Collapsing K3 surfaces and Moduli compactification

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Abstract: This note is a summary of our work [OO], which provides an explicit and global moduli-theoretic framework for the collapsing of Ricci-flat Kähler metrics and we use it to study especially the K3 surfaces case. For instance, it allows us to discuss their Gromov-Hausdorff limits along any sequences, which are even not necessarily "maximally degenerating". Our results also give a proof of Kontsevich-Soibelman [KS06, Conjecture 1] (cf., [GW00, Conjecture 6.2]) in the case of K3 surfaces as a byproduct.

Key words: Locally symmetric spaces; Satake compactification; Kähler-Einstein metrics; K3 surfaces; Moduli; tropical geometry.

1. Introduction. Our paper [OO] is a sequel to a series by the first author [Od14, Od18], which compactified both the moduli space of compact Riemann surfaces $M_g(g \geq 2)$ and that of principally polarized abelian varieties A_g. In each case, as we actually expect an analogue for any moduli of general polarized Kähler-Einstein varieties with non-positive scalar curvatures, we introduce and study two similar (non-variety) compactifications of the moduli space $\mathcal{M}$, which we denote by $\overline{\mathcal{M}}^{\mathrm{GH}}$ and $\overline{\mathcal{M}}^{\mathrm{T}}$. The former $\overline{\mathcal{M}}^{\mathrm{GH}}$ is the Gromov-Hausdorff compactification with respect to *rescaled* Kähler-Einstein metrics of *fixed diameters* and the latter "tropical geometric compactification" $\overline{\mathcal{M}}^{\mathrm{T}}$ should dominate the former $\overline{\mathcal{M}}^{\mathrm{GH}}$ as its boundary $\partial\overline{\mathcal{M}}^{\mathrm{T}}$ encodes more structure of the Gromov-Hausdorff limits (collapses) rather than just distance structure. For a precise definition of $\overline{\mathcal{M}}^{\mathrm{GH}}$ we employ the same definition as [Od14, §2.3], [Od18, §2.2]. For $\overline{\mathcal{M}}^{\mathrm{T}}$, we have a case by case definition for only particular classes of varieties. Here, we recall the structure theorem of $\overline{A_g}^{\mathrm{GH}}$ from [Od18, Theorems 2.1, 2.3 and Corollary 2.5].

Theorem 1.1 ([Od18]). *A_g can be explicitly compactified as $\overline{A_g}^{\mathrm{GH}}$ whose boundary parametrizes all flat (real) tori $\mathbf{R}^i/\mathbf{Z}^i$ of diameter 1 where $1 \leq i \leq g$. Once we attach the rescaled flat Kähler metric in the principal polarization with diameter 1 to each abelian variety, the parametrization of metric spaces on whole $\overline{A_g}^{\mathrm{GH}}$ is continuous with respect to the Gromov-Hausdorff distance.*

In the above case, we simply set $\overline{A_g}^{\mathrm{T}} := \overline{A_g}^{\mathrm{GH}}$. On the other hand, in the analogue for M_g [Od14], we distinguish $\overline{M_g}^{\mathrm{GH}}$ and $\overline{M_g}^{\mathrm{T}}$, where the boundaries of $\overline{M_g}^{\mathrm{GH}}$ (resp., $\overline{M_g}^{\mathrm{T}}$) parametrize metrized graphs (resp., metrized graphs with integer weights on the vertices). We refer the details to [Od14].

Our [OO] contains the followings

(i) We first apply the Morgan-Shalen type compactification for general Hermitian locally symmetric spaces and identify it with one of the Satake compactifications ([Sat60a], [Sat60b]).

(ii) We partially prove that the boundary of the Satake compactification of the type which appears in (i) parametrizes collapses of abelian varieties and Ricci-flat K3 surfaces. This gives a generalisation of some results in [GW00], [Tos10], [GTZ13], [GTZ16], [TZ17] for the K3 surface case. For instance, a proof of the conjecture of Kontsevich-Soibelman [KS06, Conjecture 1] (see also Gross-Wilson [GW00, Conjecture 6.2]), which is related to the Strominger-Yau-Zaslow mirror symmetry [SYZ96], for the case of K3 surfaces directly follows from our description of collapsing. We also give a conjecture for higher dimensional hyperKähler varieties.

Now we move on to a more detailed description.

2. General Hermitian symmetric domain. Let $\mathbf{G}$ be a reductive algebraic group over $\mathbf{Q}$, $G = \mathbf{G}(\mathbf{R})$, K (one of) its maximal compact subgroup, and $D := G/K$, which we suppose to

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have a Hermitian symmetric domain structure. We moreover assume D is irreducible and G is simple as a Lie group. Suppose that Γ is an arithmetic subgroup of $\mathbf{G}(\mathbf{Q})$, which acts on D. Hence we can discuss Hermitian locally symmetric space $\Gamma \backslash D$.

Satake [Sat60a], [Sat60b] constructed compactifications of Riemannian locally symmetric spaces G/K associated to irreducible projective representations $\tau : G \to PGL(\mathbf{C})$ satisfying certain conditions. They are stratified as:

$$\overline{\Gamma \backslash D}^{\mathrm{Sat},\tau} = \Gamma \backslash D \sqcup \bigsqcup_P (\Gamma \cap Q(P)) \backslash M_P / (K \cap M_P).$$

Here, P runs over all the $\mu(\tau)$-connected rational parabolic subgroups, $P = N_P A_P M_P$ denotes the Langlands decomposition, and $Q(P)$ is the $\mu(\tau)$-saturation of P. We are particularly interested in the case when τ is the adjoint representation τ_{ad}.

On the other hand, given any toroidal compactification [AMRT75] for $\Gamma \backslash D$, we can apply the Morgan-Shalen type compactification to it as [Od18, Appendix] (following [MS84,BJ17]). The Morgan-Shalen type compactification $\overline{\Gamma \backslash D}^{\mathrm{MSBJ}}$ obtained in this way is independent of the cone decomposition for the toroidal compactification [Od18, A.13, A.14].

We now compare these two compactifications.

Theorem 2.1. *Let $\Gamma \backslash D$ be a locally Hermitian symmetric space. Consider its toroidal compactification and the associated (generalised) Morgan-Shalen compactification $\overline{\Gamma \backslash D}^{\mathrm{MSBJ}}$. Then this is homeomorphic to the Satake compactification $\overline{\Gamma \backslash D}^{\mathrm{Sat},\tau_{\mathrm{ad}}}$ for the adjoint representation τ_{ad} of G.*

In the following we make an "elementary" but important observation on a rationality phenomenon of the limits along one parameter holomorphic family, which we expect to fit well with the recent approach to extend the theta functions in [GS16] etc.

Proposition 2.2. *Suppose $U \subset \overline{U}^{\mathrm{hyb}}(\mathcal{X})$ is a Morgan-Shalen-Boucksom-Jonsson compactification associated to an arbitrary dlt stacky pair $(\mathcal{X}, \mathcal{D})$ of boundary coefficients 1 ([Od18]) with $\mathcal{U} := \mathcal{X} \setminus \mathcal{D}$, its coarse moduli space $\mathcal{U} \to U$. Then for any holomorphic morphism $\Delta^* := \{z \in \mathbf{C} \mid 0 < |z| < 1\} \to \mathcal{U}$ which extend to $\Delta := \{z \in \mathbf{C} \mid |z| < 1\} \to \mathcal{X}$, it induces a continuous map $\Delta \to \overline{U}^{\mathrm{hyb}}(\mathcal{X})$, i.e., the limit exists. Furthermore, such possible*

limits in $\Delta(\mathcal{D})$ are characterized as points with rational coordinates.

Corollary 2.3 (corollary to Theorem 2.1 and Proposition 2.2)**.** *Take an arbitrary holomorphic map $f : \Delta^* \to \Gamma \backslash D$, which extends to a map to a toroidal compactification of $\Gamma \backslash D$. Then f also extends to a map $\Delta \to \overline{\Gamma \backslash D}^{\mathrm{Sat},\tau_{ad}}$ where 0 is sent to a point with rational coordinates, i.e., a point in the dense subset $(C(F) \cap U(F) \otimes \mathbf{Q})/\mathbf{Q}_{>0} \subset C(F)/\mathbf{R}_{>0}$.*

This is partially proved in the case of A_g in [Od18] by using degeneration data in [FC90].

Remark 2.4. Although we assume that G is simple in this section, our Morgan-Shalen type compactification construction [Od18, Appendix] still works for non-simple G. Thus, our construction also gives a new Satake-type compactification for non-simple G, e.g., of the Hilbert modular varieties.

3. Abelian varieties case. We identify our tropical geometric compactification $\overline{A_g}^{\mathrm{T}}$ ([Od18]) of A_g with the adjoint type Satake compactification.

Theorem 3.1. *There are canonical homeomorphisms between the three compactifications*

$$\overline{A_g}^{\mathrm{T}} \cong \overline{A_g}^{\mathrm{Sat},\tau_{\mathrm{ad}}} \cong \overline{A_g}^{\mathrm{MSBJ}},$$

extending the identity on A_g.

The second canonical homeomorphism is a special case of Theorem 2.1 and the first is essentially reduced to matrix computations.

In [OO], we also give a purely moduli-theoretic reexplanation of the structure theory of one parameter degenerations of abelian varieties in [Mum72], [FC90], after the above Theorem 3.1 as follows:

Theorem 3.2. *Take a holomorphic maximally degenerating family of principally polarized abelian varieties $\pi : (\mathcal{X}, \mathcal{L}) \to \Delta$. Consider the rescaled Gromov-Hausdorff limit $B(\mathcal{X}, \mathcal{L})$ of diameter 1 as in Theorem 1.1 ([Od18]) and its discrete Legendre transform $\check{B}(\mathcal{X}, \mathcal{L})$ ([GS11], [KS06]).*

Then we can enhance the underlying integral affine structure of $\check{B}(\mathcal{X}, \mathcal{L})$ as K-affine structure (in the sense of [KS06, §7.1]) naturally via the data of π. Furthermore, such K-affine structure recovers π up to an equivalence relation generated by base change (replace t by t^a with $a \in \mathbf{Q}_{>0}$).

4. Moduli of Algebraic K3 surfaces.

4.1. Satake compactification. Let $\mathcal{F}_{2d}$ be the moduli space of polarized K3 surfaces of degree $2d$ possibly with ADE singularities. Its structure is known as follows: Let $\Lambda_{\mathrm{K3}} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ be

the K3 lattice and fix a primitive vector λ with $(\lambda, \lambda) = 2d$ and $\Lambda_{2d} := \lambda^\perp$. The complex manifold

$$\Omega(\Lambda_{2d}) := \{[w] \in \mathbf{P}(\Lambda_{2d} \otimes \mathbf{C}) \mid (w, w) = 0, \ (w, \bar{w}) > 0\}$$

has two connected components. We choose one component and denote by $\mathcal{D}_{\Lambda_{2d}}$. Let $O(\Lambda_{\mathrm{K3}})$ denote the isomorphism group of the lattice Λ_{K3} preserving the bilinear form and set

$$\tilde{O}(\Lambda_{2d}) := \{g|_{\Lambda_{2d}} : g \in O(\Lambda_{\mathrm{K3}}), \ g(\lambda) = \lambda\}.$$

The group $\tilde{O}(\Lambda_{2d})$ naturally acts on $\Omega(\Lambda_{2d})$. We define $\tilde{O}^+(\Lambda_{2d})$ to be the index two subgroup of $\tilde{O}(\Lambda_{2d})$ consisting of the elements preserving each connected component of $\Omega(\Lambda_{2d})$. Then it is well-known that

$$\mathcal{F}_{2d} \simeq \tilde{O}^+(\Lambda_{2d}) \backslash \mathcal{D}_{\Lambda_{2d}} \simeq \tilde{O}(\Lambda_{2d}) \backslash \Omega(\Lambda_{2d}).$$

Let $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}, \tau_{\mathrm{ad}}}$ (or simply $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}}$ in our papers) be the Satake compactification of $\mathcal{F}_{2d}$ corresponding to the adjoint representation of $O(2, 19)$. It decomposes as

$$\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} = \mathcal{F}_{2d} \sqcup \bigcup_l \mathcal{F}_{2d}(l) \sqcup \bigcup_p \mathcal{F}_{2d}(p),$$

where l runs over one-dimensional isotropic subspaces of $\Lambda_{2d} \otimes \mathbf{Q}$, and p runs over two-dimensional isotropic subspaces of $\Lambda_{2d} \otimes \mathbf{Q}$. Also, we simply define the tropical geometric compactification of $\mathcal{F}_{2d}$ as this $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}}$. The boundary component $\mathcal{F}_{2d}(l)$ is given as

$$\mathcal{F}_{2d}(l) = \{v \in (l^\perp/l) \otimes \mathbf{R} \mid (v, v) > 0\}/ \sim .$$

Here $v \sim v'$ if $g \cdot v = c v'$ for some $g \in \tilde{O}^+(\Lambda_{2d})$ and $c \in \mathbf{R}^\times$. We have $\mathcal{F}_{2d}(l) = \mathcal{F}_{2d}(l')$ if $g \cdot l = l'$ for some $g \in \tilde{O}^+(\Lambda_{2d})$ and $\mathcal{F}_{2d}(l) \cap \mathcal{F}_{2d}(l') = \emptyset$ if otherwise. Since $(l^\perp/l) \otimes \mathbf{R}$ has signature $(1, 18)$, there is an isomorphism

$$\{v \in (l^\perp/l) \otimes \mathbf{R} \mid (v, v) > 0\}/\mathbf{R}^\times$$
$$\simeq O(1, 18)/O(1) \times O(18)$$

and hence $\mathcal{F}_{2d}(l)$ is an arithmetic quotient of $O(1, 18)/O(1) \times O(18)$. The other component $\mathcal{F}_{2d}(p)$ is a point and $\mathcal{F}_{2d}(p) = \mathcal{F}_{2d}(p')$ if and only if $g \cdot p = p'$ for some $g \in \tilde{O}^+(\Lambda_{2d})$. Therefore, if we take representatives of l and p from each equivalence class, we get a finite decomposition:

$$\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} = \mathcal{F}_{2d} \sqcup \bigsqcup_l \mathcal{F}_{2d}(l) \sqcup \bigsqcup_p \mathcal{F}_{2d}(p).$$

4.2. Tropical K3 surfaces. In our paper, what we mean by *tropical polarized K3 surface* is a topological space B homeomorphic to the sphere S^2, with an affine structure away from certain finite points $\mathrm{Sing}(B)$, with a metric which is Mongé-Ampere metric g with respect to the affine structure on $B \setminus \mathrm{Sing}(B)$. Studies of such object as tropical version of K3 surfaces are pioneered in well-known papers of Gross-Wilson [GW00] and Kontsevich-Soibelman [KS06].

Here we assign such tropical K3 surface to each point in the boundary component $\mathcal{F}_{2d}(l)$ as follows: Let l be an oriented one-dimensional isotropic subspace of $\Lambda_{2d} \otimes \mathbf{Q}$. Write e for the primitive element of l such that $\mathbf{R}_{>0} e$ agrees with the orientation of l. Take a vector $v \in (l^\perp/l) \otimes \mathbf{R}$ such that $(v, v) > 0$. Write $[e, v]$ for the corresponding point in $\mathcal{F}_{2d}(l)$. Then there exists a (not necessarily projective) K3 surface X and a marking α_X: $H^2(X, \mathbf{Z}) \to \Lambda_{2d}$ with

(i) $\alpha_X(H^{2,0}) \subset \mathbf{R}\lambda + \sqrt{-1}\mathbf{R}v$,
(ii) $\alpha_X^{-1}(e)$ is in the closure of Kähler cone.

Let L be a line bundle on X such that $\alpha_X([L]) = e$. By the Torelli theorem, the pair (X, L) is unique up to isomorphisms. Then by (ii) we get an elliptic fibration $f: X \to B(\simeq \mathbf{P}^1)$. Take a holomorphic volume form Ω on X such that $\alpha_X([\mathrm{Re}\,\Omega]) = \lambda$. The map f is a Lagrangian fibration with respect to the symplectic form $\mathrm{Re}\,\Omega$. Hence it gives an affine manifold structure on $B \setminus \Delta$, where Δ denotes the finite set of singular points. Similarly, the imaginary part $\mathrm{Im}\,\Omega$ gives another affine manifold structure on $B \setminus \Delta$.

We endow the base space B with the McLean metric on the base B ([ML98]), where we regard f as special Lagrangian fibration after hyperKähler rotation. A straightforward calculation shows that this coincides with the "special Kähler metric" g_{sp} introduced and studied in [DW96,Hit99,Freed99] and appears as the metric on $\mathbf{P}^1$ in [GTZ16]. We rescale the metric to make its diameter 1 and denote this obtained tropical K3 surface by $\Phi_{\mathrm{alg}}([e, v])$.

Remark 4.1. Recall the concepts of the *class of metric* (*metric class*) and the *radiance obstruction* of Mongé-Ampére manifolds B with singularities. They are introduced in [KS06] and discussed in [GS06] in more details. We denote them by $k(B) \in H^1(B, i_*\tilde{\Lambda}^\vee \otimes \mathbf{R})$ and $c(B) \in H^1(B, i_*\Lambda)$, respectively. Here, Λ is the affine structure as a $\mathbf{Z}^{dim(B)}$-local system in the tangent bundle $T(B \setminus \Delta)$, $-^\vee$ denotes $-$'s dual local system, $\tilde{\Lambda}^\vee$ is the local system of affine functions.

In particular, we naturally have a morphism of local systems $f: \tilde{\Lambda}^\vee \to \Lambda^\vee$ which induces f_*: $H^1(B, i_*\tilde{\Lambda}^\vee \otimes \mathbf{R}) \to H^1(B, i_*\Lambda^\vee \otimes \mathbf{R})$. The "linear" part $f_*k(B)$ of the metric class naturally recovers the data $\overline{v} \in (e^\perp \otimes \mathbf{R}/\mathbf{R}e)$. Namely, we have $f_*k(\Phi_{\mathrm{alg}}([e,v])) = [v]$, under the natural identification $H^1(\Phi_{\mathrm{alg}}([e,v]), \quad i_*\Lambda^\vee \otimes \mathbf{R}) \hookrightarrow (e^\perp \otimes \mathbf{R}/\mathbf{R}e)$ which comes from the Leray spectral sequence applied to the elliptic fibration $X \to \Phi_{\mathrm{alg}}([e,v])$ in §4.2. Our results in [Od18] and Theorem 3.1 for A_g can be re-interpretted similarly (but with weight 1).

Remark 4.2. *Yuto Yamamoto* [Yam] has some ongoing interesting work which seems to be related to our works, where he constructs a sphere with an integral affine structure from the tropicalization of an anticanonical hypersurface in a toric Fano 3-fold, and computes its radiance obstruction.

4.3. Gromov-Hausdorff collapse of K3 surfaces. For a point in $\mathcal{F}_{2d}$ we have a corresponding polarized K3 surface (X, L), equipped with a natural Ricci-flat metric. For $[e, v] \in \mathcal{F}_{2d}(l)$ we defined in a previous section $\Phi_{\mathrm{alg}}([e,v])$. For a point in $\mathcal{F}_{2d}(p)$ we assign a (one-dimensional) segment, which we denote by $\Phi_{\mathrm{alg}}(\mathcal{F}_{2d}(p))$. Let us normalize these metric spaces so that their diameters are one. We thus obtained a map $\Phi_{\mathrm{alg}}: \overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} \to \{$compact metric spaces with diameter one$\}$. Here, we associate Gromov-Hausdorff distance to the right-hand side (target space) and denote it by $CMet_1$.

Conjecture 4.3. *The map*

$$\Phi_{\mathrm{alg}}: \overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} \to CMet_1$$

given above is continuous.

We would like to simply set the tropical geometric compactification of $\mathcal{F}_{2d}$ as $\overline{\mathcal{F}_{2d}}^{\mathrm{T}} := \overline{\mathcal{F}_{2d}}^{\mathrm{Sat}}$. Indeed, if Conjecture 4.3 holds, we get a continuous map $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} \to \overline{\mathcal{F}_{2d}}^{\mathrm{GH}}$ and we also observe that each $\mathcal{F}_{2d}(l)$ encodes affine structure of the limit tropical K3 surface as well. (This answers a question of Prof. B. Siebert in 2016 to the first author, regarding if one can associate tropical affine structure to limit of any collapsing *sequence*). So far, we have partially confirmed the conjecture. The case of (A_1-singular flat) Kummer surfaces, with 3-dimensional moduli, are easily reduced to [Od18]. More generally, we have proved the following. In particular, Conjecture 4.3 holds at least away from finite points.

Theorem 4.4. *The map* Φ_{alg} *is continuous on* $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} \setminus (\bigcup_p \mathcal{F}_{2d}(p))$. *It is continuous also when*

restricted to the boundary $\partial\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} = \overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} \setminus \mathcal{F}_{2d}$.

The proof of the former half of the statements involves some symmetric space theory, hyperKähler geometry, algebraic geometry of moduli, and a priori analytic estimates. The estimates heavily depends on [Tos10], [GW00], [GTZ13], [GTZ16], [TZ17] and their extensions. One nontrivial part of the extension is, for instance, to make many of the C^2-estimations in *op.cit.* following methods of [Yau78] locally uniform with respect to a family of elliptic K3 surfaces even along degenerations to orbifolds.

During our work, we learnt that *Kenji Hashimoto, Yuichi Nohara, Kazushi Ueda* also studied the Gromov-Hausdorff collapses along a certain 2-dimensional subvariety of $\mathcal{F}_{2d}$, i.e., the moduli of $(E_8^{\oplus 2} \oplus U)$-polarizable K3 surfaces. Moreover, a result of Hashimoto and Ueda [HU18] implies that the restriction of Φ_{alg} to the boundary is a generically one-to-one map. We appreciate their gentle discussion with us.

Theorem 4.4 (resp., Conjecture 4.3) combined with Proposition 2.2 determines the Gromov-Hausdorff limits of Type III (resp., Type II) one parameter family of Ricci-flat algebraic K3 surfaces, which solves a conjecture of Kontsevich-Soibelman [KS06, Conjecture 1], Todorov, and Gross-Wilson (cf., e.g., [Gross13, Conjecture 6.2]) in the K3 surfaces case.

In the next section, we discuss collapsing of general Kähler K3 surfaces, which are not necessarily algebraic.

5. Moduli of Kähler K3 surfaces. It is known (cf., [Tod80], [Looi81], [KT87]) that the moduli space of all Einstein metrics on a K3 surface (including orbifold-metrics) has again a structure of the locally Riemannian symmetric space:

$$O(\Lambda_{\mathrm{K3}})\backslash SO_0(3, 19)/(SO(3) \times SO(19)),$$

which we denote by $\mathcal{M}_{\mathrm{K3}}$. An enriched version encoding also complex structures of the K3 surfaces is

$$\mathbf{R}_{>0} \times (O(\Lambda_{\mathrm{K3}})\backslash SO_0(3, 19)/(SO(2) \times SO(19))).$$

Roughly speaking, this is a union of Kähler cones of ADE K3 surfaces with marking of the minimal resolutions.

Thus we can again compare a Satake compactification of $\mathcal{M}_{\mathrm{K3}}$ with the Gromov-Hausdorff compactification. Inside the Satake compactification for the adjoint representation, we consider an

open locus (a partial compactification of $\mathcal{M}_{K3}$) $\mathcal{M}_{K3} \sqcup \mathcal{M}_{K3}(a)$, where $\mathcal{M}_{K3}(a)$ denotes the 36-dimensional boundary stratum corresponding to an isotropic rational line $l = \mathbf{Q}e$ in $\Lambda_{K3} \otimes \mathbf{Q}$, with primitive integral generator e, which are unique up to $O(\Lambda_{K3})$. Then for each point $p = \langle e, v_1, v_2 \rangle$ in the stratum $\mathcal{M}_{K3}(a)$, we take an appropriate marked (possibly ADE) elliptic K3 surface X_p with period $\langle v_1, v_2 \rangle$ and the fiber class e as in §4.2. Then we define $\Phi(p)$ as its base biholomorphic to $\mathbf{P}^1$ with the McLean metric, which only depends on $\langle v_1, v_2 \rangle$. Similarly to the projective case Theorem 4.4, [OO] proves that for non-algebraic situation:

Theorem 5.1. *The map*

$$\Phi \colon \mathcal{M}_{K3} \sqcup \mathcal{M}_{K3}(a) \to CMet_1$$

given above is continuous. Here, we put the Gromov-Hausdorff topology for the right hand side.

In [OO], we further explicitly define an extension to the whole Satake compactification $\Phi \colon \overline{\mathcal{M}_{K3}}^{\mathrm{Sat}} \to CMet_1$, and conjecture that this is still continuous with respect to the Gromov-Hausdorff topology. For the boundary strata other than $\mathcal{M}_{K3}(a)$, we assign flat tori $\mathbf{R}^i/\mathbf{Z}^i$ ($i = 1, 2, 3$) modulo (-1)-multiplication, which appear as limits of unresolved Kummer surfaces for instance. Indeed, we show that Φ restricted to the closure of such locus which parametrizes $\mathbf{R}^4/\mathbf{Z}^4$ modulo ± 1, that includes those boundary strata, is continuous. Furthermore, we also prove the restriction of Φ to the closure of $\mathcal{M}_{K3}(a)$ is continuous by using Weierstrass models.

6. Higher dimensional case. We expect that our results for K3 surfaces naturally extend to higher dimensional compact hyperKähler manifolds. Let us focus on algebraic case in this notes. We set up as follows: Fix any connected moduli M of polarized $2n$-dimensional irreducible holomorphic symplectic manifolds (X, L). By [Ver13,Mark11] ([GHS13, Theorem 3.7]), it is a Zariski open subset of a Hermitian locally symmetric space of orthogonal type $\Gamma \backslash \mathcal{D}_M$.

Then, a weaker version of our conjecture for algebraic case (in [OO]) is as follows:

Conjecture 6.1. *There is a continuous map Ψ (which we call the "geometric realization map") from the Satake compactification $\overline{\Gamma \backslash \mathcal{D}_M}^{\mathrm{Sat}, \tau_{ad}}$ with respect to the adjoint representation to the Gromov-Hausdorff compactification of M, extending the identity map on M. The $(b_2(X) - 4)$-dimensional*

boundary strata of $\overline{\Gamma \backslash \mathcal{D}_M}^{\mathrm{Sat}, \tau_{ad}}$ parametrize via Ψ the projective space $\mathbf{P}^n$ with special Kähler metrics in the sense of [Freed99] and the metric space parametrized by 0-dimensional cusps are all homeomorphic to the closed ball of dimension n.

At the moment of writing this notes, the authors have only succeeded in proving that $\Gamma \backslash \mathcal{D}_M$ is the moduli of polarized symplectic varieties with continuous (non-collapsing) weak Ricci-flat Kähler metrics, and making some progress on the necessary algebro-geometric preparations in particular for the case of K3$^{[n]}$-type.

Remark 6.2 (General K-trivial case)**.** In [OO], we also discuss a possible extension of Conjecture 4.3 for general K-trivial varieties under some technical conditions, although there is much less evidence for that generality.

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