A note on the non-vanishing of Poincaré series for the Fricke group $\Gamma_0^+(p)$

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Abstract: For each prime p, we give an upper bound in m for Poincaré series $P_k^+(z,m)$ of weight k for $\Gamma_0^+(p)$ to be non-vanishing.

Key words: Weakly holomorphic modular form.

1. Introduction. The Poincaré series $P_k(z,m)$ of weight k for $\Gamma_0(N)$ for positive integers N have played an important role in number theory. For example, it is known that the forms $P_k(z,m)$ span the space of cusp forms of weight k, and also, it is known that the Petersson inner product of $P_k(z,m)$ by a cusp form $f(z)$ of weight k determines the Fourier coefficients of $f(z)$. In fact, the coefficients of $P_k(z,m)$ are defined in terms of Bessel functions and Kloosterman sums. Nevertheless, not much is known of the properties of coefficients of the Poincaré series. For example, it is not known whether they are zero or not, they are algebraic or transcendental, etc. In particular, there is no known efficient way to determine whether poincaré series is identically zero or not, which we call "the non-vanishing property", and there are a few results on the non-vanishing property of the Poincaré series $P_k(z,m)$ for some congruent groups. As some of the known results on this, Rankin [4] and Mozzochi [3] have given upper bounds of m for non-vanishing $P_k(z,m)$ for $\Gamma_0(1)$ and $\Gamma_0(N)$, respectively. More precisely, they have showed that there exist positive constants k_0 and B (independent of N), where $B > 4\log 2$ such that for all $k \geq k_0$ and all positive integers m such that

$$k \leq m \leq k^2 \exp(-B\log k/\log\log k),$$

$P_k(z,m) \not\equiv 0$ for $\Gamma_0(N)$. In [5] Rhoades found linear relations among the Poincaré series of weight k for $\Gamma_0(N)$ given by weakly holomorphic modular forms of weight $2-k$ for $\Gamma_0(N)$. This result implies that the non-vanishing problem is related to the existence of a weakly holomorphic modular form of weight $2-k$ for $\Gamma_0(N)$ with a certain given principal part.

In this paper, for a prime p, we consider the Poincaré series $P_k^+(z,m)$ for the Fricke group $\Gamma_0^+(p)$, where $\Gamma_0^+(p)$ is generated by the Hecke congruence group $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$ and $P_k^+(z,m)$ is given by the following;

$$(1)\; P_k^+(z,m) = \sum_{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma_0^+(p)} (cz+d)^{-k} e(m\gamma z),$$

where $e(z) = e^{2\pi i z}$ and Γ_∞ is the stabilizer of the cusp ∞ in $\Gamma_0^+(p)$. Generalizing the result for $\Gamma_0(N)$ of Rhoades [5], the first author and Kim [1] have obtained all linear relations among $P_k^+(z,m)$ for $\Gamma_0^+(N)$ which give a necessary and sufficient condition related to the non-vanishing of the Poincaré series $P_k^+(z,m)$.

The purpose of this paper is to extend results given by Rankin and Mozzochi to non-vanishing of the Poincaré series $P_k^+(z,m)$ for the Fricke group $\Gamma_0^+(p)$ for a prime p, in which case the only cusp is at infinity.

The main idea of the proof is based on ones in Rankin and Mozzochi ([3,4]), but in the case of $\Gamma_0^+(p)$, we have to compute the bounds of two kinds of the product of the Bassel function and the Kloosterman sums, while Rankin and Mozzochi have one product of them. So emphasizing this, we give the details of the proof of our main theorem, which is now ready to be stated:

Theorem 1.1. *Let $P_k^+(z,m)$ be the Poincaré series of weight k for the Fricke groups $\Gamma_0^+(p)$ for primes p. Then, there exist positive constants $k_0(p)$ (depending on p) and C where $C > 8\log 2$ is a constant such that for all $k \geq k_0(p)$ and all positive integers m such that*

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$$k \leq m \leq p^{3/2} k^2 \exp(-C \log k / \log\log k),$$

$P_k^+(z,m) \not\equiv 0$.

2. The Fourier expansion of the Poincaré series for $\boldsymbol{\Gamma_0^+(p)}$. Throughout this paper, for a given prime p, we let $P_k^+(z,m)$ be the Poincaré series of weight k attached to $\Gamma_0^+(p)$. Recall (1),

$$P_k^+(z,m) = \sum_{\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0^+(p)} (cz+d)^{-k} e(m\gamma z),$$

where $e(z) = e^{2\pi i z}$ and Γ_∞ is the stabilizer of the cusp ∞ in $\Gamma_0^+(p)$.

By [2, Ch.3 (3.18), (3.19)], $P_k^+(z,m)$ has the following Fourier expansion at the cusp ∞,

$$P_k^+(z,m) = \sum_{n=1}^{\infty} P_{\infty\infty}(m,n) e(nz),$$

where

$$(2)\ P_{\infty\infty}(m,n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \times \left(\delta_{mn} + 2\pi i^{-k} \sum_{c>0} \frac{S_{\infty\infty}(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right),$$

for the Bessel function of the first kind J_{k-1} and $S_{\infty\infty}(m,n;c)$ defined by

$$(3)\, S_{\infty\infty}(m,n;c) = \sum_{\gamma=\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} e\left(\frac{ma+nd}{c}\right).$$

Recalling (2) and (3), in order to express the Fourier expansion explicitly, we need to find representatives of the double coset $\Gamma_\infty \setminus \Gamma_0^+(p)/\Gamma_\infty$. Consequently, by considering $\Gamma_0^+(p)$ as the union of $\Gamma_0(p)$ and $\Gamma_0(p)W_p$ we get the following representatives of $\Gamma_\infty \setminus \Gamma_0^+(p)/\Gamma_\infty$.

Proposition 2.1. *For $\Gamma_0^+(p)$, we have the disjoint union of double cosets,*

$$\Gamma_\infty \setminus \Gamma_0^+(p)/\Gamma_\infty = B \cup \left(\bigcup_{q\geq 1} \bigcup_{\substack{d=1 \\ (d,pq)=1 \\ ad\equiv 1 \pmod{pq}}}^{pq} B \begin{pmatrix} a & * \\ pq & d \end{pmatrix} B \right) \cup \left(\bigcup_{\substack{q \geq 1 \\ (p,q)=1}} \bigcup_{\substack{c=1 \\ (c,q)=1 \\ -bcp\equiv 1 \pmod{q}}}^{q} B \begin{pmatrix} -b\sqrt{p} & * \\ q\sqrt{p} & c\sqrt{p} \end{pmatrix} B \right),$$

where

$$B := \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}.$$

Proof. For $\gamma = \begin{pmatrix} a & b \\ cp & d \end{pmatrix} \in \Gamma_0(p)$ with $a, b, c, d \in \mathbf{Z}$ such that $ad - bcp = 1$, if $c = 0$, then $\gamma \in B$ and $B\gamma B = B$.

Now we assume that $c \neq 0$. Then for $\beta_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in B$,

$$\pm \beta_m \gamma \beta_n = \pm \begin{pmatrix} a+mcp & n(a+mcp)+b+md \\ cp & ncp+d \end{pmatrix}.$$

Hence it is uniquely determined by $c \geq 1$ and $d \pmod{cp}$, since a is determined by $ad \equiv 1 \pmod{cp}$ and b is determined by $ad - 1 = cpb$. So the double cosets for $\Gamma_0(p)$ have representatives of the form

$$\bigcup_{q\geq 1} \bigcup_{\substack{d=1 \\ (d,pq)=1 \\ ad\equiv 1 \pmod{pq}}}^{pq} B \begin{pmatrix} a & * \\ pq & d \end{pmatrix} B.$$

To compute the double cosets for $\Gamma_0(p)W_p$ with $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$, we let $\gamma = \begin{pmatrix} a & -b \\ -cp & d \end{pmatrix} \in \Gamma_0(p)$ with $a, b, c, d \in \mathbf{Z}$ such that $ad - bcp = 1$,

$$\pm \beta_m \gamma W_p \beta_n = \begin{pmatrix} (md-b)\sqrt{p} & n(md-b)\sqrt{p} - a/\sqrt{p} + mc\sqrt{p} \\ d\sqrt{p} & (nd+c)\sqrt{p} \end{pmatrix}.$$

Hence it is uniquely determined by $d \geq 1$ with $(d,p) = 1$ and $b \pmod{d}$ since c is determined by $-bcp \equiv 1 \pmod{d}$ and a is determined by $bcp + 1 = da$. So the double cosets for $\Gamma_0(p)W_p$ have representatives of the form

$$\bigcup_{\substack{q \geq 1 \\ (p,q)=1}} \bigcup_{\substack{c=1 \\ (c,q)=1 \\ -bcp\equiv 1 \pmod{q}}}^{q} B \begin{pmatrix} -b\sqrt{p} & * \\ q\sqrt{p} & c\sqrt{p} \end{pmatrix} B.$$

□

Therefore, referring (2), by Proposition 2.1,

(4) $$P_{\infty\infty}(m,n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}}\left(\delta_{mn} + 2\pi i^{-k}\left(\sum_{q\geq 1}\frac{1}{pq}\sum_{\substack{d=1\\(d,pq)=1\\ \bar{d}d\equiv 1 \pmod{pq}}}^{pq} \times e\left(\frac{m\bar{d}+nd}{pq}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{pq}\right) + \sum_{\substack{q\geq 1\\ p\nmid q}}\frac{1}{q\sqrt{p}}\sum_{\substack{c=1\\(c,q)=1\\ -bcp\equiv 1 \pmod q}}^{q} e\left(\frac{-mb+nc}{q}\right) \times J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q\sqrt{p}}\right)\right)\right).$$

Since

$$\sum_{\substack{d=1\\(d,pq)=1\\ \bar{d}d\equiv 1 \pmod{pq}}}^{pq} e\left(\frac{m\bar{d}+nd}{pq}\right) = S(m,n;pq),$$

where $S(m,n;pq)$ is the Kloosterman sum, and if we take $\bar{p_q}\in\mathbf{Z}$ such that $\bar{p_q}p\equiv 1 \pmod q$, letting $b' = -bp$,

$$\sum_{\substack{c=1\\(c,q)=1\\ -bcp\equiv 1 \pmod q}}^{q} e\left(\frac{-m\bar{p_q}pb+nc}{q}\right) = \sum_{\substack{c=1\\ b'c\equiv 1 \pmod q}}^{q} e\left(\frac{m\bar{p_q}b'+nc}{q}\right) = S(m\bar{p_q},n;q),$$

we have that

(5) $$P_{\infty\infty}(m,n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}}\left(\delta_{mn} + 2\pi i^{-k}\left(\sum_{q\geq 1}\frac{S(m,n;pq)}{pq} \times J_{k-1}\left(\frac{4\pi\sqrt{mn}}{pq}\right) + \sum_{\substack{q\geq 1\\ p\nmid q}}\frac{S(m\bar{p_q},n;q)}{q\sqrt{p}}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q\sqrt{p}}\right)\right)\right).$$

3. The proof of the main theorem. We note that by [2, Ch.3, Corollary 3.4],

(6) $P_k^+(z,m)\neq 0$ if and only if $P_{\infty\infty}(m,m)\neq 0$.

From (5) with $m=n$, we have that

$$P_{\infty\infty}(m,m) = 1 + 2\pi i^{-k}(S_{m,p}+S_{m,p}^W),$$

where

$$S_{m,p} := \sum_{q\geq 1}\frac{S(m,m;pq)}{pq}J_{k-1}\left(\frac{4\pi m}{pq}\right) \text{ and}$$

$$S_{m,p}^W := \sum_{\substack{q\geq 1\\ p\nmid q}}\frac{S(m\bar{p_q},m;q)}{q\sqrt{p}}J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right).$$

Hence if we show that for large k,

(7) $$|S_{m,p}| < \frac{1}{4\pi} \text{ and } |S_{m,p}^W| < \frac{1}{4\pi},$$

then

$$|P_{\infty\infty}(m,m)| \geq 1 - 2\pi(|S_{m,p}|+|S_{m,p}^W|) > 0,$$

hence $P_{\infty\infty}(m,m)\neq 0$, and so $P_k^+(z,m)\neq 0$.

Therefore, in this section we will show that (7) holds for large k and complete the proof of Theorem 1.1.

First, we give the detailed proof of $|S_{m,p}^W| < \frac{1}{4\pi}$ for large k. Following the notations of [4], we let

$$\nu = k-1, \sigma = \nu^{-1/6}, \text{ and } Q = \frac{4\pi m}{\nu},$$

and as in [4, Lemma 3.1], we let

$$d = (m\bar{p_q},m,q) \text{ and } q = rd \text{ for } r\geq 1.$$

Then, since $p\nmid q$ and $(\bar{p_q},q)=1$, we have that $d = (m\bar{p_q},m,q) = (m,q)$.

We consider the sum $S_{m,p}^W$ as the sum of two cases when $q < Q/\sqrt{p}$ or $q\geq Q/\sqrt{p}$ and for the sum when $q < Q/\sqrt{p}$, we need an upper bound of the absolute value of the Kloosterman sum by applying [4, Lemma 3.1] as follows.

Lemma 3.1. ([4, Lemma 3.1]) *Under the notations defined above,*

$$|S(m\bar{p_q},m;q)| \leq 2^{w(r)}r^{1/2}d,$$

where $w(p) = \begin{cases}\frac{3}{2}, & \text{if } p=2,\\ 1, & \text{if } p \text{ is an odd prime,}\end{cases}$ *and for each positive integer* $n\geq 2$, $w(n) = \sum_{\text{a prime } p|n} w(p)$.

Note that if $k\geq 14$, then $\nu\geq 13 > 4\pi$, so we have that

$$Q = \frac{4\pi}{\nu} m \le m. \tag{8}$$

We have that

$$\begin{aligned}|S^W_{m,p}| &\le \sum_{\substack{q\ge 1\\ p\nmid q}} \frac{\left|S(m\bar p_q, m; q)\right|}{q\sqrt{p}} \left| J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right)\right| \\ &\le \sum_{\substack{q<Q/\sqrt{p}\\ p\nmid q}} \frac{\left|S(m\bar p_q, m; q)\right|}{q\sqrt{p}} \left| J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right)\right| \\ &\quad + \sum_{\substack{q\ge Q/\sqrt{p}\\ p\nmid q}} \frac{\left|S(m\bar p_q, m; q)\right|}{q\sqrt{p}} \left| J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right)\right|.\end{aligned} \tag{9}$$

If $q < Q/\sqrt{p}$, then by the first inequality of Lemma 3.1,

$$\begin{aligned}S_1 &:= \sum_{\substack{q<Q/\sqrt{p}\\ p\nmid q}} \frac{\left|S(m\bar p_q, m; q)\right|}{q\sqrt{p}} \left| J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right)\right| \\ &\le \sum_{d|m} \sum_{r<Q/(d\sqrt{p})} 2^{w(r)} (rp)^{-1/2} \left| J_{k-1}\left(\nu \frac{Q}{rd\sqrt{p}}\right)\right|.\end{aligned}$$

Moreover, if $k \ge 14$, by (8), we have that $r < Q/(d\sqrt{p}) < Q \le m$. So by [4, (3.7)], $2^{w(r)} \le M(m) = \exp(B_1 \log m/\log\log 2m)$ for some constant $B_1 > \log 2$. And since $\frac{Q}{rd\sqrt{p}} > 1$, by [4, Lemma 4.4 (4.18)], we have that $|(\frac{Q}{rd\sqrt{p}})^{1/2} J_{k-1}(\nu\frac{Q}{rd\sqrt{p}})| \le A_5 g(\frac{Q}{rd\sqrt{p}})$ for some positive constant A_5, where g is as defined in [4, (4.15)]. Hence,

$$\begin{aligned}S_1 &\le \frac{M(m)}{p^{1/4}Q^{1/2}} \sum_{\substack{d|m\\ d<Q/\sqrt{p}}} d^{1/2} \sum_{1\le r<Q/(d\sqrt{p})} \\ &\quad \times \left| \left(\frac{Q}{rd\sqrt{p}}\right)^{1/2} J_{k-1}\left(\nu\frac{Q}{rd\sqrt{p}}\right)\right| \\ &\le \frac{M(m)}{p^{1/4}Q^{1/2}} \sum_{\substack{d|m\\ d<Q/\sqrt{p}}} d^{1/2} A_5 \sum_{1\le r<Q/(d\sqrt{p})} g\left(\frac{Q}{rd\sqrt{p}}\right).\end{aligned} \tag{10}$$

Here by the definition of g as in [4, (4.15)] and as in [4, p. 158], we show that

$$\begin{aligned}&\sum_{1\le r<Q/(d\sqrt{p})} g\left(\frac{Q}{rd\sqrt{p}}\right) \\ &\le \int_1^{Q/d\sqrt{p}} g\left(\frac{Q}{ud\sqrt{p}}\right) du + 3g(y_\nu) \\ &= \frac{Q}{d\sqrt{p}} \int_1^{Q/d\sqrt{p}} x^{-2} g(x)dx + 3g(y_\nu) \\ &\le \frac{Q}{d\sqrt{p}} \int_1^{\infty} x^{-2} G(x)dx + 3\sigma^2 y_\nu^{1/2} \\ &\qquad \text{(by [4, (4.16), (4.15)])} \\ &\le \frac{Q}{d\sqrt{p}} \sigma^3 \int_1^\infty \frac{x^{-3/2}}{(x^2-1)^{1/4}} dx + 4\sigma^2 \\ &= \frac{1}{\sqrt{p}} \frac{2(2\pi)^{5/2}}{\Gamma^2(1/4)} \frac{m\sigma^9}{d} + 4\sigma^2.\end{aligned} \tag{11}$$

The last inequality is from [4, (4.16), (4.3)] and that $y_\nu = (1+\sigma^4)^{1/2}$ and so $3\sigma^2 y_\nu^{1/2} = 3(\sigma^8+\sigma^{12})^{1/4} \le 3(2\sigma^8)^{1/4} \le 4\sigma^2$, since $0<\sigma<1$.

Hence from (10) and (11), letting $c = \frac{2(2\pi)^{5/2}}{\Gamma^2(1/4)}$,

$$\begin{aligned}S_1 &\le \frac{A_5 M(m)}{p^{1/4}Q^{1/2}} \\ &\quad \times \sum_{\substack{d|m\\ d<Q/\sqrt{p}}} d^{1/2} \left(\frac{1}{\sqrt{p}} \frac{2(2\pi)^{5/2}}{\Gamma^2(1/4)} \frac{m\sigma^9}{d} + 4\sigma^2\right) \\ &\le A_6 M(m) \\ &\quad \times \sum_{\substack{d|m\\ d<Q/\sqrt{p}}} \left(\left(\frac{m}{d}\right)^{1/2} \sigma^6 p^{-3/4} \right. \\ &\qquad \left. + 4/c\left(\frac{d}{m}\right)^{1/2} \sigma^{-1} p^{-1/4}\right), \\ &\qquad \text{for a positive constant } A_6, \\ &\le A_6 M(m)(\sigma^6 m^{1/2} \sigma_{-1/2}(m) p^{-3/4} \\ &\quad + 4/c(4\pi)^{1/2} \sigma^2 \sigma_0(m) p^{-1/2}), \\ &\left(\text{since } \frac{d}{m} \le \frac{4\pi}{\sqrt{p}\nu} = 4\pi\sigma^6 p^{-1/2}\right) \\ &\le A_6 M(m) \sigma^6 m^{1/2} \sigma_{-1/2}(m) p^{-3/4} \\ &\quad + A_7 M^2(m) \sigma^2 p^{-1/2}, \\ &\qquad \text{for a positive constant } A_7.\end{aligned} \tag{12}$$

The last inequality follows from [4, (3.9)].

Next, if $q \ge Q/\sqrt{p}$, then we recall the trivial bound,

$$|S(m\bar p_q, m, q)| \le \phi(q) \le q,$$

and by [4, Lemma 4.4 (4.17)], for some positive constant A_5,

$$\left|J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right)\right| = \left|J_{k-1}\left(\nu\frac{Q}{q\sqrt{p}}\right)\right| \leq A_5 f\left(\frac{Q}{q\sqrt{p}}\right),$$

where f is defined as in [4, (4.13)]. So

$$\text{(13)}\quad S_2 := \sum_{\substack{q\geq Q/\sqrt{p}\\ p\nmid q}} \frac{|S(m\bar{p}_q, m; q)|}{q\sqrt{p}}\left|J_{k-1}\left(\frac{4\pi m}{q\sqrt{p}}\right)\right|$$
$$\leq \sum_{q\geq Q/\sqrt{p}} \frac{q}{q\sqrt{p}} A_5 f\left(\frac{Q}{q\sqrt{p}}\right)$$
$$\leq A_9 p^{-1/2}\left(\int_{Q/\sqrt{p}}^{\infty} f\left(\frac{Q}{u\sqrt{p}}\right)du + f(1)\right)$$
$$\leq A_9 p^{-1/2}\left(\frac{Q}{p^{1/2}}\int_0^1 x^{-2}f(x)dx + \sigma^2\right),$$

by [4, (4.13)],

$$\leq A_9 p^{-1/2}\left(\frac{Q}{p^{1/2}}\int_0^1 x^{-2}F(x)dx + \sigma^2\right),$$

by [4, (4.14)],

$$\leq A_{10} m\sigma^{12}p^{-1} + A_9\sigma^2 p^{-1/2},$$

for positive constants A_9, A_{10} by modifying the proof of [4, (5.12), (5.13)].

Finally, by (9), (10) and (13), we have that

$$|S^W_{m,p}| \leq S_1 + S_2 \leq A_6 M(m)\sigma^6 m^{1/2}\sigma_{-1/2}(m)p^{-3/4} + A_7 M^2(m)\sigma^2 p^{-1/2} + A_{10} m\sigma^{12}p^{-1} + A_9\sigma^2 p^{-1/2},$$

and the last three terms are all $o(1)$ for sufficiently large k. For the first term, we show that there exists a positive constant C such that if $k \leq m \leq p^{3/2}k^2\exp(-C\log k/\log\log k)$, then the first term is also $o(1)$.

By [4, (3.7) and (3.10)], for some positive constants $B_1 > \log 2$ and $B_2 > 2$, if we let $B = \max(B_1, B_2)$,

$$\text{(14)}\quad A_6 M(m)\sigma^6 m^{1/2}\sigma_{-1/2}(m)p^{-3/4}$$
$$\leq A_6\frac{k}{\nu}\exp(-C\log k/2\log\log k)$$
$$\times\exp((B_1\log m + B_2(\log m)^{1/2})/\log\log 2m)$$
$$\leq A_6\frac{k}{\nu}\exp(-C\log k/2\log\log k + 2B\log m/\log\log 2m).$$

Then since $k \leq m \leq p^{3/2}k^2\exp(-C\log k/\log\log k)$, we have that $\log\log k < \log\log 2m$, and $\log m \leq 3/2\log p + 2\log k - C\log k/\log\log k < 3/2\log p + 2\log k$, so

$$\text{(15)}\quad -C\log k/2\log\log k + 2B\log m/\log\log 2m$$
$$\leq (-C\log k + 4B(3/2\log p + 2\log k))/2\log\log k$$
$$\leq (-C+8B)\log k/2\log\log k + (3B\log p)/\log\log k.$$

Hence, the first term of (14) is

$$\text{(16)}\quad \leq A_6\frac{k}{\nu}\exp((-C+8B)\log k/2\log\log k + (3B\log p)/\log\log k)$$
$$= o(1),$$

for sufficiently large k, if $C > 8B > 8\log 2$. Hence we have shown that for sufficiently large k depending on p, there exists a positive constant $C > 8\log 2$ such that if $k \leq m \leq p^{3/2}k^2\exp(-C\log k/\log\log k)$, then

$$|S^W_{m,p}| < \frac{1}{4\pi}.$$

Next, in order to prove $|S_{m,p}| < \frac{1}{4\pi}$, for large k, this part is for $\Gamma_0(p)$ so we can prove in a similar way by dividing q into two cases when $q < Q/p$ or $q \geq Q/p$ instead of $q < Q/\sqrt{p}$ or $q \geq Q/\sqrt{p}$ as in the proof of [3, Theorem 2] for $\Gamma_0(p)$ and by modifying the proof of [3, Theorem 2] leaving rational powers of p out of the constant parts in the bounds, and we can derive that for some positive constants $A'_5, A'_6, A'_7, A'_{10}$,

$$\text{(17)}\quad |S_{m,p}| \leq \sum_{1\leq q<Q/p}\frac{|S(m,m;pq)|}{pq}\left|J_{k-1}\left(\frac{4\pi m}{pq}\right)\right| + \sum_{q\geq Q/p}\frac{|S(m,m;pq)|}{pq}\left|J_{k-1}\left(\frac{4\pi m}{pq}\right)\right|$$
$$\leq A'_6 M(m)m^{1/2}\sigma^6\sigma_{-1/2}(m) + A'_{10}M^2(m)\sigma^2 + A'_5\sigma^2 + A'_7 m\sigma^{12}p^{-1},$$

and the last three terms are $o(1)$ for large k. Then if $k \leq m \leq p^{3/2}k^2\exp(-C\log k/\log\log k)$, then as we have shown above, the first term satisfies that

$$\text{(18)}\quad A'_6 M(m)m^{1/2}\sigma^6\sigma_{-1/2}(m)$$
$$\leq A_6 p^{3/4}\frac{k}{\nu}\exp((-C+8B + o(1))\log k/2\log\log k)$$
$$= o(1),$$

for sufficiently large k, if $C > 8B > 8\log 2$. Hence we have shown that for sufficiently large k, there exists a positive constant $C > 8\log 2$ such that if $k \leq m \leq p^{3/2}k^2\exp(-C\log k/\log\log k)$, then

$$|S_{m,p}| < \frac{1}{4\pi}\,.$$

So (7) has been proved and this completes the proof of Theorem 1.1.

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