Symmetry breaking operators for the restriction of representations of indefinite orthogonal groups $O(p,q)$

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Abstract: For the pair $(G, G') = (O(p+1, q+1), O(p, q+1))$, we construct and give a complete classification of intertwining operators (*symmetry breaking operators*) between most degenerate spherical principal series representations of G and those of the subgroup G', extending the work initiated by Kobayashi and Speh [Mem. Amer. Math. Soc. 2015] in the real rank one case where $q = 0$. Functional identities and residue formulæ of the regular symmetry breaking operators are also provided explicitly. The results contribute to Program C of branching problems suggested by the first author [Progr. Math. 2015].

Key words: Representation theory; reductive group; branching law; broken symmetry; conformal geometry; symmetry breaking operator.

1. Branching problem. Suppose $G \supset G'$ are reductive groups and π is an irreducible representation of G. The restriction of π to the subgroup G' is no more irreducible in general as a representation of G'. If G is compact, then any irreducible π is finite-dimensional and splits into a finite direct sum

$$\pi|_{G'} = \bigoplus_{\pi' \in \widehat{G'}} m(\pi, \pi')\pi'$$

of irreducibles π' of G' with multiplicities $m(\pi, \pi')$. These multiplicities have been studied by various techniques including combinatorial algorithms.

However, for noncompact G' and for infinite-dimensional π, the restriction $\pi|_{G'}$ is not always a direct sum of irreducible representations, see [5,6] for details. In order to define the "multiplicity" in this generality, we recall that, associated to a continuous representation π of a Lie group on a Banach space $\mathcal{H}$, a continuous representation π^∞ is defined on the Fréchet space $\mathcal{H}^\infty$ of C^∞-vectors of $\mathcal{H}$. Given another representation π' of a subgroup G', we consider the space of continuous G'-intertwining operators (*symmetry breaking operators*)

$$\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, (\pi')^\infty). \tag{1.1}$$

If both π and π' are admissible representations of finite length of reductive Lie groups G and G', respectively, then the dimension of the space (1.1) is determined by the underlying $(\mathfrak{g}, K)$-module π_K of π and the $(\mathfrak{g}', K')$-module $\pi'_{K'}$ of π', and is independent of the choice of Banach globalizations by the Casselman–Wallach theory [17, Chap. 11]. We denote by $m(\pi, \pi')$ the dimension of (1.1), and call it the *multiplicity* of π' in the restriction $\pi|_{G'}$.

The above definition of the multiplicity $m(\pi, \pi')$ makes sense for nonunitary representations, too.

In general, $m(\pi, \pi')$ may be infinite, even when G' is a maximal reductive subgroup of G (*e.g.* symmetric pairs). By using the theory of real spherical spaces [14], the geometric criterion for finite multiplicities was proved in [7] and [14] as follows.

Fact 1.1. *Let (G, G') be a pair of real reductive Lie groups with complexification $(G_{\mathbf{C}}, G'_{\mathbf{C}})$.*

(1) *The multiplicity $m(\pi, \pi')$ is finite for all irreducible representations π of G and all irreducible representations π' of G' if and only if a minimal parabolic subgroup of G' has an open orbit on the real flag variety of G.*
(2) *The multiplicity $m(\pi, \pi')$ is uniformly bounded if and only if a Borel subgroup of $G'_{\mathbf{C}}$ has an open orbit on the complex flag variety of $G_{\mathbf{C}}$.*

The complete classification of symmetric pairs (G, G') satisfying the above geometric criteria was accomplished in Kobayashi–Matsuki [11].

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On the other hand, switching the order in (1.1), we may also consider another space

$$\mathrm{Hom}_{G'}((\pi')^\infty, \pi^\infty|_{G'}) \text{ or } \mathrm{Hom}_{\mathfrak{g}',K'}(\pi'_{K'}, \pi_K|_{\mathfrak{g}',K'}).$$

The study of these objects is closely related to the theory of discretely decomposable restrictions [5,6].

Notation. We adopt the same convention as in [16] for the following notation. $\mathbf{N} := \{0, 1, 2, \cdots\}$. $(x)_j := x(x+1)\cdots(x+j-1)$. For two subsets A and B of a set, we write $A - B := \{a \in A : a \notin B\}$ rather than the usual notation $A \setminus B$. The symbols $//$, $\backslash\backslash$, $||$, and $|||$ are defined to be subsets of $\mathbf{C}^2$, and are not binary relations.

2. $\mathcal{ABC}$ program for branching. In [8] the first author suggested a program for studying the restriction of representations of reductive groups, which may be summarized as follows:

($\mathcal{A}$) Abstract features of the restriction;
($\mathcal{B}$) Branching law of $\pi|_{G'}$;
($\mathcal{C}$) Construction of symmetry breaking operators.

Program $\mathcal{A}$ aims for establishing the general theory of the restrictions $\pi|_{G'}$ (*e.g.* spectrum, multiplicity), which would single out the *good* triples (G, G', π). In turn, we could expect concrete and detailed study of those restrictions $\pi|_{G'}$ in Programs $\mathcal{B}$ and $\mathcal{C}$.

The current work concerns Program $\mathcal{C}$ for certain standard representations with focus on symmetry breaking operators (SBOs for short) as follows:

($\mathcal{C}1$) Construct SBOs explicitly;
($\mathcal{C}2$) Classify all SBOs;
($\mathcal{C}3$) Find residue formulæ for SBOs;
($\mathcal{C}4$) Study functional equations among SBOs;
($\mathcal{C}5$) Determine the images of subquotients by SBOs.

The subprogram ($\mathcal{C}1$)–($\mathcal{C}5$) was proposed by Kobayashi–Speh in their book [16] with a complete answer for the pair $(G, G') = (O(n+1,1), O(n,1))$ of real rank one groups.

In this note we treat degenerate spherical principal series representations $\pi = I(\lambda)$ of G and $\pi' = J(\nu)$ of G' for the pair of higher real rank groups

$$(G, G') = (O(p+1, q+1), O(p, q+1)), \tag{2.1}$$

and give an answer to ($\mathcal{C}1$)–($\mathcal{C}4$). The subprogram ($\mathcal{C}5$) will be discussed in a separate paper.

Concerning Program $\mathcal{A}$, Fact 1.1 assures the following *a priori* estimate:

$$m(\pi, \pi') \text{ is uniformly bounded}$$

if the pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ is a real form of $(\mathfrak{sl}(n+1, \mathbf{C}), \mathfrak{gl}(n, \mathbf{C}))$ or $(\mathfrak{o}(n+1, \mathbf{C}), \mathfrak{o}(n, \mathbf{C}))$, in particular, if (G, G') is of the form (2.1).

3. Settings. Let $G = O(p+1, q+1)$ be the automorphism group of the quadratic form on $\mathbf{R}^{p+q+2}$ of signature $(p+1, q+1)$ defined by

$$Q_{p+1,q+1}(x) = x_0^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2.$$

A degenerate spherical principal series representation $I(\lambda) := \mathrm{Ind}_P^G(\mathbf{C}_\lambda)$ with parameter $\lambda \in \mathbf{C}$ of G is induced from a character $\mathbf{C}_\lambda$ of a maximal parabolic subgroup $P = MAN_+$ with Levi part $MA \simeq O(p,q) \times \{\pm 1\} \times \mathbf{R}$. We realize $I(\lambda)$ on the space of C^∞ sections of the G-equivariant line bundle

$$\mathcal{L}_\lambda = G \times_P \mathbf{C}_\lambda \to G/P$$

so that $I(\lambda)$ itself is the smooth Fréchet globalization of moderate growth. Our parametrization is chosen in a way that $I(\lambda)$ contains a finite-dimensional submodule if $-\lambda \in 2\mathbf{N}$ and a finite-dimensional quotient if $\lambda - (p+q) \in 2\mathbf{N}$ (*cf.* [3]).

Let $G' = O(p, q+1)$ be the stabilizer of the basis element e_p. Similarly to $I(\lambda)$, we denote by $J(\nu) := \mathrm{Ind}_{P'}^{G'}(\mathbf{C}_\nu)$ the representation of G' induced from a character $\mathbf{C}_\nu$ of a maximal parabolic subgroup P' of G' with Levi part $O(p-1, q) \times \{\pm 1\} \times \mathbf{R}$.

The representation $I(\lambda)$ arises from conformal geometry as follows. We endow the direct product manifold $\mathbf{S}^p \times \mathbf{S}^q$ with the pseudo-Riemannian structure $g_{\mathbf{S}^p} \oplus (-g_{\mathbf{S}^q})$ of signature (p, q). Then the group $G = O(p+1, q+1)$ acts as conformal diffeomorphisms on $\mathbf{S}^p \times \mathbf{S}^q$, and also on its quotient space $X = (\mathbf{S}^p \times \mathbf{S}^q)/\mathbf{Z}_2$ by identifying the direct product of antipodal points. By the general theory of conformal groups, one has a natural family of representations ϖ_λ on $C^\infty(X)$ with parameter $\lambda \in \mathbf{C}$ [12, Sect. 2]. Then X identifies with G/P, and ϖ_λ identifies with $I(\lambda)$. Thus the branching problem in our setting arises from the conformal construction of representations for the pair

$$(X, Y) = ((\mathbf{S}^p \times \mathbf{S}^q)/\mathbf{Z}_2, (\mathbf{S}^{p-1} \times \mathbf{S}^q)/\mathbf{Z}_2).$$

4. Multiplicity formulæ. In this section we determine explicitly the multiplicity

$$m(I(\lambda), J(\nu)) = \dim \mathrm{Hom}_{G'}\big(I(\lambda)|_{G'}, J(\nu)\big).$$

We shall find $m(I(\lambda), J(\nu)) > 0$ for all $\lambda, \nu \in \mathbf{C}$. Following [16], we define four subsets of $\mathbf{C}^2$ as below:

$$|||:=\{(\lambda,\nu)\in\mathbf{C}^2 \mid \nu\in -2\mathbf{N}\cup(q+1+2\mathbf{Z})\},$$
$$\backslash\backslash:=\{(\lambda,\nu)\in\mathbf{C}^2 \mid n-1-\lambda-\nu\in 2\mathbf{N}\},$$
$$//:=\{(\lambda,\nu)\in\mathbf{C}^2 \mid \nu-\lambda\in 2\mathbf{N}\},$$
$$||:=\{(\lambda,\nu)\in\mathbf{C}^2 \mid \nu\in 1+2\mathbf{N}\},$$

and two subsets of $\mathbf{Z}^2$ by

$$\mathcal{A}:=//\cap||| \text{ and } \mathcal{X}:=||\cap\backslash\backslash.$$

Theorem 4.1. *Let (G,G') be as in (2.1) with $p,q\geq 1$. Then*

$$m(I(\lambda),J(\nu))\in\{1,2\}$$

for all $\lambda,\nu\in\mathbf{C}$. Furthermore, $m(I(\lambda),J(\nu))=2$ if and only if one of the following conditions holds:

Case 1. $p>1$. $(\lambda,\nu)\in\mathcal{A}$.

Case 2. $p=1$ *and q is odd.* $(\lambda,\nu)\in\mathcal{A}\cup\mathcal{X}$.

Case 3. $p=1$ *and q is even.* $(\lambda,\nu)\in\mathcal{A}\cup\mathcal{X}-\mathcal{X}\cap//$.

We shall construct explicitly all the symmetry breaking operators in Section 6.

5. Double coset space $P'\backslash G/P$. In general, as is seen in Fact 1.1 (and Fact 6.2 below), the double coset space $P'\backslash G/P$ plays a fundamental role in analyzing symmetry breaking operators

$$\mathrm{Ind}_P^G(\sigma)\to\mathrm{Ind}_{P'}^{G'}(\tau),$$

where σ is a representation of a parabolic subgroup P of G and τ is that of a parabolic subgroup P' of G'. The description of the double coset space $P'\backslash G/P$ is nothing but the Bruhat decomposition if $G'=G$; the Iwasawa decomposition if G' is a maximal compact subgroup K of G where P' automatically equals K.

In this section we give a description of $P'\backslash G/P$ together with its closure relation in the setting where (G,G',P,P') is given as in Section 3. Then the natural action of $G=O(p+1,q+1)$ on $\mathbf{R}^{p+q+2}$ preserves the isotropic cone

$$\Xi:=\{x\in\mathbf{R}^{p+q+2}-\{0\}\mid Q_{p+1,q+1}(x)=0\},$$

inducing the G-action on its quotient space

$$X:=\Xi/\mathbf{R}^\times\simeq(\mathbf{S}^p\times\mathbf{S}^q)/\mathbf{Z}_2.$$

We define the subvarieties of X by

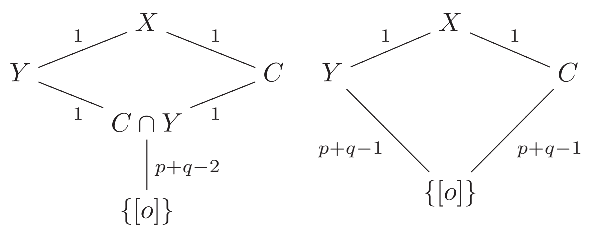
$$Y:=\{[x]\in X\mid x_p=0\},$$
$$C:=\{[x]\in X\mid x_0=x_{p+q+1}\}.$$

Let P be the stabilizer of the point

$$o:=[1:0:\cdots:0:1]\in X\simeq\Xi/\mathbf{R}^\times,$$

and $P':=P\cap G'$. Then X and Y are identified with the real flag varieties G/P and G'/P', respectively.

Theorem 5.1 (description of $P'\backslash G/P$)**.** *Suppose $p,q\geq 1$. The left P'-invariant closed subsets of G/P are described in the following Hasse diagram. Here $\overset{A}{\underset{B}{|_m}}$ means that $A\supset B$ and that the subvariety B is of codimension m in A.*



(a) when $p>1$ (b) when $p=1$

6. Construction of SBOs. Let $n:=p+q$. The slice of Ξ by the hyperplane $x_0+x_{p+q+1}=2$ defines the coordinates $(x_1,\ldots,x_n)\in\mathbf{R}^n$ of the open Bruhat cell U of G/P, and induces the N-picture of the representation $I(\lambda)$, $\iota_\lambda^*:I(\lambda)\hookrightarrow C^\infty(\mathbf{R}^n)$ via the trivialization $\mathcal{L}_\lambda|_U\simeq\mathbf{R}^n\times\mathbf{C}$. Likewise, $x'=(x_1,\ldots,\widehat{x_p},\ldots,x_n)\in\mathbf{R}^{n-1}$ give the coordinates of the Bruhat cell of G'/P', and we have the N-picture $\iota_\nu^*:J(\nu)\hookrightarrow C^\infty(\mathbf{R}^{n-1})$.

We shall realize a symmetry breaking operator T in the N-pictures of $I(\lambda)$ and $J(\nu)$, and find a distribution $K_T\in\mathcal{D}'(\mathbf{R}^n)$ such that for all $f\in I(\lambda)$

$$\iota_\nu^*(Tf)(x')=\mathrm{Rest}_{x_p=0}\circ\int_{\mathbf{R}^n}K_T(x-y)(\iota_\lambda^*f)(y)dy.$$

In order to analyze the distribution kernels K_T of symmetry breaking operators T, we begin with:

Definition 6.1. We let $O(p-1,q)$ act on $\mathbf{R}^n$ $(n=p+q)$ by leaving x_p invariant. We define $\mathcal{S}ol(\mathbf{R}^{p,q};\lambda,\nu)$ to be the space of distributions $F\in\mathcal{D}'(\mathbf{R}^n)$ satisfying the following three conditions:

(1) F is $O(p-1,q)$-invariant and $F(x)=F(-x)$;

(2) F is homogeneous of degree $\lambda-\nu-n$;

(3) F is invariant by $N'_+:=N_+\cap G'$.

Applying the general results proven in [16, Chap. 3] to our particular setting, we get the following.

Fact 6.2 ([16, Thm. 3.16])**.** *Recall $n=p+q$ $(p,q\geq 1)$. Then the following diagram commutes*:

$$\begin{array}{ccc} \operatorname{Hom}_{G'}\left(I(\lambda)|_{G'}, J(\nu)\right) & \xrightarrow{\simeq} & \left(\mathcal{D}'(G/P, \mathcal{L}_{n-\lambda}) \otimes \mathbf{C}_\nu\right)^{P'} \\ {\scriptstyle Op}\uparrow{\scriptstyle \simeq} & \swarrow_{\mathrm{Rest}}^{\simeq} & \\ \mathcal{S}ol(\mathbf{R}^{p,q}; \lambda, \nu) \subset \mathcal{D}'(\mathbf{R}^n) & & \end{array}$$

For $T \in \operatorname{Hom}_{G'}(I(\lambda)|_{G'}, J(\nu))$, a closed P'-invariant subset $\mathcal{S}upp(T)$ in $X = G/P$ is defined to be the support of the distribution kernel $K_T \in (\mathcal{D}'(G/P, \mathcal{L}_{n-\lambda}) \otimes \mathbf{C}_\nu)^{P'}$. By [15, Lem. 2.22], T is a *differential* symmetry breaking operator if and only if $\mathcal{S}upp(T)$ is a singleton.

Conversely, for each P'-invariant closed subset $S = \{o\}, C, Y$ or X itself, we define a subset D_S of $\mathbf{C}^2$ which is either the whole $\mathbf{C}^2$ or a countable union of one-dimensional complex affine spaces, and construct a family of SBOs, $R^S_{\lambda,\nu}\colon I(\lambda) \to J(\nu)$, such that

- $R^S_{\lambda,\nu}$ depends holomorphically on $(\lambda, \nu) \in D_S$;
- $\mathcal{S}upp(R^S_{\lambda,\nu}) \subset S$ for every $(\lambda, \nu) \in D_S$, and the equality holds for generic points in D_S.

The distribution kernels $K^S_{\lambda,\nu}$ of the operators $R^S_{\lambda,\nu}$ will be given explicitly in Theorems 6.3–6.6 and Fact 6.7. The relations among them are discussed in Section 8 as "residue formulæ". The space of SBOs is generated by these operators, as we shall see the classification results in Theorem 6.9.

Here is a summary of the symmetry breaking operators that we construct below.

$R^S_{\lambda,\nu} = \operatorname{Op}(K^S_{\lambda,\nu})$	D_S	
$R^X_{\lambda,\nu} = \operatorname{Op}(K^X_{\lambda,\nu})$	$\mathbf{C}^2$	Theorem 6.3
$\tilde{R}^X_{\lambda,\nu} = \operatorname{Op}(\tilde{K}^X_{\lambda,\nu})$	$\vert\vert\vert$	Theorem 6.4
$R^Y_{\lambda,\nu} = \operatorname{Op}(K^Y_{\lambda,\nu})$	$\backslash\backslash$	Theorem 6.5
$R^C_{\lambda,\nu} = \operatorname{Op}(K^C_{\lambda,\nu})$	$\vert\vert$	Theorem 6.6
$R^{\{o\}}_{\lambda,\nu} = \operatorname{Op}(K^{\{o\}}_{\lambda,\nu})$	$//$	Fact 6.7

Theorem 6.3 (regular symmetry breaking operator)**.** *Suppose $n = p + q$ with $p, q \geq 1$.*

(1) *There exists a family of symmetry breaking operators $R^X_{\lambda,\nu} \in \operatorname{Hom}_{G'}(I(\lambda)|_{G'}, J(\nu))$ that depends holomorphically on (λ, ν) in the entire $\mathbf{C}^2$ with the distribution kernel $K^X_{\lambda,\nu}(x)$ given by*

$$\frac{1}{\Gamma\left(\frac{\lambda-\nu}{2}\right)\Gamma\left(\frac{\lambda+\nu-n+1}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)} |x_p|^{\lambda+\nu-n} |Q_{p,q}|^{-\nu}.$$

(2) *$R^X_{\lambda,\nu}$ vanishes if and only if (λ, ν) belongs to the discrete set $\mathcal{A}$ for $p > 1$, $\mathcal{A} \cup \mathcal{X}$ for $p = 1, q$ odd and $\mathcal{A} \cup \mathcal{X} - \mathcal{X} \cap //$ for $p = 1, q$ even.*

(3) *$\mathcal{S}upp(R^X_{\lambda,\nu}) \subset Y, C$ or $\{o\}$ if $(\lambda, \nu) \in \backslash\backslash, ||$ or $//$, respectively, and $\mathcal{S}upp(R^X_{\lambda,\nu}) = X$ otherwise.*

The above normalization of $R^X_{\lambda,\nu}$ is optimal in the sense that the zeros of $R^X_{\lambda,\nu}$ form a subset of codimension two in $\mathbf{C}^2$. Next, we renormalize $R^X_{\lambda,\nu}$ in the places where $R^X_{\lambda,\nu}$ vanishes. For this, we observe that $\Gamma(\frac{\lambda-\nu}{2})$ is holomorphic in $\mathbf{C}^2 - //$, and therefore

$$\tilde{K}^X_{\lambda,\nu} := \Gamma\left(\frac{\lambda-\nu}{2}\right) K^X_{\lambda,\nu} = \frac{|x_p|^{\lambda+\nu-n} |Q_{p,q}|^{-\nu}}{\Gamma\left(\frac{\lambda+\nu-n+1}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}$$

makes sense if $(\lambda, \nu) \in \mathbf{C}^2 - //$. Moreover, in light of the fact that $K^X_{\lambda,\nu}$ vanishes on $\mathcal{A} = ||| \cap //$, we obtain its analytic continuation on $|||$ as follows.

Theorem 6.4 (renormalized operator $\tilde{R}^X_{\lambda,\nu}$)**.**

(1) *The renormalized symmetry breaking operator*

$$\tilde{R}^X_{\lambda,\nu} := \operatorname{Op}(\tilde{K}^X_{\lambda,\nu}) \in \operatorname{Hom}_{G'}\left(I(\lambda)|_{G'}, J(\nu)\right)$$

is defined for $(\lambda, \nu) \in |||$ that depends holomorphically on λ in the entire $\mathbf{C}$ for each fixed ν.

(2) *$\tilde{R}^X_{\lambda,\nu}$ vanishes if and only if $p = 1$, q even and $(\lambda, \nu) \in \mathcal{X} - //$.*

Let $N\colon \mathbf{R} \to \mathbf{Z}$ be a discontinuous function defined by $N(x) := x$ if $x \in \mathbf{N}$; $= 0$ otherwise.

Associated to closed subsets Y and C in $P' \backslash G/P$ we introduce families of *singular* SBOs. For later purpose, we discuss only the case $p = 1$.

Theorem 6.5 (singular symmetry breaking operators $R^Y_{\lambda,\nu}$)**.** *Suppose $p = 1$ and $q \geq 1$. For $(\lambda, \nu) \in \backslash\backslash$, we fix $k := \frac{1}{2}(q - \lambda - \nu) \in \mathbf{N}$. Then there exists a family of symmetry breaking operators $R^Y_{\lambda,\nu}$ that depends holomorphically on ν in the entire plane $\mathbf{C}$ with the distribution kernel $K^Y_{\lambda,\nu}$ given by*

$$\frac{1}{\Gamma\left(\frac{\lambda-\nu}{2} + N\left(k - \frac{q}{2}\right)\right)} \delta^{(2k)}(x_p) |Q_{p,q}|^{-\nu}.$$

Theorem 6.6 (singular symmetry breaking operators $R^C_{\lambda,\nu}$)**.** *Suppose $p = 1$ and $q \geq 1$. For $(\lambda, \nu) \in ||$, we fix $m := \frac{1}{2}(\nu - 1) \in \mathbf{N}$. Then there exists a family of symmetry breaking operators $R^C_{\lambda,\nu}$ that depends holomorphically on λ in the entire plane $\mathbf{C}$ with the distribution kernel $K^C_{\lambda,\nu}$ given by*

$$\frac{1}{\Gamma\left(\frac{\lambda-\nu}{2} + N\left(\nu - \frac{q}{2}\right)\right)} |x_p|^{\lambda+\nu-n} \delta^{(2m)}(Q_{p,q}).$$

The differential symmetry breaking operators $R^{\{o\}}_{\lambda,\nu}\colon C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^{n-1})$ were previously found in [4, Thms. 5.1.1 and 5.2.1] for $q = 0$ and in [13, Thm. 4.3] for general p, q by a different approach. See also [9,10] for further generalization.

Fact 6.7. *Suppose $(\lambda, \nu) \in //$. We set $l :=$*

$\frac{1}{2}(\nu-\lambda) \in \mathbf{N}$. *We define* $R^{\{o\}}_{\lambda,\nu}$ *by*

$$\mathrm{Rest}_{x_p=0} \circ \sum_{j=0}^{l} a_j(\lambda,\nu)(-\Delta_{\mathbf{R}^{p-1,q}})^j \left(\frac{\partial}{\partial x_p}\right)^{2l-2j}$$

where $a_j(\lambda,\nu)$ *is given by*

$$a_j(\lambda,\nu) = \frac{(-1)^j 2^{2l-2j}}{j!(2l-2j)!} \prod_{i=1}^{l-j} \left(\frac{\lambda+\nu-n-1}{2} + i\right).$$

Then $R^{\{o\}}_{\lambda,\nu} \in \mathrm{Hom}_{G'}(I(\lambda)|_{G'}, J(\nu))$. *The coefficients* $a_j(\lambda,\nu)$ *give rise to a Gegenbauer polynomial*

$$\tilde{C}_{2l}^{\lambda+\frac{n-1}{2}}(t) = \sum_{j=0}^{l} a_j(\lambda,\nu) t^{2l-2j}$$

renormalized as $\tilde{C}_{2l}^{\lambda+\frac{n-1}{2}}(0) = (-1)^l/l!$.

Its distribution kernel is given by

$$K^{\{o\}}_{\lambda,\nu} := \sum_{j=0}^{l} a_j(\lambda,\nu)(-\Delta_{\mathbf{R}^{p-1,q}})^j \delta_{\mathbf{R}^{p+q-1}} \delta^{(2l-2j)}(x_p).$$

Remark 6.8. The operators $R^Y_{\lambda,\nu}$, $R^C_{\lambda,\nu}$ and $R^{\{o\}}_{\lambda,\nu}$ do not vanish.

The SBOs are not always linearly independent, but exhaust all SBOs. We provide explicit basis for $\mathrm{Hom}_{G'}(I(\lambda)|_{G'}, J(\nu))$ for every $(\lambda,\nu) \in \mathbf{C}^2$:

Theorem 6.9 (classification of SBOs). *The vector space* $\mathrm{Hom}_{G'}(I(\lambda)|_{G'}, J(\nu))$ *is spanned by the operators as below.*

(1) *Suppose* $p = 1$ *and* $q \geq 1$.

$$\begin{cases} R^X_{\lambda,\nu}, & \text{if } (\lambda,\nu) \notin \mathcal{A} \cup \mathcal{X}, \\ \tilde{R}^X_{\lambda,\nu}, R^{\{o\}}_{\lambda,\nu}, & \text{if } (\lambda,\nu) \in \mathcal{A} - \mathcal{X}, \\ R^Y_{\lambda,\nu}, R^C_{\lambda,\nu}, & \text{if } (\lambda,\nu) \in \mathcal{X} - //, \\ R^{\{o\}}_{\lambda,\nu}, & \text{if } (\lambda,\nu) \in || \cap \backslash\backslash \cap //. \end{cases}$$

(2) *Suppose* $p \geq 2$ *and* $q \geq 1$.

$$\begin{cases} \tilde{R}^X_{\lambda,\nu}, R^{\{o\}}_{\lambda,\nu}, & \text{if } (\lambda,\nu) \in \mathcal{A}, \\ R^X_{\lambda,\nu}, & \text{otherwise}. \end{cases}$$

7. Spectrum of SBOs. The representation $I(\lambda)$ of G contains a one-dimensional subspace of spherical vectors (*i.e.* K-fixed vectors), and likewise $J(\nu)$ of G'. Let $\mathbf{1}_\lambda \in I(\lambda), \mathbf{1}_\nu \in J(\nu)$ be the spherical vectors normalized by $\mathbf{1}_\lambda(e) = \mathbf{1}_\nu(e) = 1$. With this normalization, we have:

Theorem 7.1 (spectrum for spherical vectors). *Let* $n = p + q$ $(p, q \geq 1)$ *as before.*

$$R^X_{\lambda,\nu} \mathbf{1}_\lambda = \frac{2^{1-\lambda}\pi^{n/2}}{\Gamma(\frac{\lambda}{2})\Gamma(\frac{\lambda+1-q}{2})\Gamma(\frac{q-\nu+1}{2})} \mathbf{1}_\nu.$$

Remark 7.2. Theorem 7.1 was known in Bernstein–Reznikov [1] for $p = q = 1$ and in [16, Prop. 7.4] for $q = 0$. Another generalization was given in [2, Thm. 1.1] for higher dimensional cases.

8. Residue formulæ of SBOs. The regular symmetry breaking operators $R^X_{\lambda,\nu}$ have two complex parameters $(\lambda,\nu) \in \mathbf{C}^2$, whereas the singular operators $R^Y_{\lambda,\nu}$, $R^C_{\lambda,\nu}$, and $R^{\{o\}}_{\lambda,\nu}$ are defined for $(\lambda,\nu) \in \backslash\backslash$, $||$ and $//$, respectively. We find the relationship among these operators as explicit residue formulæ.

Proposition 8.1. *Suppose* $p = 1$.

(1) *For* $(\lambda,\nu) \in \backslash\backslash$, *we set* $k = \frac{1}{2}(q-\lambda-\nu) \in \mathbf{N}$. *Then*

$$R^X_{\lambda,\nu} = \frac{(-1)^k k!}{(2k)!} \frac{\left(\frac{\lambda-\nu}{2}\right)_{N(k-\frac{q}{2})}}{\Gamma\left(\frac{1-\nu}{2}\right)} R^Y_{\lambda,\nu} \ \text{if } (\lambda,\nu) \in \backslash\backslash.$$

(2) *For* $(\lambda,\nu) \in ||$, *we set* $m := \frac{1}{2}(\nu - 1) \in \mathbf{N}$. *Then*

$$R^X_{\lambda,\nu} = \frac{(-1)^m m!}{(2m)!} \frac{\left(\frac{\lambda-\nu}{2}\right)_{N(\nu-\frac{q}{2})}}{\Gamma\left(\frac{\lambda+\nu-n+1}{2}\right)} R^C_{\lambda,\nu} \ \text{if } (\lambda,\nu) \in ||.$$

Theorem 8.2 (residue formula). *Let* $n = p + q$ $(p, q \geq 1)$. *For* $(\lambda,\nu) \in //$, *we set* $l := \frac{1}{2}(\nu - \lambda) \in \mathbf{N}$. *Then we have for* $(\lambda,\nu) \in //$

$$R^X_{\lambda,\nu} = \frac{(-1)^l l! \pi^{(n-2)/2}}{2^{\nu+2l-1}} \cdot \frac{\sin\left(\frac{1+q-\nu}{2}\pi\right)}{\Gamma\left(\frac{\nu}{2}\right)} R^{\{o\}}_{\lambda,\nu}.$$

Proposition 8.1 treats easier cases as the subvarieties Y and C are of codimension one in X (see Theorem 5.1), whereas Theorem 8.2 is more involved.

Remark 8.3. The residue formula in the case $q = 0$ was given in [16, Thm. 12.2].

9. Functional identities among SBOs. Let $n := p + q$ as before. We recall that there exist nonzero Knapp–Stein intertwining operators

$$\tilde{\mathbf{T}}^G_\lambda : I(\lambda) \to I(n-\lambda)$$

with holomorphic parameter $\lambda \in \mathbf{C}$ by the distribution kernel in the N-picture normalized as follows:

$$\frac{1}{\Gamma\left(\frac{\lambda-n+1}{2}\right)\Gamma\left(\frac{2\lambda-n+2}{4}\right)\Gamma\left(\frac{2\lambda-n}{4}\right)} \cdot |Q_{p,q}|^{\lambda-n}$$

$$\times \begin{cases} \Gamma\left(\dfrac{\lambda-n+2}{2}\right), & \text{if } \min(p,q) = 0, \\ 1, & \text{if } p, q > 0, p \not\equiv q \bmod 2 \\ \Gamma\left(\dfrac{2\lambda-n}{4}\right), & \text{if } p, q > 0,\ p - q \equiv 2 \bmod 4 \\ \Gamma\left(\dfrac{2\lambda-n+2}{4}\right), & \text{if } p, q > 0,\ p - q \equiv 0 \bmod 4 \end{cases}$$

Similarly, we write $\tilde{\mathbf{T}}_\nu^{G'} : J(\nu) \to J(n-1-\nu)$ for the Knapp–Stein intertwining operator for G'.

Theorem 9.1 (functional identities)**.**

$$\tilde{\mathbf{T}}^{G'}_{n-1-\nu} \circ R^X_{\lambda,n-1-\nu} = \frac{\pi^{\frac{n-3}{2}} \sin\left(\frac{p-\nu}{2}\pi\right)}{\Gamma\left(\frac{n-1-\nu}{2}\right)} a(\lambda,\nu) R^X_{\lambda,\nu},$$

$$R^X_{n-\lambda,\nu} \circ \tilde{\mathbf{T}}^G_\lambda = \frac{\pi^{-\frac{n}{2}-1} \sin\left(\frac{p-\lambda+1}{2}\pi\right)}{2^{n-2\lambda}\Gamma\left(\frac{n-\lambda}{2}\right)} b(\lambda,\nu) R^X_{\lambda,\nu},$$

for any $\lambda, \nu \in \mathbf{C}$, *where*

$$a(\lambda,\nu) = \begin{cases} 2^{\frac{1-n}{2}}\Gamma\left(\frac{1-\nu}{2}\right), & \text{if } p = 1, \\ 2^{\frac{1-n}{2}}, & \text{if } p > 1,\ p \equiv q \bmod 2, \\ \Gamma\left(\frac{n-2\nu}{2}\right), & \text{if } p > 1,\ p - q \equiv 1 \bmod 4, \\ \Gamma\left(\frac{n-2\nu-2}{4}\right), & \text{if } p > 1,\ p - q \equiv 3 \bmod 4, \end{cases}$$

$$b(\lambda,\nu) = \begin{cases} 2^{-\frac{n}{2}}, & \text{if } p \equiv q + 1 \bmod 2, \\ \Gamma\left(\frac{2\lambda-n+2}{4}\right), & \text{if } p - q \equiv 0 \bmod 4, \\ \Gamma\left(\frac{2\lambda-n}{4}\right), & \text{if } p - q \equiv 2 \bmod 4. \end{cases}$$

Remark 9.2. The functional identities in the case $q = 0$ were proven in [8, Thm. 12.6].

We have given all the constants in this note as *multiplicative formulæ* so that we can tell the zeros explicitly. Their representation-theoretic interpretation serves as a clue in the subprogram ($\mathcal{C}$5).

A detailed proof will appear elsewhere.

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