Relationship between orbit decomposition on the flag varieties and multiplicities of induced representations

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Abstract: Let G be a real reductive Lie group and H a closed subgroup. T. Kobayashi and T. Oshima established a finiteness criterion of multiplicities of irreducible G-modules occurring in the regular representation $C^\infty(G/H)$ by a geometric condition, referred to as *real sphericity*, namely, H has an open orbit on the real flag variety G/P. This note discusses a refinement of their theorem by replacing a minimal parabolic subgroup P with a general parabolic subgroup Q of G, where a careful analysis is required because the finiteness of the number of H-orbits on the partial flag variety G/Q is not equivalent to the existence of H-open orbit on G/Q.

Key words: Degenerate principal series; multiplicity; spherical variety; intertwining operators; real spherical; reductive group.

1. Introduction. In the field of global analysis on homogeneous spaces, a rich theory has been developed by the I. M. Gelfand school and Harish-Chandra for group manifolds, by S. Helgason for Riemannian symmetric spaces, and in the framework of semisimple symmetric spaces by Flensted-Jensen, T. Oshima and P. Delorme among others. In the late 80s, T. Kobayashi raised a problem on what is the "most general framework" in which we could expect reasonable and detailed analysis of function spaces on G/H. As a solution to this problem, Kobayashi and Oshima established a finiteness criterion for multiplicities of the regular representation on a homogeneous space G/H.

Fact 1.1 ([10, Thm. A])**.** Let G be a real reductive Lie group and H its closed subgroup. Suppose that G and H are defined algebraically over $\mathbf{R}$. Then the following two conditions on the pair (G, H) are equivalent:

(i) $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for any $(\pi, \tau) \in \hat{G}_{\mathrm{smooth}} \times \hat{H}_{\mathrm{alg}}$;

(ii) G/H is real spherical.

Here $\hat{G}_{\mathrm{smooth}}$ denotes the set of equivalence classes of irreducible smooth admissible Fréchet representations of G with moderate growth, and $\hat{H}_{\mathrm{alg}}$ that of algebraic irreducible finite-dimensional representations of H. Given $\tau \in \hat{H}_{\mathrm{alg}}$, we write $C^\infty(G/H, \tau)$ for the Fréchet space of smooth sections of the G-homogeneous vector bundle over G/H associated to τ. The terminology *real sphericity* was introduced by Kobayashi [8] in his study of a broader framework for global analysis on homogeneous spaces than the usual (e.g., semisimple symmetric spaces).

Definition 1.2. A homogeneous space G/H is *real spherical* if a minimal parabolic subgroup P of G has an open orbit on G/H.

The following is an equivalent definition of real spherical homogeneous spaces. This is a consequence of the rank one reduction of T. Matsuki [12] and the classification of real spherical homogeneous spaces of real rank one by B. Kimelfeld [4].

Fact 1.3 ([3])**.** For the pair (G, H), the following two conditions are equivalent:

(ii) G/H is real spherical;

(iii) $\#(H\backslash G/P) < \infty$.

Therefore, for a minimal parabolic subgroup P, the three conditions (i), (ii), and (iii) are equivalent by Facts 1.1 and 1.3 (see Figure 1.1 below).

Remark 1.4. The above condition (iii) is not equivalent to the following analogous statement for complexifications:

(iv) $\#(H_{\mathbf{C}}\backslash G_{\mathbf{C}}/P_{\mathbf{C}}) < \infty$.

In this note, we consider a refinement of the equivalence (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). We recall that

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P : minimal parabolic

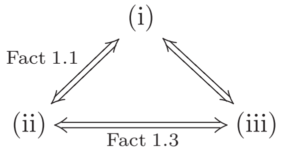

Figure 1.1.

Q : general parabolic

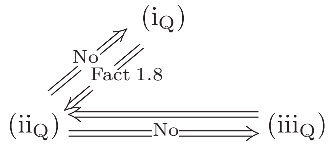

Figure 1.2.

the regular representation $L^2(G/H)$ may be decomposed into irreducible tempered representations when H is "small", but may need more singular representations such as unitarily induced representations from general parabolic subgroups when H is "large", see [2] for the precise criterion. Thus we ask a question what will happen to the relationship among the three conditions, if we replace P by a general parabolic subgroup Q of G. There is an obvious extension of the conditions (ii) and (iii) to a general parabolic subgroup Q (see Definition 1.6 below). In order to formulate a variant of (i) for a parabolic subgroup Q of G, we review the notion of Q-series.

Definition 1.5 ([9, Def. 6.6])**.** Let $\pi \in \hat{G}_{\mathrm{smooth}}$. We say that π belongs to *Q-series* if π occurs as a subquotient of the degenerate principal series representation $C^\infty(G/Q, \tau)$ for some $\tau \in \hat{Q}_{\mathrm{f}}$.

Here $\hat{Q}_{\mathrm{f}}$ is the set of equivalence classes of irreducible finite-dimensional representations of Q. Set $\hat{G}^Q_{\mathrm{smooth}} := \{\pi \in \hat{G}_{\mathrm{smooth}} \mid \pi \text{ belongs to } Q\text{-series.}\}$. Obviously, $\hat{G}^Q_{\mathrm{smooth}} \supset \hat{G}^{Q'}_{\mathrm{smooth}}$ if $Q \subset Q'$. Moreover, $\hat{G}^Q_{\mathrm{smooth}}$ is equal to $\hat{G}_{\mathrm{smooth}}$ if $Q = P$ (minimal parabolic) by Harish-Chandra's subquotient theorem [5] and to $\hat{G}_{\mathrm{f}}$ if $Q = G$.

Definition 1.6. For a parabolic subgroup Q of G, we define three conditions (i_Q), (ii_Q), and (iii_Q) as follows:

(i_Q) $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for any $(\pi, \tau) \in \hat{G}^Q_{\mathrm{smooth}} \times \hat{H}_{\mathrm{alg}}$.

(ii_Q) Q has an open orbit on G/H.

(iii_Q) $\#(H\backslash G/Q) < \infty$.

Then we consider the following question.

Question. Determine whether or not there is an implication among (i_Q), (ii_Q) and (iii_Q).

The conditions (i_Q), (ii_Q), and (iii_Q) reduce to (i), (ii), and (iii), respectively, if $Q = P$ (minimal parabolic), and we have seen in Facts 1.1 and 1.3 that the following equivalences hold for $Q = P$,

$$(\mathrm{i}_P) \Longleftrightarrow (\mathrm{ii}_P) \Longleftrightarrow (\mathrm{iii}_P).$$

Furthermore, if $Q = G$, the condition (i_Q) automatically holds by the Frobenius reciprocity, while (ii_Q) and (iii_Q) are obvious. Hence

$$(\mathrm{i}_G) \Longleftrightarrow (\mathrm{ii}_G) \Longleftrightarrow (\mathrm{iii}_G).$$

For a general parabolic subgroup Q, clearly, (iii_Q) implies (ii_Q). However there is an easy counterexample for the implication (ii_Q) $\Rightarrow$ (iii_Q) as follows:

Example 1.7. The projective space $\mathbf{RP}^2 = SL(3, \mathbf{R})/Q$ splits into an open orbit and continuously many fixed points of the unipotent radical H of Q.

On the other hand, the implication (i_Q) $\Rightarrow$ (ii_Q) holds for a general parabolic Q. To see this, we define a subset $\hat{H}_{\mathrm{f}}(G)$ of $\hat{H}_{\mathrm{f}}$ by

$$\hat{H}_{\mathrm{f}}(G) := \{\tau \in \hat{H}_{\mathrm{f}} \mid \tau \text{ appears as a quotient of some element of } \hat{G}_{\mathrm{f}}\}.$$

The implication (i_Q) $\Rightarrow$ (ii_Q) is derived from the following stronger assertion.

Fact 1.8 ([9, Cor. 6.8])**.** If there exists $\tau \in \hat{H}_{\mathrm{f}}(G)$ such that for all $\pi \in \hat{G}^Q_{\mathrm{smooth}}$, $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$, then Q has an open orbit on G/Q, namely, (ii_Q) holds.

Figures 1.1 and 1.2 summarize the known relationship among the three conditions.

2. Main Theorems. Figure 1.2 indicates that the relationship between the conditions (i_Q) and (iii_Q) is unsettled for a general parabolic Q of G. This note discusses the remaining implication in the following two theorems. Theorem 2.2 below shows that the implication (iii_Q) $\Rightarrow$ (i_Q) does not hold, hence the implication (ii_Q) $\Rightarrow$ (i_Q) does not always hold, too.

Definition 2.1. Let χ be a one-dimensional representation of Q. We say χ is a class-one character if χ is trivial on a maximal compact subgroup of Q.

Theorem 2.2. *Let Q be a maximal parabolic subgroup of $G = SL(2n, \mathbf{R})$ such that $G/Q \simeq$*

$\mathbf{RP}^{2n-1}$. *Then if $n \geq 2$, there exists an algebraic subgroup H of G satisfying the following two conditions*:

(I) $\#(H\backslash G/Q) < \infty$,

(II) $\dim \mathrm{Hom}_G(C^\infty(G/Q, \chi), C^\infty(G/H)) = \infty$ *for some class-one character χ of Q.*

Furthermore, if $n \geq 3$, H satisfies the following condition:

(II′) $\dim \mathrm{Hom}_G(C^\infty(G/Q, \chi), C^\infty(G/H)) = \infty$ *for any class-one character χ of Q.*

Remark 2.3. Let $G_{\mathbf{C}}, Q_{\mathbf{C}}$ and $H_{\mathbf{C}}$ be complexifications of G, Q and H, respectively. In our setting, $\#(H\backslash G/Q) < \infty$ (Theorem 2.2), but $\#(H_{\mathbf{C}}\backslash G_{\mathbf{C}}/Q_{\mathbf{C}}) = \infty$. The latter claim could be verified directly, but is derived also from the comparison of Theorem 2.2 with the general theory of holomorphic systems [7]. In fact, if $\#(H_{\mathbf{C}}\backslash G_{\mathbf{C}}/Q_{\mathbf{C}})$ were finite, then the space $\mathcal{D}'(G/Q, \chi)^H$ of H-invariant distribution sections would be finite-dimensional by [7, Thms. 5.1.7 and 5.1.12], contradicting to Theorem 2.2.

On the other hand, the implication $(\mathrm{i}_Q) \Rightarrow (\mathrm{iii}_Q)$ (See Figure 1.2) holds under a mild assumption as follows.

Theorem 2.4. *Let G be a real reductive algebraic group, and H a real algebraic subgroup, and Q a parabolic subgroup of G.*

(1) *If the number of orientable p-dimensional H-orbits on G/Q is infinite for some p, then*

$$\mathrm{Hom}_G(C^\infty(G/Q, \wedge^p(\mathfrak{g}/\mathfrak{q})^\vee), C^\infty(G/H))$$

is infinite-dimensional.

(2) *If the number of* transverse orientable *p-dimensional H-orbits on G/Q is infinite for some p, then*

$$\mathrm{Hom}_G(C^\infty(G/Q, \wedge^p(\mathfrak{g}/\mathfrak{q})^\vee \otimes or), C^\infty(G/H))$$

is infinite-dimensional.

Here $(\mathfrak{g}/\mathfrak{q})^\vee$ is the contragredient representation of $\mathfrak{g}/\mathfrak{q}$ and *or* is the one-dimensional representation of Q defined as the composition of the $(\dim G/Q)$-th exterior power representation $Q \to GL(\wedge^{\dim G/Q}(\mathfrak{g}/\mathfrak{q})^\vee) \simeq GL(1, \mathbf{R})$ and the signature $GL(1, \mathbf{R}) \to \{\pm 1\}$.

Figure 2.1 below summarizes a consequence of Theorems 2.2 and 2.4 on the relationship among the three conditions (i_Q), (ii_Q), and (iii_Q) for a general parabolic subgroup Q. In Figure 2.1, the symbol Δ on the arrow means that the implication is proved under an additional assumption of orientation.

Q : general parabolic

$$\begin{tikzcd}
& (\mathrm{i}_{\mathrm{Q}}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow, "\Delta", "\text{Thm. 2.4}"'] & \\
(\mathrm{ii}_{\mathrm{Q}}) \arrow[ur, Rightarrow, "\text{No}"] \arrow[rr, Rightarrow, "\text{No}"', shift right] & & (\mathrm{iii}_{\mathrm{Q}}) \arrow[ul, Rightarrow, "\text{No}", "\text{Thm. 2.2}"'] \arrow[ll, Rightarrow, shift right]
\end{tikzcd}$$

Figure 2.1.

3. Outline of the proof. A key fact of the proof of Theorems 2.2 and 2.4 is Fact 3.2 below. We construct intertwining operators by using Fact 3.2. In what follows, our normalization of the parameters is based on the interpretation of distributions as generalized functions, namely, functionals on smooth compactly supported density as in [11].

Definition 3.1. Let G be a real Lie group and H a closed subgroup of G. For $\tau \in \hat{H}_{\mathrm{f}}$, we define the finite-dimensional representation of H by $\tau^\vee_{2\rho} := \tau^\vee \otimes \mathbf{C}_{2\rho}$ where $\mathbf{C}_{2\rho}$ denotes the one-dimensional representation of H given by $h \mapsto |\det(\mathrm{Ad}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})|^{-1}$.

Fact 3.2 ([11, Prop. 3.2])**.** Let G be a real Lie group. Suppose that G' and H are closed subgroups of G and that H' is a closed subgroup of $G' \cap H$. Let τ and τ' be finite-dimensional representations of H and H', respectively.

(1) There is a natural injective map:

$$(1) \quad \mathrm{Hom}_{G'}(C^\infty(G/H, \tau), C^\infty(G'/H', \tau')) \hookrightarrow (\mathcal{D}'(G/H, \tau^\vee_{2\rho}) \otimes \tau')^{H'}.$$

Here $\mathcal{D}'(G/H, \tau^\vee_{2\rho})$ denotes the space of distribution sections of the G-homogeneous vector bundle over G/H associated to the H-module $\tau^\vee_{2\rho}$ and $(\mathcal{D}'(G/H, \tau^\vee_{2\rho}) \otimes \tau')^{H'}$ is the space of H'-fixed vectors under the diagonal action.

(2) If H is cocompact in G (e.g., a parabolic subgroup of G or a uniform lattice), then the injective map (1) is surjective.

Outline of the proof of Theorem 2.2. We only sketch the proof of Theorem 2.2 in the case $n = 2$. We define a three-dimensional subgroup H of $SL(4, \mathbf{R})$ by

$$H := \left\{ \begin{pmatrix} \cos\theta & \sin\theta & a & b \\ -\sin\theta & \cos\theta & b & -a \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix} \middle| \begin{array}{c} \theta \in \mathbf{R} \\ a, b \in \mathbf{R} \end{array} \right\}.$$

Then one can easily check that H satisfies the condition (I) in Theorem 2.2.

On the other hand, we have

$$\begin{aligned}(2)\qquad &\mathrm{Hom}_G(C^\infty(G/Q,\chi_\lambda), C^\infty(G/H))\\ &\simeq \mathcal{D}'(G/Q,\chi_{4-\lambda})^H\\ &\simeq \mathcal{D}'(\mathbf{R}^4\backslash\{0\})^H_{even,\lambda-4},\end{aligned}$$

where χ_λ is a class-one character of Q defined by $g \mapsto |\det(\mathrm{Ad}(g) : \mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{q})|^{\frac{-\lambda}{4}}$ and $\mathcal{D}'(\mathbf{R}^4\backslash\{0\})_{even,\lambda-4}$ is the space of even homogeneous distributions of degree $\lambda - 4$ on $\mathbf{R}^4\backslash\{0\}$. This follows from Fact 3.2 because $\mathbf{C}_{2\rho} = \chi_4$ as representations of Q and $\mathcal{D}'(G/Q,\chi_\lambda) \simeq \mathcal{D}'(\mathbf{R}^4\backslash\{0\})_{even,-\lambda}$ in the setting of Theorem 2.2.

We define a nonzero distribution on $\mathbf{R}^4$ by

$$T^l_2(x,y,z,w) := \left((x+\sqrt{-1}y)\left(\frac{\partial}{\partial z} - \sqrt{-1}\frac{\partial}{\partial w}\right)\right)^l \delta(z)\delta(w)$$

for $l \in \mathbf{N}$. Here (x,y,z,w) are coordinates of $\mathbf{R}^4$ and $\delta(z)$ and $\delta(w)$ are the Dirac delta functions on $\mathbf{R}^4$ supported on the hyperplanes $z = 0$ and $w = 0$, respectively. Then the restriction of T^l_2 to $\mathbf{R}^4\backslash\{0\}$ is an element of $\mathcal{D}'(\mathbf{R}^4\backslash\{0\})^H_{even,-2}$ for every $l \in \mathbf{N}$ and the operators T^l_2 ($l \in \mathbf{N}$) are linearly independent. Thus we have $\dim \mathcal{D}'(\mathbf{R}^4\backslash\{0\})^H_{even,\lambda-4} = \infty$. This implies that H satisfies the condition (II) of Theorem 2.2 via the isomorphism (2). □

Remark 3.3. The support of the distribution kernel T^l_2 is of codimension two in three-dimensional manifold G/Q (when $n = 2$) if we regard $T^l_2 \in \mathcal{D}'(G/Q,\chi_2)^H$ via the isomorphism (2).

Outline of the proof of Theorem 2.4. We sketch the proof only for Theorem 2.4 (1). Let n be the dimension of the real partial flag variety G/Q. By Fact 3.2, we have

$$\begin{aligned}(3)\qquad &\mathrm{Hom}_G(C^\infty(G/Q,\wedge^p(\mathfrak{g}/\mathfrak{q})^\vee), C^\infty(G/H))\\ &\simeq \mathcal{D}'(G/Q,\wedge^p(\mathfrak{g}/\mathfrak{q})\otimes\wedge^n(\mathfrak{g}/\mathfrak{q})^\vee\otimes or)^H\\ &\simeq \mathcal{D}'(G/Q,\wedge^{n-p}(\mathfrak{g}/\mathfrak{q})^\vee\otimes or)^H,\end{aligned}$$

by the Q-isomorphism $\mathbf{C}_{2\rho} \simeq \wedge^n(\mathfrak{g}/\mathfrak{q})^\vee \otimes or$. The right-hand side of (3) is nothing but the space of impair currents as below.

Definition 3.4. Let X be a manifold. Then we write $\mathcal{D}^p(X)$ for the space of compactly supported p-forms, and we write $\underline{\mathcal{D}}'_p(X)$ for the topological dual space of $\mathcal{D}^p(X)$ and call its elements p-dimensional impair currents on X.

In our setting, we have a natural G-isomorphism:

$$\underline{\mathcal{D}}'_p(G/Q) \simeq \mathcal{D}'(G/Q,\wedge^{n-p}(\mathfrak{g}/\mathfrak{q})^\vee\otimes or).$$

Therefore, (3) shows that the space of intertwining operators

$$\mathrm{Hom}_G(C^\infty(G/Q,\wedge^p(\mathfrak{g}/\mathfrak{q})^\vee), C^\infty(G/H))$$

is isomorphic to $\underline{\mathcal{D}}'_p(G/Q)^H$. Hence Theorem 2.4 (1) is derived from the following proposition.

Proposition 3.5. *Let H be a real algebraic subgroup of a real reductive algebraic group G. Suppose that the number of orientable p-dimensional H-orbits on a generalized flag variety G/Q is infinite. Then we have*

$$\dim \underline{\mathcal{D}}'_p(G/Q)^H = \infty.$$

Because each H-orbits on G/Q is a regular semianalytic set of G/Q, Proposition 3.5 follows from Fact 3.6 below. □

Fact 3.6 ([6, Thm. 2.1])**.** Let X be a real analytic manifold and Y an orientable regular p-dimensional semianalytic set of X. Then for any $\alpha \in \mathcal{D}^p(X)$, $T_Y(\alpha) := \int_Y \alpha$ converges, giving rise to a nonzero element T_Y in $\underline{\mathcal{D}}'_p(X)$.

A part of the results is given in [13], and detailed proof of the other part will appear elsewhere.

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