have the following proposition.

Proposition 2.1. *In $H^*(B\mathrm{Spin}(10))$, we have*

$$\begin{aligned}
\mathrm{Sq}(w_4) &= w_4 + w_6 + w_7 + w_4^2,\\
\mathrm{Sq}(w_6) &= w_6 + w_7 + (w_{10} + w_4 w_6) + w_4 w_7 + w_6^2,\\
\mathrm{Sq}(w_7) &= w_7 + w_4 w_7 + w_6 w_7 + w_7^2,\\
\mathrm{Sq}(w_8) &= w_8 + w_{10} + w_4 w_8 + (w_4 w_{10} + w_6 w_8)\\
&\quad + w_7 w_8 + w_8^2,\\
\mathrm{Sq}(w_{10}) &= w_{10} + w_4 w_{10} + w_6 w_{10} + w_8 w_{10} + w_{10}^2.
\end{aligned}$$

We recall the Chern classes of the representations λ_1 and Δ_+. According to [8, Theorem 5.11 in Chapter III], the Chern classes of the representation λ_1 are given by

$$c_i(\lambda_1) = \begin{cases} w_i^2 & (i = 4, 6, 7, 8, 10), \\ 0 & (i = 1, 2, 3, 5, 9). \end{cases}$$

According to [7, p. 159], the Chern classes of the representation Δ_+ are given by

$$\begin{aligned}
c_8(\Delta_+) &= w_8^2 + w_6 w_{10} + w_4^4,\\
c_{12}(\Delta_+) &= w_4 w_{10}^2 + w_6 w_8 w_{10} + w_4^2 w_6 w_{10} + w_4^2 w_8^2\\
&\quad + w_6^4,\\
c_{14}(\Delta_+) &= w_8 w_{10}^2 + w_4^2 w_{10}^2 + w_4 w_6 w_8 w_{10} + w_6^3 w_{10}\\
&\quad + w_6^2 w_8^2 + w_7^4,\\
c_{15}(\Delta_+) &= w_{10}^3 + w_4 w_6 w_{10}^2 + w_6^2 w_8 w_{10} + w_7^2 w_8^2,\\
c_{16}(\Delta_+) &= u_{32},
\end{aligned}$$

and $c_i(\Delta_+) = 0$ for $i \neq 8, 12, 14, 15, 16$.

Using the Whitney sum formula

$$c_i(1 + \lambda_1 + \Delta_+) = \sum_{k=0}^{i} c_{i-k}(\lambda_1) c_k(\Delta_+),$$

we obtain the following proposition.

Proposition 2.2. *The 2^i-th ($i = 0, 1, 2, 3, 4$), 24-th and 27-th Chern classes of the restriction of ρ_6 to* Spin(10) *are given as follows*:

$$\begin{aligned}
j^*_{\mathrm{Spin}(10)} c_1(\rho_6) &= 0,\\
j^*_{\mathrm{Spin}(10)} c_2(\rho_6) &= 0,\\
j^*_{\mathrm{Spin}(10)} c_4(\rho_6) &= w_4^2,\\
j^*_{\mathrm{Spin}(10)} c_8(\rho_6) &= w_6 w_{10} + w_4^4,\\
j^*_{\mathrm{Spin}(10)} c_{16}(\rho_6) &= u_{32} + w_8^4 + w_6 w_8^2 w_{10}\\
&\quad + w_4^2 w_6 w_8 w_{10} + w_4^3 w_{10}^2\\
&\quad + w_4^2 w_6^4 + w_4^4 w_6 w_{10},\\
j^*_{\mathrm{Spin}(10)} c_{24}(\rho_6) &= w_8^2 u_{32} + w_8 w_{10}^4 + w_6^2 w_8^2 w_{10}^2\\
&\quad + w_6^3 w_{10}^3 + w_4 w_6 w_8 w_{10}^3 + w_4^2 w_{10}^4,\\
j^*_{\mathrm{Spin}(10)} c_{27}(\rho_6) &= 0.
\end{aligned}$$

3. The choice of generators of $H^*(BE_6)$. In this section, we fix the algebra generators of $H^*(BE_6)$. First, we adopt the generators y_4, y_6, y_7, y_{10}, y_{18} defined in [5]. Namely, y_4 is the unique generator in $H^4(BE_6) \cong \mathbf{Z}/2$, and y_6, y_7, y_{10} and y_{18} are defined as follows:

$$\text{(3.1)} \qquad y_6 := \mathrm{Sq}^2 y_4, \qquad y_7 := \mathrm{Sq}^1 y_6,$$
$$y_{10} := \mathrm{Sq}^4 y_6 + y_4 y_6, \quad y_{18} := \mathrm{Sq}^8 y_{10}.$$

In [5, Proposition 6.10], Kono and Mimura showed that $Sq^{16} y_{18}$ can be taken as an algebra generator of degree 34. In this paper, we adopt the following element y_{34} as a generator in order to simplify our presentation of the homomorphism $j^*_{\mathrm{Spin}(10)}$:

$$\text{(3.2)} \qquad y_{34} := \mathrm{Sq}^{16} y_{18} + y_6 y_{10} y_{18} + y_4 y_{10}^3.$$

We shall define the generators of degrees 32 and 48. Since we have $j^*_{\mathrm{Spin}(10)} c_4(\rho_6) = w_4^2$ and $H^8(BE_6) \cong \mathbf{Z}/2\{y_4^2\}$, we obtain $j^*_{\mathrm{Spin}(10)} y_4 = w_4$. Using the squaring operations, we obtain

$$(3.3) \quad \begin{aligned} j^*_{\mathrm{Spin}(10)}(y_i) &= w_i \quad \text{for } i = 4, 6, 7, 10, \\ j^*_{\mathrm{Spin}(10)}(y_{18}) &= w_8 w_{10}, \\ j^*_{\mathrm{Spin}(10)}(y_{34}) &= w_8^3 w_{10}. \end{aligned}$$

Using the Whitney sum formula

$$c_i(1+\lambda_1+\Delta_+) = \sum_{k=0}^{i} c_{i-k}(\lambda_1)c_k(\Delta_+),$$

we have

$$\begin{aligned} j^*_{\mathrm{Spin}(10)}(c_{16}(\rho_6)) = u_{32} &+ w_8^4 + w_6 w_8^2 w_{10} + w_4^3 w_{10}^2 \\ &+ w_4^2 w_6 w_8 w_{10} + w_4^4 w_6 w_{10} \\ &+ w_4^2 w_6^4, \end{aligned}$$

which is an indecomposable element. Thus we obtain the following proposition.

Proposition 3.1 (Toda [10])**.** *The Chern class $c_{16}(\rho_6)$ is an indecomposable element.*

Therefore we can take

$$(3.4) \qquad y_{32} := \rho_6^*(c_{16} + c_4 c_{12} + c_4^2 c_8)$$

as an algebra generator of degree 32.

Furthermore, we define an element y_{48} as follows:

$$(3.5) \quad y_{48} := \rho_6^*(c_{24} + c_{10}c_{14} + c_6 c_{18} + c_6^2 c_{12} + c_4 c_6 c_{14}).$$

One can show that y_{48} is also an indecomposable element. According to Adams' book [1], there exists an inclusion $j_{F_4} : F_4 \hookrightarrow E_6$. Then the induced homomorphism $j^*_{F_4} : H^*(BE_6) \to H^*(BF_4)$ is computed in Appendix A. The equations (A.4) show that the element $j^*_{F_4}(y_{48})$ is an indecomposable element in $j^*_{F_4}H^*(BE_6)$. Therefore y_{48} is also an indecomposable element as well. By Proposition 2.2, we have

$$\begin{aligned} j^*_{\mathrm{Spin}(10)}(y_{32}) &= u_{32} + w_8^4 + w_6 w_8^2 w_{10}, \\ j^*_{\mathrm{Spin}(10)}(y_{48}) &= w_8^2 u_{32}. \end{aligned}$$

It follows easily from Theorem 1.1 that the module structure of $H^*(BE_6)$ is given as follows:

$$(3.6) \quad \begin{aligned} H^*(BE_6) = \mathbf{Z}/2[y_4, y_6, y_{10}, y_{18}, y_{32}, y_{48}]\{1, y_{34}\} \\ \oplus \mathbf{Z}/2[y_4, y_6, y_7, y_{32}, y_{48}]\{y_7\}. \end{aligned}$$

Consider the following submodule M_1 of $H^*(BE_6)$:

$$\begin{aligned} M_1 = \mathbf{Z}/2[y_4, y_6, y_{10}, y_{32}, y_{48}]\{1, y_{18}, y_{18}^2, y_{34}, \\ y_{18}y_{34}, y_{18}^2 y_{34}\} \\ \oplus \mathbf{Z}/2[y_4, y_6, y_7, y_{32}, y_{48}]\{y_7\}. \end{aligned}$$

Then the following result holds:

Proposition 3.2. *The composition of maps*

$$M_1 \hookrightarrow H^*(BE_6) \longrightarrow H^*(B\mathrm{Spin}(10))$$

is injective.

Proof. We replace the generator $u_{32} \in H^*(B\mathrm{Spin}(10))$ by $j^*_{\mathrm{Spin}(10)}(y_{32})$ which is also denoted by y_{32}. We define a partial ordering for the monomial of $H^*(B\mathrm{Spin}(10))$ as follows:

$$w_4^{k_4} w_6^{k_6} w_7^{k_7} w_8^{k_8} w_{10}^{k_{10}} y_{32}^{k_{32}} < w_4^{l_4} w_6^{l_6} w_7^{l_7} w_8^{l_8} w_{10}^{l_{10}} y_{32}^{l_{32}}$$

if and only if $k_8 < l_8$.

Then the leading term of

$$j^*_{\mathrm{Spin}(10)} y_4^{n_4} y_6^{n_6} y_{10}^{n_{10}} y_{18}^{n_{18}} y_{34}^{n_{34}} y_{32}^{n_{32}} y_{48}^{n_{48}}$$

for $0 \le n_{18} < 3$ and $0 \le n_{34} < 2$ is

$$w_4^{n_4} w_6^{n_6} w_8^{n_{18}+3n_{34}+6n_{48}} w_{10}^{n_{10}+n_{18}+n_{34}} y_{32}^{n_{32}},$$

and the leading term of

$$j^*_{\mathrm{Spin}(10)} y_4^{n_4} y_6^{n_6} y_7^{n_7} y_{32}^{n_{32}} y_{48}^{n_{48}}$$

is

$$w_4^{n_4} w_6^{n_6} w_7^{n_7} w_8^{6n_{48}} y_{32}^{n_{32}}.$$

Since all the leading terms are different, the proposition is proved. □

By (3.3), we see that the element $y_{18}^3 + y_{10}^2 y_{34}$ is in the kernel of $j^*_{\mathrm{Spin}(10)}$, and we obtain the following corollary by using the module structure of $H^*(BE_6)$.

Corollary 3.3. *The kernel of the homomorphism $j^*_{\mathrm{Spin}(10)}$ is the ideal generated by $y_{18}^3 + y_{10}^2 y_{34}$. In particular, $j^*_{\mathrm{Spin}(10)}$ is injective for $* < 54$.*

Using Corollary 3.3, we can determine the action of the total Steenrod square on $y_4, y_6, y_7, y_{10}, y_{18}$.

Proposition 3.4. *In $H^*(BE_6)$, we have*

$$\begin{aligned} \mathrm{Sq}(y_4) &= y_4 + y_6 + y_7 + y_4^2, \\ \mathrm{Sq}(y_6) &= y_6 + y_7 + (y_{10} + y_4 y_6) + y_4 y_7 + y_6^2, \\ \mathrm{Sq}(y_7) &= y_7 + y_4 y_7 + y_6 y_7 + y_7^2, \\ \mathrm{Sq}(y_{10}) &= y_{10} + y_4 y_{10} + y_6 y_{10} + y_{18} + y_{10}^2, \\ \mathrm{Sq}(y_{18}) &= y_{18} + y_{10}^2 + (y_6 y_{10}^2 + y_4^2 y_{18}) + y_4^2 y_{10}^2 \\ &\quad + (y_{10}^3 + y_6^2 y_{18} + y_4 y_6 y_{10}^2) \\ &\quad + (y_{34} + y_6 y_{10} y_{18} + y_4 y_{10}^3) + y_{18}^2. \end{aligned}$$

Note that, using Corollary 3.3, we can calculate the action of the total Steenrod square on y_{34}, y_{32} and y_{48} up to degree $* < 54$.

4. Chern classes of the 27-dimensional representation $\boldsymbol{\rho_6}$. In this section, we compute the Chern classes of the complex representation $\rho_6 : E_6 \to SU(27)$. The result is needed in determining the relation r_{68} (Proposition 5.1). It is well-known that $H^*(BSU(27))$ is a polynomial ring in the universal Chern classes $c_2, c_3, \dots, c_{27}$. The action of Steenrod squares on $H^*(BSU(27))$ is given by the Wu formula:

$$\text{(4.1)} \qquad \mathrm{Sq}^{2i+1} c_j = 0,$$
$$\mathrm{Sq}^{2i} c_j = \sum_{t=0}^{i} \binom{j-i-1+t}{t} c_{i-t} c_{j+t},$$

where $c_0 = 1$ and $c_i = 0$ for $i \notin \{2, 3, \dots, 27\}$. As an algebra over the Steenrod algebra, $H^*(BSU(27))$ is generated by c_2, c_4, c_8, c_{16}.

Proposition 4.1. *The* 2^i*-th* ($i = 0, 1, 2, 3, 4$), *24-th and 27-th Chern classes of the representation* ρ_6 *are given as follows*:

$$\begin{aligned}
c_1(\rho_6) &= 0,\\
c_2(\rho_6) &= 0,\\
c_4(\rho_6) &= y_4^2,\\
c_8(\rho_6) &= y_6 y_{10} + y_4^4,\\
c_{16}(\rho_6) &= y_{32} + y_4^2 y_6 y_{18} + y_4^3 y_{10}^2 + y_4^2 y_6^4 + y_4^4 y_6 y_{10},\\
c_{24}(\rho_6) &= y_{48} + y_{10}^3 y_{18} + y_6^2 y_{18}^2 + y_4 y_6 y_{10}^2 y_{18}\\
&\quad + y_6^3 y_{10}^3 + y_4^2 y_{10}^4,\\
c_{27}(\rho_6) &= y_{18}^3 + y_{10}^2 y_{34}.
\end{aligned}$$

Proof. Using Corollary 3.3 and Proposition 2.2, we can compute all the Chern classes except for $c_{27}(\rho_6)$. We have $c_{27}(\rho_6) = \mathrm{Sq}^2(c_{26}(\rho_6))$ by the Wu formula $\mathrm{Sq}^2 c_{26} = c_{27} + c_1 c_{26}$ and $c_1(\rho_6) = 0$. The right-hand side can be computed by using $\mathrm{Sq}^2 y_{32} = 0$ and $\mathrm{Sq}^2 y_{34} = y_{18}^2$ by the remark below Proposition 3.4. □

We end this section by computing the action of Steenrod squares on y_{32}, y_{48}. By definition, we have $y_{32} = \rho_6^*(c_{16} + c_4 c_{12} + c_4^2 c_8)$. By computing $\rho_6^*(\mathrm{Sq}(c_{16} + c_4 c_{12} + c_4^2 c_8))$ using Propositions 3.4 and 4.1 and the Wu formula, we have the following proposition:

Proposition 4.2. *In* $H^*(BE_6)$, *we have*

$$\begin{aligned}
\mathrm{Sq}^1 y_{32} &= 0,\\
\mathrm{Sq}^2 y_{32} &= 0,\\
\mathrm{Sq}^4 y_{32} &= y_{18}^2 + y_6 y_{10}^3,\\
\mathrm{Sq}^8 y_{32} &= y_4 y_{18}^2 + y_6 y_{34},\\
\mathrm{Sq}^{16} y_{32} &= y_{48} + y_{10}^3 y_{18} + y_6^2 y_{18}^2 + y_6 y_{10} y_{32}\\
&\quad + y_4 y_6 y_{10}^2 y_{18} + y_4^3 y_{18}^2 + y_4^2 y_6 y_{34}\\
&\quad + y_6^3 y_{10}^3 + y_4^2 y_{10}^4 + y_4^4 y_{32},\\
\mathrm{Sq}^{32} y_{32} &= y_{32}^2.
\end{aligned}$$

In a similar way, since we have $y_{48} = \rho_6^*(c_{24} + c_{10} c_{14} + c_6 c_{18} + c_6^2 c_{12} + c_4 c_6 c_{14})$, we have the following proposition by computing $\rho_6^*(\mathrm{Sq}(c_{24} + c_{10} c_{14} + c_6 c_{18} + c_6^2 c_{12} + c_4 c_6 c_{14}))$:

Proposition 4.3. *In* $H^*(BE_6)$, *we have*

$$\begin{aligned}
\mathrm{Sq}^1 y_{48} &= 0,\\
\mathrm{Sq}^2 y_{48} &= 0,\\
\mathrm{Sq}^4 y_{48} &= y_{18} y_{34} + y_6 y_{10} y_{18}^2 + y_{10}^2 y_{32},\\
\mathrm{Sq}^8 y_{48} &= y_4^2 y_{48},\\
\mathrm{Sq}^{16} y_{48} &= y_{10} y_{18}^3 + y_6 y_{10} y_{48} + y_4 y_6 y_{18}^3 + y_{10}^3 y_{34}\\
&\quad + y_4 y_6 y_{10}^2 y_{34} + y_4^4 y_{48},\\
\mathrm{Sq}^{32} y_{48} &= y_{32} y_{48} + y_{10} y_{18}^2 y_{34} + y_6 y_{10}^2 y_{18}^3\\
&\quad + y_4 y_6 y_{18}^2 y_{34} + y_{10}^3 y_{18} y_{32} + y_6^2 y_{10}^2 y_{48}\\
&\quad + y_4^2 y_6 y_{18} y_{48} + y_6^3 y_{10} y_{18} y_{34} + y_4^2 y_{10}^2 y_{18} y_{34}\\
&\quad + y_4 y_6 y_{10}^2 y_{18} y_{32} + y_4^3 y_{10}^2 y_{48} + y_6^4 y_{10}^2 y_{18}^2\\
&\quad + y_4^2 y_6 y_{10}^3 y_{18}^2 + y_4^2 y_6^3 y_{18}^3 + y_4^4 y_{10} y_{18}^3\\
&\quad + y_4 y_6^2 y_{10}^3 y_{34} + y_6^3 y_{10}^3 y_{32} + y_4^2 y_{10}^4 y_{32}\\
&\quad + y_4^2 y_6^4 y_{48} + y_4^4 y_6 y_{10} y_{48} + y_4^5 y_6 y_{18}^3\\
&\quad + y_4^2 y_6^3 y_{10}^2 y_{34} + y_4^4 y_{10}^3 y_{34} + y_4^5 y_6 y_{10}^2 y_{34}.
\end{aligned}$$

5. The last relation r_{68} and the remaining action of the cohomology operations.

First, we determine the relation $r_{68} = 0$. For the sake of notational simplicity, we write ϕ, ϕ_1 for $\mathrm{Sq}^{32}\mathrm{Sq}^{16}\mathrm{Sq}^8\mathrm{Sq}^4$, $\mathrm{Sq}^{16}\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2$, respectively. On the one hand,

$$\phi(y_4^2) = (\phi_1(y_4))^2$$

and both sides can be computed by using Proposition 3.4. On the other hand, $\rho_6^*(\phi(c_4))$ can be computed by Proposition 4.1 and the Wu formula (4.1). Then, by the naturality of the cohomology operations $\phi(\rho_6^*(c_4)) = \rho_6^*(\phi(c_4))$, we obtain the desired relation r_{68} immediately.

Proposition 5.1. *In* $H^*(BE_6)$, *the following relation holds*:

$$\begin{aligned}
&y_{34}^2 + y_{18}^2 y_{32} + y_{10}^2 y_{48} + y_6 y_{10} y_{18} y_{34}\\
&\quad + y_4 y_{10} y_{18}^3 + y_4 y_{10}^3 y_{34} = 0.
\end{aligned}$$

In the rest of this section, we compute the action of Steenrod squares on y_{34}. We put $F := \rho_6^*(c_{20} + c_4 c_{16}) + y_6 y_{34}$. Then, by using Proposition 4.1, we have

$$F = y_4y_{18}^2 + y_6^2y_{10}y_{18} + y_4y_6^3y_{18} + y_6^2y_7^4 + y_6^5y_{10} + y_4^2y_6^2y_{10}^2 + y_4^4y_6y_{18} + y_4^5y_{10}^2 + y_4^4y_6^4 + y_4^6y_6y_{10}.$$

Note that F is a polynomial in $y_4, y_6, y_7, y_{10}, y_{18}$. Hence, we can compute $\mathrm{Sq}(F)$ by using Proposition 3.4. We put

$$F_1 := \mathrm{Sq}(F) + \rho_6^*(\mathrm{Sq}(c_4c_{16} + c_{20})) = \mathrm{Sq}(y_6)\mathrm{Sq}(y_{34}),$$
$$F_2 := \mathrm{Sq}(y_6) + y_6.$$

It follows from the module structure (3.6) that the multiplication by y_6 is injective in $H^*(BE_6)$. Therefore we obtain

$$\mathrm{Sq}(y_{34}) = \{F_1 + F_2\mathrm{Sq}(y_{34})\}/y_6.$$

Using this equality, we can compute $\mathrm{Sq}^i y_{34}$ for $i = 1, 2, \ldots$, inductively.

Proposition 5.2. *In $H^*(BE_6)$, we have*

$$\begin{aligned}
\mathrm{Sq}^1 y_{34} &= 0,\\
\mathrm{Sq}^2 y_{34} &= y_{18}^2,\\
\mathrm{Sq}^4 y_{34} &= y_{10}^2y_{18},\\
\mathrm{Sq}^8 y_{34} &= y_6y_{18}^2,\\
\mathrm{Sq}^{16} y_{34} &= y_4y_{10}y_{18}^2 + y_6y_{10}y_{34} + y_{10}^5 + y_6^2y_{10}^2y_{18}\\
&\quad + y_4^2y_6y_{18}^2 + y_4y_6y_{10}^4 + y_4^4y_{34},\\
\mathrm{Sq}^{32} y_{34} &= y_{18}y_{48} + y_{34}y_{32} + y_4y_{10}y_{18}y_{34}.
\end{aligned}$$

Thus we have determined the structure of $H^*(BE_6)$ as an algebra over the Steenrod algebra.

A. The homomorphism $H^*(BE_6) \to H^*(BF_4)$. In this appendix, we fix the algebra generators of $H^*(BF_4)$ which are needed to show the indecomposability of $y_{48} \in H^*(BE_6)$, and calculate the homomorphism $j_{F_4}^* : H^*(BE_6) \to H^*(BF_4)$ induced from the inclusion $j_{F_4} : F_4 \hookrightarrow E_6$.

The algebra structure of mod 2 cohomology ring of BF_4 is well-known, and given as follows (see [3, Proposition 19.2], [8, Theorem 6.6 in Chapter VII], [11, Section 2] or [5, (1.12)]):

(A.1) $\quad H^*(BF_4) = \mathbf{Z}/2[y_4, y_6, y_7, y_{16}, y_{24}].$

In order to fix the algebra generators of $H^*(BF_4)$, we consider the representation $\rho_4 : F_4 \to SO(26)$ and the inclusion $i_{\mathrm{Spin}(8)} : \mathrm{Spin}(8) \hookrightarrow F_4$. The maps ρ_4 and $i_{\mathrm{Spin}(8)}$ induce the homomorphisms in cohomology:

$$\rho_4^* : H^*(BSO(26)) \longrightarrow H^*(BF_4),$$
$$i_{\mathrm{Spin}(8)}^* : H^*(BF_4) \longrightarrow H^*(B\mathrm{Spin}(8)).$$

The restriction of ρ_4 to $\mathrm{Spin}(8)$ is

$$\rho_4 \circ i_{\mathrm{Spin}(8)} = 2 + \lambda_1 + \Delta_+ + \Delta_-$$

where $\lambda_1 : \mathrm{Spin}(8) \longrightarrow SO(8)$ is the standard representation and $\Delta_\pm : \mathrm{Spin}(8) \longrightarrow SO(8)$ are the spin representations.

The mod 2 cohomology of $B\mathrm{Spin}(8)$ is

$$H^*(B\mathrm{Spin}(8)) = \mathbf{Z}/2[w_4, w_6, w_7, w_8, u_8],$$

where u_8 is the 8-th Stiefel-Whitney class $w_8(\Delta_+)$ of the spin representation Δ_+. The total Stiefel-Whitney classes of these representations are

$$\begin{aligned}
w(\lambda_1) &= 1 + w_4 + w_6 + w_7 + w_8,\\
w(\Delta_+) &= 1 + w_4 + w_6 + w_7 + u_8,\\
w(\Delta_-) &= 1 + w_4 + w_6 + w_7 + (w_8 + u_8).
\end{aligned}$$

See [7] for the details.

Using Whitney sum formula, the total Stiefel-Whitney class of the representation $\rho_4 \circ i_{\mathrm{Spin}(8)}$ can be obtained as follows:

(A.2)

$$\begin{aligned}
i_{\mathrm{Spin}(8)}^* w(\rho_4) &= 1 + w_4 + w_6 + w_7 + w_4^2 + (w_6^2 + w_4^3)\\
&\quad + (w_7^2 + w_4^2w_6) + w_4^2w_7\\
&\quad + (w_8^2 + w_8u_8 + u_8^2 + w_4w_6^2)\\
&\quad + (w_6^3 + w_4w_7^2) + w_6^2w_7\\
&\quad + (w_6w_7^2 + w_4w_8^2 + w_4w_8u_8 + w_4u_8^2)\\
&\quad + w_7^3 + (w_6w_8^2 + w_6w_8u_8 + w_6u_8^2)\\
&\quad + (w_7w_8^2 + w_7w_8u_8 + w_7u_8^2)\\
&\quad + (w_8^2u_8 + w_8u_8^2).
\end{aligned}$$

We will define all the generators of $H^*(BF_4)$ in terms of the Stiefel-Whitney classes of the representation ρ_4. On the one hand, by (A.2), $w_i(\rho_4)$ are nonzero for $i = 4, 6, 7$. On the other hand, by (A.1), $\dim H^i(BF_4) = 1$ for $i = 4, 6, 7$. Therefore, the elements $y_i := w_i(\rho_4)$ $(i = 4, 6, 7)$ can be taken as algebra generators. We define

$$y_{16} := w_{16}(\rho_4) + y_4y_6^2,$$

since $i_{\mathrm{Spin}(8)}^*(w_{16}(\rho_4) + y_4y_6^2) = w_8^2 + w_8u_8 + u_8^2$ is not in $i_{\mathrm{Spin}(8)}^*\mathbf{Z}/2[y_4, y_6, y_7]$. In a similar way, we define

$$y_{24} := w_{24}(\rho_4),$$

since $i_{\mathrm{Spin}(8)}^*(w_{24}(\rho_4)) = w_8^2u_8 + w_8u_8^2$ is not in $i_{\mathrm{Spin}(8)}^*\mathbf{Z}/2[y_4, y_6, y_7, y_{16}]$. We remark that y_{16} and y_{24} correspond to the 2-nd and 3-rd elementary symmetric polynomials of the Stiefel-Whitney classes $w_8(\lambda_1)$, $w_8(\Delta_+)$ and $w_8(\Delta_-)$ respectively, and the generators of $H^*(BF_4)$ defined as above are the same as those of [11].

The action of Steenrod squares on $H^*(BF_4)$ can be calculated from that on $H^*(B\mathrm{Spin}(8))$, since the homomorphism $i^*_{\mathrm{Spin}(8)} : H^*(BF_4) \to H^*(B\mathrm{Spin}(8))$ is the monomorphism (see [11, Theorem 2.5]). Thus we obtain

$$\text{(A.3)}\quad \begin{aligned} w(\rho_4) &= 1 + y_4 + y_6 + y_7 + y_4^2 + (y_6^2 + y_4^3) \\ &\quad + (y_7^2 + y_4^2 y_6) + y_4^2 y_7 + (y_{16} + y_4 y_6^2) \\ &\quad + (y_6^3 + y_4 y_7^2) + y_6^2 y_7 + (y_4 y_{16} + y_6 y_7^2) \\ &\quad + y_7^3 + y_6 y_{16} + y_7 y_{16} + y_{24}. \end{aligned}$$

Since the restriction of ρ_6 to F_4 is given by

$$\rho_6 \circ j_{F_4} = 1 + (\rho_4)_{\mathbf{C}},$$

we obtain $j^*_{F_4} c(\rho_6) = w(\rho_4)^2$. Then the induced homomorphism is given as follows:

$$\text{(A.4)}\quad \begin{aligned} &j^*_{F_4}(y_i) = y_i \text{ (for } i = 4, 6, 7), \\ &j^*_{F_4}(y_i) = 0 \text{ (for } i = 10, 18, 34), \\ &j^*_{F_4}(y_{2i}) = y_i^2 \text{ (for } i = 16, 24). \end{aligned}$$

B. Comparison with Toda's generators. Finally, we compare our generators with Toda's generators in [10]. Using our generators, Toda's generators are given as follows:

$$\text{(B.1)}\quad \begin{aligned} &x_i = y_i, \quad \text{for } i = 4, 6, 7, 10, 18, \\ &x_{34} = y_{34} + y_6 y_{10} y_{18} + y_4 y_{10}^3, \\ &x_{32} \equiv c_{16}(\rho_6) \quad \text{mod decomp}, \\ &x_{48} \equiv c_{24}(\rho_6) \quad \text{mod decomp}. \end{aligned}$$

Note that x_{34} is the same as the generator $\bar{y}_{34}$ defined in [5, Definition 6.11]. Then Toda announced the following relations without proof:

$$\text{(B.2)}\quad \begin{aligned} &x_7 x_{10}, \quad x_7 x_{18}, \quad x_7 x_{34}, \\ &x_{34}^2 + x_{18}^2 x_{32} + x_{10}^2 x_{48} + x_6 x_{10} x_{18} x_{34}. \end{aligned}$$

There are several choices of generators that satisfy both (B.1) and (B.2). For example, if we define

$$\begin{aligned} &x_{32} = y_{32} + y_4 y_{10} y_{18}, \\ &x_{48} = y_{48} + y_4 y_{10} y_{34} + y_4 y_6 y_{10}^2 y_{18} + y_4^2 y_{10}^4, \end{aligned}$$

they satisfy (B.1) and (B.2).

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