Hitting times to spheres of Brownian motions with drifts starting from the origin

Dedicated to Professor Hiroyuki Matsumoto on the occasion of his 60th birthday

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Abstract: We investigate the first hitting times to spheres of Brownian motions with constant drifts. In the case when the Brownian motion starts from a point in $\mathbf{R}^d$ except for the origin, an explicit formula for the density function of the hitting time has been obtained. When the starting point is the origin, we represent the density function by means of the density of the hitting time of the Brownian motion without the drift.

Key words: Brownian motion with drift; first hitting time; modified Bessel functions.

1. Introduction. This article deals with the first passage problem of a Brownian motion with a constant drift. Let $\{B_t\}_{t\geqq 0}$ be a standard Brownian motion on $\mathbf{R}^d$ starting from a given point $x \in \mathbf{R}^d$. For a constant vector $\varrho \in \mathbf{R}^d$ a Brownian motion with a drift ϱ, denoted by $\{B_t^{(\varrho)}\}_{t\geqq 0}$, is defined as $B_t^{(\varrho)} = B_t + \varrho t$. For $\varrho \in \mathbf{R}^d$ and $r > 0$ let

$$\tau_r^{(\varrho)} = \inf\{t > 0; |B_t^{(\varrho)}| = r\},$$

which implies the first hitting time of $\{B_t^{(\varrho)}\}_{t\geqq 0}$ to the sphere S_r^{d-1} with radius r and centered at the origin.

In this paper we will discuss the probability density function of $\tau_r^{(\varrho)}$, for which we write $p_r^{(\varrho)}(\cdot; x)$ in the case when $d \geqq 2$. Explicit forms of $p_r^{(0)}(\cdot; x)$ are obtained in [1,6,7] for $|x| < r$ and in [4] for $|x| > r$, where $|y|$ is the Euclidean distance between $y \in \mathbf{R}^d$ and the origin. In the case $\varrho \neq 0$, formulas for $p_r^{(\varrho)}(\cdot; x)$ have been deduced for $x \neq 0$. One of the formulas is given in [5, Theorem 1.1] and expressed as an infinite sum of which each summand consists of the modified Bessel functions, the Gegenbauer polynomials and the densities $p_r^{(0)}(\cdot; x)$. Other form is represented in [11] by an integral involving the Bessel functions. We should remark that a general framework for discussing the distribution of the hitting time is provided in [10].

One of our purposes of this paper is to give an explicit form of $p_r^{(\varrho)}(\cdot; 0)$ when $\varrho \neq 0$. For simplicity we use the notation $p_r^{(\varrho)}(\cdot)$ instead of $p_r^{(\varrho)}(\cdot; 0)$. We obtain that $p_r^{(\varrho)}$ is represented by the density $p_r^{(0)}$ and the modified Bessel function I_μ of the first kind of order μ. For convenience we put $\nu = d/2 - 1$.

Theorem 1.1. *Let $d \geqq 2$ and $\varrho \neq 0$. We have that*

$$p_r^{(\varrho)}(t) = \frac{2^\nu \Gamma(\nu+1) I_\nu(r|\varrho|)}{(r|\varrho|)^\nu} e^{-|\varrho|^2 t/2} p_r^{(0)}(t)$$

for any $t > 0$.

The idea of the proof is to represent the Laplace transform of $\tau_r^{(\varrho)}$ as an integral with respect to the distribution of (τ, B_τ) by the Cameron-Martin formula, which is similar to the calculation used in [5]. Here the notation τ has been used instead of $\tau_r^{(0)}$ for simplicity. A proof of Theorem 1.1 will be given in the next section. We should mention that the formula for $p_r^{(\varrho)}(t; 0)$ can not be simply deduced by taking a limit of $p_r^{(\varrho)}(t; x)$, given in [5, Theorem 1.1], as x tends to 0 since the formula for $p_r^{(\varrho)}(t; x)$ has terms which contain $\langle \varrho, x\rangle / (|\varrho| \cdot |x|)$, where $\langle \varrho, x\rangle$ is the standard inner product of ϱ and x. In addition, we remark that the explicit form of $p_r^{(0)}$ is provided in the following way:

$$(1.1) \qquad p_r^{(0)}(t) = \frac{1}{2^\nu \Gamma(\nu+1) r^2} \times \sum_{n=1}^\infty \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} e^{-j_{\nu,n}^2 t/2r^2},$$

where J_μ is the Bessel function of the first kind of order μ and $\{j_{\mu,n}\}_{n=1}^\infty$ is the increasing sequence of positive zeros of J_μ (cf. [1, Theorem 2]).

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Another purpose of this paper is to give the asymptotic behavior of the tail probability of $\tau_r^{(\varrho)}$ for $\varrho \neq 0$. The following theorem can be deduced from (1.1) and Theorem 1.1.

Theorem 1.2. *Let $d \geqq 2$ and $\varrho \neq 0$. We have that*

$$P(\tau_r^{(\varrho)} > t) = \frac{2I_\nu(r|\varrho|)}{(r|\varrho|)^\nu} \frac{j_{\nu,1}^{\nu+1}}{J_{\nu+1}(j_{\nu,1})} \frac{1}{j_{\nu,1}^2 + r^2|\varrho|^2} \times e^{-(j_{\nu,1}^2+r^2|\varrho|^2)t/2r^2}(1+o[1])$$

as $t \to \infty$.

We will prove the theorem in Section 3.

2. The density function. In this section we give a proof of Theorem 1.1 with the help of the Laplace transform of $\tau_r^{(\varrho)}$. When $x \neq 0$, the Laplace transform of $\tau_r^{(\varrho)}$ is represented in [5, p. 5391]. In the same way we can deduce that

$$E[e^{-\lambda\tau_r^{(\varrho)}}] = E[e^{-(\lambda+|\varrho|^2/2)\tau+\langle\varrho,B_\tau\rangle}; \tau < \infty]$$

for $x = 0$ and the right-hand side is equal to

$$\int_0^\infty \int_{\mathbf{R}^d} e^{-(\lambda+|\varrho|^2/2)t+\langle\varrho,y\rangle} P(\tau \in dt, B_\tau \in dy). \tag{2.1}$$

We omit the detailed calculation. It is known that

$$P(\tau \leqq t, B_\tau \in A) = P(\tau \leqq t)\sigma_r(A)$$

for $t \geqq 0$ and a Borel set A in S_r^{d-1}, where the notation σ_r has been used to denote the uniform distribution on S_r^{d-1} (cf. [8, p. 27]). This implies that (2.1) can be represented by

$$\int_0^\infty e^{-(\lambda+|\varrho|^2/2)t} p_r^{(0)}(t)dt \int_{S_r^{d-1}} e^{\langle\varrho,y\rangle} d\sigma_r(y). \tag{2.2}$$

Hence it is sufficient to calculate the integral on y in (2.2). We have that

$$\int_{S_r^{d-1}} e^{\langle\varrho,y\rangle} d\sigma_r(y) = \int_{S_1^{d-1}} e^{r\langle\varrho,y\rangle} d\sigma_1(y). \tag{2.3}$$

Let $w = (1, 0, \dots, 0) \in \mathbf{R}^d$. We take an orthogonal matrix T of order d such that $T\varrho = |\varrho|w$. Since σ_1 is preserved under orthogonal linear transformations on $\mathbf{R}^d$, the right-hand side of (2.3) is equal to

$$\int_{S_1^{d-1}} e^{r\langle\varrho,{}^tTy\rangle} d\sigma_1(y) = \int_{S_1^{d-1}} e^{r|\varrho|\langle w,y\rangle} d\sigma_1(y). \tag{2.4}$$

It is easy to see that, if $d \geqq 3$, the right-hand side of (2.4) coincides with the product of the following two integrals:

$$\int_0^\pi e^{r|\varrho|\cos\theta_1} \sin^{d-2}\theta_1 d\theta_1, \tag{2.5}$$

$$\frac{1}{S_{d-1}} \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{d-2} \times \int_0^{2\pi} d\theta_{d-1} \sin^{d-3}\theta_2 \cdots \sin\theta_{d-2}, \tag{2.6}$$

where S_{d-1} is used for the surface area of S_1^{d-1}. We can find that (2.5) coincides with

$$\sqrt{\pi}\left(\frac{2}{r|\varrho|}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) I_\nu(r|\varrho|)$$

in [3, p. 491] and it is obvious that the integral in (2.6) is equal to S_{d-2}. By the well-known formula $S_{m-1} = 2\pi^{m/2}/\Gamma(m/2)$ for each $m \geqq 2$, we obtain that the right-hand side of (2.4) and also (2.3) are equal to

$$2^\nu \Gamma(\nu+1) \frac{I_\nu(r|\varrho|)}{(r|\varrho|)^\nu}.$$

This yields that (2.2), which is the Laplace transform of $\tau_r^{(\varrho)}$, can be expressed by

$$\int_0^\infty e^{-\lambda t} \left\{ 2^\nu \Gamma(\nu+1) \frac{I_\nu(r|\varrho|)}{(r|\varrho|)^\nu} \times e^{-|\varrho|^2 t/2} p_r^{(0)}(t) \right\} dt \tag{2.7}$$

in the case $d \geqq 3$.

Note that $\nu = 0$ if $d = 2$. The calculation in the two dimensional case is easy. Indeed, we have that the right-hand side of (2.4) is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{r|\varrho|\cos\theta} d\theta = \frac{1}{\pi}\int_0^\pi e^{r|\varrho|\cos\theta} d\theta = I_0(r|\varrho|)$$

(cf. [3, p. 491]). This implies that (2.2) has the same form as (2.7) for $\nu = 0$.

We complete the proof of Theorem 1.1.

3. Asymptotics of the tail probability. This section is devoted to showing Theorem 1.2. Theorem 1.1 gives that

$$\int_0^\infty p_r^{(\varrho)}(t)dt = 2^\nu \Gamma(\nu+1) \frac{I_\nu(r|\varrho|)}{(r|\varrho|)^\nu} \times \int_0^\infty e^{-|\varrho|^2 t/2} p_r^{(0)}(t)dt. \tag{3.1}$$

In addition, it is known that

$$\int_0^\infty e^{-\lambda t} p_r^{(0)}(t)dt = \frac{(r\sqrt{2\lambda})^\nu}{2^\nu \Gamma(\nu+1) I_\nu(r\sqrt{2\lambda})}$$

for $\lambda > 0$ (cf. [2,6]). Thus we immediately conclude that the right-hand side of (3.1) is equal to 1, which implies that $P(\tau_r^{(\varrho)} < \infty) = 1$. By Theorem 1.1 and (1.1) we have that $P(\tau_r^{(\varrho)} > t)$ is equal to

$$\frac{I_\nu(r|\varrho|)}{r^{\nu+2}|\varrho|^\nu} \int_t^\infty \sum_{n=1}^\infty \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \times e^{-(r^2|\varrho|^2 + j_{\nu,n}^2)s/2r^2} ds. \tag{3.2}$$

In order to prove Theorem 1.2, we should justify changing the order of summation and integration in (3.2).

It is well-known that

$$j_{\nu,n} = \pi n + O[1] \tag{3.3}$$

for large n (cf. [9, p. 506]). Moreover, combining (3.3) and the asymptotic behavior of the Bessel function of the first kind (cf. [9, p. 199]), we can obtain that

$$J_{\nu+1}(j_{\nu,n}) = \frac{(-1)^{n+1}\pi}{\sqrt{2}} \sqrt{n} \left(1 + O\left[\frac{1}{n}\right]\right) \tag{3.4}$$

as $n \to \infty$ (cf. [7, p. 318]). It is easy to show by (3.3) and (3.4) that

$$\begin{aligned} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} &= \frac{\sqrt{2}(\pi n)^{\nu+1}(1 + O[1/n])^{\nu+1}}{(-1)^{n+1}\pi\sqrt{n}(1 + O[1/n])} \\ &= (-1)^{n+1}\sqrt{2}\pi^\nu n^{\nu+\frac{1}{2}} \left(1 + O\left[\frac{1}{n}\right]\right), \end{aligned}$$

which implies that there exists a constant C such that

$$\left| \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \right| \leqq C n^{\nu+1/2} \tag{3.5}$$

for each $n \geqq 1$. Hence we deduce from (3.3) and (3.5) that

$$\sum_{n=1}^\infty \int_t^\infty \left| \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \right| e^{-(r^2|\varrho|^2 + j_{\nu,n}^2)s/2r^2} ds$$

converges for each $t > 0$. We can change the order of the summation and the integral in (3.2) and thus it follows that $P(\tau_r^{(\varrho)} > t)$ is equal to

$$\begin{aligned} &\frac{I_\nu(r|\varrho|)}{r^{\nu+2}|\varrho|^\nu} \sum_{n=1}^\infty \int_t^\infty \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} e^{-(r^2|\varrho|^2 + j_{\nu,n}^2)s/2r^2} ds \\ &= \frac{2I_\nu(r|\varrho|)}{r^\nu|\varrho|^\nu} \sum_{n=1}^\infty \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})(j_{\nu,n}^2 + r^2|\varrho|^2)} \times e^{-(r^2|\varrho|^2 + j_{\nu,n}^2)t/2r^2}. \end{aligned}$$

Using (3.5) again, we obtain the claim of Theorem 1.2.

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