On products of cyclic and abelian finite p-groups (p odd)

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Abstract: For an odd prime p, it is shown that if $G = AB$ is a finite p-group, for subgroups A and B such that A is cyclic and B is abelian of exponent at most p^k, then $\Omega_k(A)B \trianglelefteq G$, where $\Omega_k(A) = \langle g \in A \mid g^{p^k} = 1\rangle$.

Key words: Products of groups; factorised groups; finite p-groups.

Much of what is known about finite p-groups that are the product of a cyclic subgroup and an abelian subgroup is limited to the case where both "factors" are cyclic. Products of two cyclic p-groups were investigated for odd primes by Huppert [5], and for $p = 2$ by Itô [7], Itô and Ôhara [8,9], and Blackburn [2]. Huppert showed in particular that if p is an odd prime and if the finite p-group G is the product of two cyclic subgroups, then G possesses a normal cyclic subgroup N such that G/N is cyclic ([5] Hauptsatz I).

Apart from products of cyclic subgroups, little is known about the detailed structure of finite p-groups of the form $G = AB$, where A is cyclic and B is abelian. Such products are, of course, metabelian by the celebrated Theorem of Itô ([6] Satz 1); while a result of Howlett ([4] Theorem A) shows that $\exp(G) \le \exp(A)\exp(B)$, where $\exp(G)$ denotes the exponent of a finite group G. The only other relevant result appears to be that of Conder and Isaacs ([3] Corollary C), which states that if $G = AB$ for abelian subgroups A and B such that B is finite and either A or B is cyclic, then $G'/(G' \cap A)$ is isomorphic to a subgroup of B.

The present note considers the case where p is an odd prime and $G = AB$ is a finite p-group, where A is a cyclic subgroup and B is an abelian subgroup of exponent at most p^k. For such a group Theorem 6 shows that $\Omega_k(A)B \trianglelefteq G$, where the characteristic subgroup, $\Omega_k(H)$, of a finite p-group H is defined by $\Omega_k(H) = \langle h \in H \mid h^{p^k} = 1\rangle$. For $|A| = p^n$ $(n > k)$ and $N = \Omega_k(A)B$, it can then be seen that G/N is cyclic of order p^{n-k}, a result that can be viewed as a partial analogue to that of Huppert cited above. Theorem 6 also generalises a recent result of the author ([10] Lemma 2.5), which deals with the case where A is cyclic and B is elementary abelian.

The following notation is used. The cyclic group of order p^n is denoted by C_{p^n}. U_G denotes the core of the subgroup U of a group G. Thus $U_G = \bigcap_{g\in G} U^g$. The normal closure of U in G is denoted by U^G, so $U^G = \langle U^g \mid g \in G\rangle$. We first derive some elementary results which will be used in the proof of Theorem 6.

Lemma 1. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is abelian. Let N be a normal subgroup of G and let $s \ge 0$ and $t \ge 0$ be such that*:
(i) $N \leqslant \Omega_{s+t}(A)B \leqslant G$;
(ii) $\Omega_s(AN/N)BN/N \trianglelefteq G/N$;
(iii) $A \cap N \leqslant \Omega_t(A)$.
Then $\Omega_{s+t}(A)B \trianglelefteq G$.

Proof. We let $\widetilde{A}/N = \Omega_s(AN/N)$. Since A is abelian and $A \cap N \leqslant \Omega_t(A)$, we have $\Omega_s(AN/N) \leqslant \Omega_{s+t}(A)N/N$, so $\widetilde{A} \leqslant \Omega_{s+t}(A)N$. Now $\widetilde{A}BN = \widetilde{A}B \trianglelefteq G$ and $\widetilde{A}B \leqslant \Omega_{s+t}(A)NB = \Omega_{s+t}(A)B$. Since $G/\widetilde{A}B$ is abelian, it follows that $\Omega_{s+t}(A)B \trianglelefteq G$. □

Lemma 2. *Let $G = AB$ be a finite group for subgroups A and B such that A is the cyclic group $\langle x\rangle$. Then $B^G = \langle B, B^x\rangle$.*

Proof. We have $\langle B, B^x\rangle = (\langle B, B^x\rangle \cap A)B$ and so $\langle B, B^x\rangle^x = (\langle B, B^x\rangle \cap A)^x B^x = (\langle B, B^x\rangle \cap A)B^x \leqslant \langle B, B^x\rangle$. Hence x normalises $\langle B, B^x\rangle$ and thus $B^G = \langle B, B^x\rangle$. □

Lemma 3. *Let $G = AB$ be a finite p-group for subgroups A and B such that A is the cyclic group $\langle x\rangle$ and B is a proper subgroup of G. Let s be such that $A \cap B^G = \Omega_s(A)$. If t is such that $\Omega_t(A) \leqslant B$,*

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then $t \le s$ *and* $|B : B \cap B^x| \le p^{s-t}$.

Proof. Since G is a finite p-group and B is a proper subgroup of G, we have $B^G \neq G$. Hence $\Omega_{s+1}(A) \nleqslant B^G$, so $\Omega_s(A) \neq \Omega_{s+1}(A)$. But $\Omega_t(A) \leqslant A \cap B \leqslant A \cap B^G = \Omega_s(A)$, so $t \le s$. Now $B^G = \Omega_s(A)B$, so

$$|BB^x| = \frac{|B||B^x|}{|B \cap B^x|} \le |B^G| = \frac{|\Omega_s(A)||B|}{|\Omega_s(A) \cap B|}.$$

Since $\Omega_t(A) \leqslant \Omega_s(A) \cap B$, we have $|\Omega_s(A) \cap B| \ge |\Omega_t(A)|$. Hence

$$\frac{|B||B^x|}{|B \cap B^x|} \le \frac{|\Omega_s(A)||B|}{|\Omega_t(A)|} = p^{s-t}|B|,$$

and it follows that $|B : B \cap B^x| \le p^{s-t}$. □

Lemma 4. *Let p be an odd prime and let $G = HK$ be a finite p-group for subgroups H and K such that $[H,K] \leqslant Z(G)$ and $\exp(K) \le p^t$. Then*

(i) $\exp([H,K]) \le p^t$;

(ii) $\Omega_t(G) = \Omega_t(H)[H,K]K = \langle \Omega_t(H), K\rangle$.

Proof. For (i) we let $h \in H$ and $k \in K$, and let $z = [h,k]$. Then $h = h^{k^{p^t}} = hz^{p^t}$, so $z^{p^t} = 1$. But $[H,K] \leqslant Z(G)$, so $[H,K]$ is abelian. Hence $\exp([H,K]) \le p^t$.

For (ii) we note first that $K^G = [H,K]K$, so by (i), we have $\langle \Omega_t(H), K\rangle \leqslant \Omega_t(H)[H,K]K \leqslant \Omega_t(G)$. Conversely, let $g = hk \in G$ be such that $g^{p^t} = 1$, where $h \in H$ and $k \in K$. Letting $z = [h,k] \in Z(G)$, we see that

$$1 = g^{p^t} = (hk)^{p^t} = k^{p^t} h^{p^t} z^{\frac{(p^t+1)p^t}{2}}.$$

Since p is odd and $\exp([H,K]) \le p^t$, we have $z^{\frac{(p^t+1)p^t}{2}} = 1$. In addition $k^{p^t} = 1$. Hence $h^{p^t} = 1$, so $\Omega_t(G) \leqslant \langle \Omega_t(H), K\rangle \leqslant \Omega_t(H)[H,K]K$. □

Corollary 5. *Let p be an odd prime and let G be a finite p-group such that $G = HZK$ for subgroups H, Z and K such that*

(i) $Z \leqslant Z(G)$;

(ii) $[H,K] \leqslant Z$;

(iii) $\exp(K) \le p^t$.

Then $\Omega_t(G) = \Omega_t(HZ)K$.

Proof. Since $Z \leqslant Z(G)$, we have $[HZ,K] = [H,K] \leqslant Z(G)$. In addition, K normalises HZ, so $\langle \Omega_t(HZ), K\rangle = \Omega_t(HZ)K$. The result then follows from Lemma 4. □

We now come to our main result.

Theorem 6. *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and B is abelian. If $\exp(B) \le p^k$, then $\Omega_k(A)B \trianglelefteq G$.*

Proof. We use induction on $|G|$. We may assume that G is non-cyclic, $G \neq B$ and $\Omega_k(A) \neq G$. Thus $A \neq 1$ and $B \neq 1$, and hence $\Omega_1(A) \neq 1$ and $k \ge 1$. Moreover, let $|A| = p^n$. If $k \ge n$, then $\Omega_k(A) = A$ and $\Omega_k(A)B = AB = G$. Thus we can also assume that $k \le n-1$. Since A is cyclic and G is a finite p-group, we note that $\Omega_t(A)B \leqslant G$ for all values of t.

We have $Z(G) = (Z(G) \cap A)(Z(G) \cap B)$ by, say, [1] Lemma 2.1.2. If $A \cap Z(G) = 1$, then $1 \neq Z(G) \leqslant B$. By induction, we have

$$\Omega_k(AZ(G)/Z(G))B/Z(G) \trianglelefteq G/Z(G).$$

Since $A \cap Z(G) = 1$, we apply Lemma 1 to see that $\Omega_k(A)B \trianglelefteq G$. We thus may assume that

$$\Omega_1(A) \leqslant Z(G).$$

Moreover, letting $\widetilde{B} = \Omega_1(A)B$, we have $\exp(\widetilde{B}) = \exp(B)$ and $\Omega_k(A)B = \Omega_k(A)\widetilde{B}$. Thus if we can show that $\Omega_k(A)\widetilde{B} \trianglelefteq G$, then we also have $\Omega_k(A)B \trianglelefteq G$. Hence we may assume that

$$\Omega_1(A) \leqslant B.$$

We next show that the result holds for $k = 1$. In this case B is elementary abelian. By induction, we have

$$\Omega_1(A/\Omega_1(A))B/\Omega_1(A) \trianglelefteq G/\Omega_1(A).$$

But $\Omega_1(A/\Omega_1(A)) = \Omega_2(A)/\Omega_1(A)$, so

$$\Omega_2(A)B \trianglelefteq G.$$

Now $\Omega_1(A) \neq A$, so $|\Omega_2(A)B : B| = |\Omega_2(A) : \Omega_1(A)| = p$ and $B \trianglelefteq \Omega_2(A)B$. If $B \ntrianglelefteq G$ then, letting $g \in G \backslash N_G(B)$, we see, by comparison of orders, that

$$\Omega_2(A)B = BB^g.$$

Thus $\Omega_2(A)B$ is the product of two elementary abelian normal subgroups. Since p is odd, we see that $\Omega_2(A)B$ has exponent p, which is a contradiction. We thus conclude that $B = \Omega_1(A)B \trianglelefteq G$.

We now assume that $k \ge 2$. We let M be a maximal proper subgroup of G such that $A \leqslant M$. Then $|G : M| = p$ and $M = A(B \cap M)$. Since $B \nleqslant M$, we have $|B : B \cap M| = p$. We let $B_1 = B \cap M$. By induction, we have $\Omega_k(A)B_1 \trianglelefteq M$. Since B normalises B_1, we note further that $B_1^G = B_1^M \leqslant \Omega_k(A)B_1$.

We have $B \nleqslant B_1^G$, as otherwise $G = AB_1^G = M$. Since $|B : B_1| = p$, we further have $BB_1^G/B_1^G \cong C_p$. Now $AB_1^G/B_1^G = M/B_1^G$ is a non-trivial, normal cyclic subgroup of index p in G/B_1^G and G/B_1^G is the extension of AB_1^G/B_1^G by BB_1^G/B_1^G. Since p is odd, we have

$$\Omega_1(G/B_1^G) = \Omega_1(AB_1^G/B_1^G)BB_1^G/B_1^G \trianglelefteq G/B_1^G.$$

Now $A \cap B_1^G \leqslant A \cap \Omega_k(A)B_1 = \Omega_k(A)(A \cap B_1)$. But $\exp(B) \le p^k$, so $A \cap B_1 \leqslant \Omega_k(A)$. Hence $A \cap B_1^G \leqslant \Omega_k(A)$.

We consider the case where $A \cap B_1^G \neq \Omega_k(A)$. Then $A \cap B_1^G \leqslant \Omega_{k-1}(A)$. Now $B_1^G \leqslant \Omega_k(A)B_1 \leqslant \Omega_k(A)B$. Hence, by Lemma 1, we have $\Omega_k(A)B \trianglelefteq G$.

We thus assume that $A \cap B_1^G = \Omega_k(A)$, so $B_1^G = \Omega_k(A)B_1 \leqslant \Omega_{k+1}(A)B$. By Lemma 1, we have $\Omega_{k+1}(A)B \trianglelefteq G$. Since $\exp(B) \le p^k$, we have $A \cap B \leqslant \Omega_k(A)$, so $\Omega_k(A) \cap B = \Omega_{k+1}(A) \cap B = A \cap B$. Hence $|\Omega_{k+1}(A)B : \Omega_k(A)B| = p$ and $\Omega_k(A)B_1 = B_1^G \leqslant B_1^G B = \Omega_k(A)B \leqslant B^G \leqslant \Omega_{k+1}(A)B \trianglelefteq G$.

Since $BB_1^G/B_1^G \cong C_p$, we have $\Phi(B) \leqslant B_1^G$. Now $k \ge 2$, so $g^{p^{k-1}} = (g^p)^{p^{k-2}} \in \Omega_1(\Phi(B)) \leqslant \Omega_1(B_1^G)$ for all $g \in B$. Hence $\exp(B\Omega_1(B_1^G)/\Omega_1(B_1^G)) \le p^{k-1}$. But $1 \neq \Omega_1(A) \leqslant \Omega_1(B_1^G)$ so, by induction $\Omega_{k-1}(A\Omega_1(B_1^G)/\Omega_1(B_1^G))B\Omega_1(B_1^G)/\Omega_1(B_1^G) \trianglelefteq G/\Omega_1(B_1^G)$. Now if B_1^G is abelian, then $\Omega_1(B_1^G)$ is elementary abelian, so $A \cap \Omega_1(B_1^G) = \Omega_1(A)$. In addition, we have $\Omega_1(B_1^G) \leqslant B_1^G \leqslant \Omega_k(A)B$, so, by Lemma 1, $\Omega_k(A)B \trianglelefteq G$.

We can thus assume that B_1^G is non-abelian. We let $Z = Z(B^G)$ and note that $\Omega_1(A) \leqslant B^G \cap Z(G) \leqslant Z$. We show that $Z \leqslant \Omega_k(A)B$. If not, then, by comparison of orders, $B^G = \Omega_{k+1}(A)B = \Omega_k(A)BZ$. Now $\Omega_k(A) = \Phi(\Omega_{k+1}(A))$, so $B^G = \Omega_{k+1}(A)B = BZ$. But B is abelian, so B^G is abelian. Then B_1^G is abelian, which is a contradiction. Therefore

$$Z \leqslant \Omega_k(A)B.$$

We note further that $\Omega_k(A) \nleqslant Z$, as otherwise $B_1^G = \Omega_k(A)B_1$ is abelian.

We let $A = \langle x \rangle$ and see, by Lemma 2, that $B^G = \langle B, B^x \rangle$. Now B is abelian, so $B \cap B^x \leqslant Z$. Since $B^G \leqslant \Omega_{k+1}(A)B$ and $\Omega_1(A) \leqslant B$, we apply Lemma 3 to see that $|B : B \cap B^x| \le p^k$. It follows that

$$|B : B \cap Z| \le p^k.$$

Now suppose that $\exp(B\Omega_1(Z)/\Omega_1(Z)) \le p^{k-1}$. Then, by induction, we see that $\Omega_{k-1}(A\Omega_1(Z)/\Omega_1(Z))B\Omega_1(Z)/\Omega_1(Z) \trianglelefteq G/\Omega_1(Z)$. But $\Omega_1(Z) \leqslant \Omega_k(A)B$ and $A \cap \Omega_1(Z) = \Omega_1(A)$. Hence, by Lemma 1, $\Omega_k(A)B \trianglelefteq G$.

We thus may assume that there exists $y \in B$ such that $y^{p^{k-1}} \notin \Omega_1(Z)$. Since $\exp(B) \le p^k$, it follows that $y^{p^{k-1}} \notin Z$. Thus $o(y) = p^k$ and $\langle y \rangle \cap (B \cap Z) = 1$. But $|B : B \cap Z| \le p^k$, so $B = \langle y \rangle (B \cap Z)$. Hence $BZ = \langle y \rangle Z$ and $BZ/Z \cong \langle y \rangle/(\langle y \rangle \cap Z) \cong \langle y \rangle \cong C_{p^k}$. Thus G/Z is the product of the non-trivial cyclic subgroups AZ/Z and BZ/Z.

Now G/Z is a finite p-group, so AZ/Z is normalised by a non-trivial subgroup of BZ/Z. Hence $\Omega_1(BZ/Z)$ normalises AZ/Z. But AZ/Z is cyclic and $\Omega_1(BZ/Z) \cong C_p$. Since p is odd we see, by considering the action of $\Omega_1(BZ/Z)$ on AZ/Z, that $\Omega_1(AZ/Z)\Omega_1(BZ/Z) \trianglelefteq \Omega_1(BZ/Z)AZ/Z$. We similarly have $\Omega_1(AZ/Z)\Omega_1(BZ/Z) \trianglelefteq \Omega_1(AZ/Z)BZ/Z$. Hence

$$\Omega_1(AZ/Z)\Omega_1(BZ/Z) \trianglelefteq G/Z.$$

In addition, since BZ is abelian, we have $A \cap BZ \leqslant Z(G) \cap BZ \leqslant Z$. It follows that $AZ/Z \cap BZ/Z = 1_{G/Z}$.

We let r be such that $\Omega_1(AZ/Z) = \Omega_r(A)Z/Z$. Since $\Omega_1(A) \leqslant Z$ and $\Omega_k(A) \nleqslant Z$, we have $2 \le r \le k$. We further let $y_1 = y^{p^{k-1}}$. Then $\langle y_1 \rangle = \Omega_1(\langle y \rangle)$ and $\Omega_1(BZ/Z) = \langle y_1 \rangle Z/Z$. From above, we then have

$$\Omega_r(A)Z\langle y_1 \rangle \trianglelefteq G.$$

But $\Omega_1(AZ/Z)$ and $\Omega_1(BZ/Z)$ both centralise each other and $AZ/Z \cap BZ/Z = 1_{G/Z}$, so

$$\Omega_r(A)Z\langle y_1 \rangle/Z = \Omega_r(A)Z/Z \times \langle y_1 \rangle Z/Z \cong C_p \times C_p.$$

Now $\Omega_r(A)Z\langle y_1 \rangle/Z$ is abelian, so $[\Omega_r(A), \langle y_1 \rangle] \leqslant Z$. Since $r \le k$, we have $\Omega_r(A)Z\langle y_1 \rangle \leqslant B^G$, so $Z \leqslant Z(\Omega_r(A)Z\langle y_1 \rangle)$. Hence, by Corollary 5, we have

$$\Omega_1(\Omega_r(A)Z\langle y_1 \rangle) = \Omega_1(\Omega_r(A)Z)\langle y_1 \rangle.$$

In addition, $\Omega_1(B)Z/Z \leqslant \Omega_1(BZ/Z) = \langle y_1 \rangle Z/Z$, so

$$\Omega_1(B) \leqslant \langle y_1 \rangle Z.$$

We let $N = \Omega_1(\Omega_r(A)Z\langle y_1 \rangle)$ and note that $\Omega_1(A) \leqslant N$. We let $\Omega_2(A) = \langle x_1 \rangle$, where $o(x_1) = p^2$. Now $\Omega_r(A)Z$ is abelian, so $\Omega_1(\Omega_r(A)Z)$ is elementary abelian. Hence $x_1 \notin \Omega_1(\Omega_r(A)Z)$. If $A \cap N \neq \Omega_1(A)$, then $x_1 \in N$. Thus there exist $g \in \Omega_1(\Omega_r(A)Z)$ and $1 \neq \widetilde{y} \in \langle y_1 \rangle$ such that $x_1 = g\widetilde{y}$. It follows that $\widetilde{y} = g^{-1}x_1 \in \Omega_1(\Omega_r(A)Z)\Omega_2(A) \leqslant \Omega_r(A)Z$. Since $\widetilde{y} \neq 1$, we have $\langle y_1 \rangle = \langle \widetilde{y} \rangle \leqslant \Omega_r(A)Z$, which is a contradiction since the order of $\Omega_r(A)Z\langle y_1 \rangle/Z$ is p^2.

We thus have $A \cap N = \Omega_1(A)$. Since $\Omega_r(A)Z\langle y_1\rangle \trianglelefteq G$, we have $N \trianglelefteq G$. From above, we have $\Omega_1(B) \leqslant \Omega_1(\langle y_1\rangle Z) \leqslant N$, so $\exp(BN/N) \leq p^{k-1}$. We once more apply induction to see that $\Omega_{k-1}(AN/N)BN/N \trianglelefteq G/N$. Noting that $\Omega_r(A)Z\langle y_1\rangle \leqslant \Omega_k(A)B$, a final application of Lemma 1 allows us to conclude that $\Omega_k(A)B \trianglelefteq G$. $\square$

Example 7. Letting p be an odd prime and $n > k \geq 1$, we let G be the semi-direct product of a cyclic group of order p^n by a cyclic group of order p^k as follows:

$$G = \langle x, y, \mid x^{p^n} = y^{p^k} = 1,\ x^y = x^{1+p^{n-k}}\rangle.$$

Then $G = AB$, where $A = \langle x\rangle \cong C_{p^n}$ and $B = \langle y\rangle \cong C_{p^k}$. This example shows that Theorem 6 is the best one can expect, in the sense that $B^G = \langle x^{p^{n-k}}, y\rangle = \Omega_k(A)B$, so $\Omega_s(A)B \ntrianglelefteq G$ for $s < k$.

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