Twisted Alexander invariants and hyperbolic volume

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Abstract: We give a volume formula of hyperbolic knot complements using twisted Alexander invariants.

Key words: Twisted Alexander polynomial; hyperbolic knot; volume.

1. Introduction. The purpose of this note is to give a formula of the hyperbolic volume of a knot complement using twisted Alexander invariants.

A twisted Alexander polynomial was first defined in [3] for knots in the 3-sphere, and Wada ([10]) generalized this work and showed how to define a twisted Alexander polynomial given only a presentation of a group and representations to \(\mathbb{Z}\) and \(\text{GL}(V)\) where \(V\) is a finite dimensional vector space over a field. In [2], Kitano proved that in the case of knot groups the twisted Alexander polynomial can be regarded as a Reidemeister torsion.

Let \(M\) be a compact and oriented 3-manifold whose interior admits a finite volume hyperbolic structure. Porti ([8]) has investigated the case of knot groups the twisted Alexander polynomial given only a presentation of a group and representations to \(\mathbb{Z}\) and \(\text{GL}(V)\) where \(V\) is a finite dimensional vector space over a field. In [2], Kitano proved that in the case of knot groups the twisted Alexander polynomial can be regarded as a Reidemeister torsion.

Let \(\pi_1(M)\) be a knot in the 3-sphere, and Wada’s notation ([10]). For the integer \(k\), we have a formula of the hyperbolic volume of a knot complement using twisted Alexander invariants. Let \(\Delta_{K,\rho}(t)\) be the twisted Alexander invariant of Wada’s notation ([10]). For the integer \(k(>1)\), set \(\Lambda_{K,2k}(t) := \frac{\Delta_{K,2k}(t)}{\Delta_{K,2k-1}(t)}\) and \(\Lambda_{K,2k+1}(t) := \frac{\Delta_{K,2k+1}(t)}{\Delta_{K,2k}(t)}\).

Theorem 1.1. Let \(K\) be a hyperbolic knot in the 3-sphere. Then

\[
\lim_{k \to \infty} \frac{\log |\Lambda_{K,2k+1}(1)|}{(2k+1)^2} = \lim_{k \to \infty} \frac{\log |\Lambda_{K,2k}(1)|}{(2k)^2} = \frac{\text{Vol}(K)}{4\pi}.
\]

In the last section, we give some calculations for the figure eight knot. The details, including link case, will be given elsewhere.

2. Reidemeister torsions and twisted Alexander invariants. Following [9] and [13], we review some definitions and conventions in this section.

Let \(F\) be a field and \(C_* = (C_*, \partial)\) a chain complex of finite dimensional \(F\)-vector spaces:

\[
0 \to C_d \to C_{d-1} \to \cdots \to C_0 \to 0.
\]

For each \(i\), we denote by \(B_i = \text{Im}(C_{i+1} \to C_i)\), \(Z_i = \ker(C_i \to C_{i-1})\), and the homology is denoted by \(H_i = Z_i / B_i\). By the definition of \(Z_i\) and \(B_i\) and \(H_i\), we obtain the following exact sequence:

\[
0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0,
\]

\[
0 \to B_i \to Z_i \to H_i \to 0.
\]
Let \( \tilde{B}_{i-1} \) be a lift of \( B_{i-1} \) to \( C_i \), and \( \tilde{H}_i \) a lift of \( H_i \) to \( Z_i \). Then we can decompose \( C_i \) as follows:
\[
C_i = Z_i \oplus \tilde{B}_{i-1} = B_{i-1} \oplus \tilde{H}_i \oplus \tilde{B}_{i-1}.
\]
Let \( c^i \) be a basis for \( C_i \) and \( c \) the collection \( \{c^i\}_{i \geq 0} \). Similarly, let \( h^i \) be a basis for \( H_i \), if nonzero, and \( h \) the collection \( \{h^i\}_{i \geq 0} \). We choose \( b^i \) a basis of \( B_i \). Let \( \tilde{b}^{i-1} \) be a lift of \( b^{i-1} \) to \( C_i \), and \( \tilde{h}^i \) a lift of \( h^i \) to \( Z_i \), then we have a new basis \( b^i \sqcup \tilde{b}^{i-1} \sqcup \tilde{h}^i \) of \( C_i \), where \( \sqcup \) means a disjoint union. We denote by \([b^i, \tilde{b}^{i-1}, \tilde{h}^i/c^i] \) the determinant of the transformation matrix from the basis \( c^i \) to \( b^i \sqcup \tilde{b}^{i-1} \sqcup \tilde{h}^i \).

**Definition 2.1.** The torsion of the chain complex \( C_* \) with basis \( c \) and \( h \) for \( H_i \) is:
\[
\text{tor}(C_*, c, h) = \prod_{i=0}^d [b^i, \tilde{b}^{i-1}, \tilde{h}^i/c^i]^{-1} \in F^*/\{\pm 1\}.
\]

It is known that \( \text{tor}(C_*, c, h) \) is independent of the choice of \( b^i \) and the lifts \( b^{i-1} \) and \( \tilde{h}^i \).

**Remark 2.2.** In [5], Menal-Ferrer and Porti use \((-1)^i\) instead of \((-1)^{i+1}\) in Definition 2.1. Then the sign of the right-hand side of the equation in Theorem 7.1 in [5] becomes opposite. See Remark 2.2 and Theorem 4.5 in [9].

Let \( W \) be a finite CW-complex, and \( \rho : \pi_1(W, *) \to \text{SL}(n, F) \) a representation of its fundamental group. Consider the chain complex of vector spaces
\[
C_*(W, \rho) := F^n \otimes \rho C_*(\tilde{W}; Z)
\]
where \( C_*(\tilde{W}; Z) \) denotes the simplicial complex of the universal covering of \( W \) and \( \otimes \rho \) means that one takes the quotient of \( F^n \otimes Z \) by \( Z \)-module generated by
\[
\rho(\gamma)^{-1} v \otimes c - v \otimes \gamma \cdot c.
\]
Here, \( v \in F^n \) and \( c \in C_*(\tilde{W}; Z) \).

Namely,
\[
v \otimes \gamma \cdot c = \rho(\gamma)^{-1} v \otimes c \quad \forall \gamma \in \pi_1(W, *).
\]
The boundary operator is defined by linearity and \( \partial(v \otimes c) = \text{Id} \otimes \partial(v \otimes c) = v \otimes \partial c \). We denote by \( H_*(W, \rho) \) the homology of this complex.

Let \( \{v_1, \ldots, v_n\} \) be a basis of \( F^n \) and let \( c_1, \ldots, c_k \) denote the set of \( i \)-dimensional cells of \( W \). We take a lift \( \bar{c}_j \) of the cell \( c_j \) in \( \tilde{W} \). Then, for each \( i, \bar{c}^i = \{\bar{c}_1, \ldots, \bar{c}_k\} \) is a basis of the \( \mathbb{Z}[-\pi_1(W)] \)-module \( C_i(\tilde{W}; Z) \). Thus we have the following basis of \( C_i(W, \rho) \):
\[
c^i = \{v_1 \otimes \bar{c}_1, v_2 \otimes \bar{c}_1, \ldots, v_n \otimes \bar{c}_k\}.
\]
Suppose \( H_*(W, \rho) \neq 0 \), and let \( h^i \) be a basis of \( H_i(W, \rho) \). We denote by \( h \) the basis \( \{h^0, \ldots, h^{\dim W}\} \) of \( H_* \). Then \( \text{tor}(C_*(W, \rho), c, h) \in F^*/\{\pm 1\} \) is well defined. Note that it does not depend on the lifts of the cells \( \bar{c}^i \) since \( \det \rho = 1 \). Further, if the Euler characteristic of \( W \) is equal to zero (e.g. the case that \( W \) corresponds to a knot exterior), it does not depend on the choice of a basis \( \{v_1, \ldots, v_n\} \) (cf. Lemma 2.4.2 in [13]).

**Remark 2.3.** The Reidemeister torsion is independent of the choice of a base point \( * \) of the fundamental group \( \pi_1(W, *) \). Furthermore, it is known that the Reidemeister torsion is an invariant under subdivision of the cell decomposition of \( W \) with \( \rho \)-coefficients up to factor \( \pm 1 \).

**Remark 2.4.** Let \( K \) be a knot in the 3-sphere \( S^3 \) and \( M_K = S^3 - \text{int}N(K) \). We denote by \( G(K) \) the fundamental group of \( M_K \). From the result of Waldhausen ([11]), the Whitehead group \( \text{Wh}(G(K)) \) is trivial. In such a case, the Reidemeister torsion does not depend on the choice of its CW-structure. Suppose \( H_* (M_K, \rho) = 0 \). Then the Reidemeister torsion does not depend on \( h = 0 \). In this case we denote by \( \text{tor}(M_K, \rho) \) the Reidemeister torsion.

Let \( \alpha \) be a surjective homomorphism from \( \pi_1(W, *) \) to the multiplicative group \( \{t\} \). Instead of a representation \( \rho : \pi_1(W, *) \to \text{SL}(n, F) \), consider the twisted representation:
\[
\alpha \otimes \rho : \pi_1(W, *) \to \text{GL}(F(t)),
\]
where \( F(t) \) is the field of fraction of the polynomial ring \( F[t] \). By the same method as above, we can define \( \text{tor}(C_*(W, \alpha \otimes \rho), 1 \otimes c, h) \in F^*(t)/\{\pm t^Z\} \). As the determinant is not one, there is an independency factor \( t^m \), for some integer \( m \). More precisely, we define:
\[
C_*(W, \alpha \otimes \rho) = F(t) \otimes F^n \otimes \rho C_*(\tilde{W}; Z),
\]
where the action is given by \( f \otimes v \otimes (\gamma \cdot c) = f \cdot t^{\gamma} \otimes (\gamma \cdot c) \). The boundary operator is defined by linearity and \( \partial(f \otimes v \otimes c) = f \cdot \partial c \).

Kitano ([2]) investigated the relationship between the Reidemeister torsions and the twisted Alexander invariants for knots. Namely, he proved

**Theorem 2.5 ([2]).** Let \( K \) be a knot in the 3-sphere \( S^3 \) and \( M_K = S^3 - \text{int}N(K) \). Suppose \( \rho \) is a
non-trivial representation such that $H_{\ast}(M_K, \rho) = 0$. Then, $H_{\ast}(M_K, \alpha \otimes \rho) = 0$ and $\text{tor}(M_K, \alpha \otimes \rho) = \Delta_{K, \rho}(t)$, where $\Delta_{K, \rho}(t)$ is the twisted Alexander invariant.}

See also Theorem 2.13 in [9]. The twisted Alexander invariant can be computed using the Fox calculus ([1, 2, 10]).

3. Representations of the fundamental groups of hyperbolic 3-manifolds. Let $M$ be an oriented, complete, hyperbolic 3-manifold of finite volume. Then $M$ has the holonomy representation: $\text{Hol}_M : \pi_1(M, \ast) \rightarrow \text{Isom}^+ \mathbb{H}^3$, where $\text{Isom}^+ \mathbb{H}^3$ is the orientation preserving isometry group of hyperbolic 3-space $\mathbb{H}^3$. Using the upper half-space model, $\text{Isom}^+ \mathbb{H}^3$ is identified with $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \{ \pm 1 \}$. It is known that $\text{Hol}_M$ can be lifted to $\text{SL}(2, \mathbb{C})$, and such lifts are in canonical one-to-one correspondence with spin structures on $M$. Thus, attached to a fixed spin structure $\eta$ on $M$, we get a representation:

$\text{Hol}_{(M, \eta)} : \pi_1((M, \eta), \ast) \rightarrow \text{SL}(2, \mathbb{C})$.

Let $W$ be a finite CW-complex and $\rho$ a representation of $\pi_1(W, \ast)$ to $\text{SL}(2, \mathbb{C})$. Then the pair $(C^2, \rho)$ is an SL(2, C)-representation of $\pi_1(W, \ast)$ by the standard action $\text{SL}(2, \mathbb{C})$ to $C^2$. It is known that the pair of the symmetric product $\text{Sym}^{n-1}(C^2)$ and the induced action by $\text{SL}(2, \mathbb{C})$ gives an $n$-dimensional irreducible representation of $\text{SL}(2, \mathbb{C})$. More precisely, let $V_n$ be the vector space of homogeneous polynomials on $C^2$ with degree $n - 1$, that is,

$$V_n = \text{span}_C(x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}).$$

Then the symmetric product $\text{Sym}^{n-1}(C^2)$ can be identified with $V_n$ and the action of $A \in \text{SL}(2, \mathbb{C})$ is expressed as

$$A \cdot p\left(\begin{array}{c} x \\ y \end{array}\right) = p\left(\begin{array}{c} A^{-1}x \\ A^{-1}y \end{array}\right)$$

where $p\left(\begin{array}{c} x \\ y \end{array}\right)$ is a homogeneous polynomial and the right-hand side is determined by the action of $A^{-1}$ on the column vector as a matrix multiplication.

We denote by $(V_n, \sigma_n)$ the representation given by this action of $\text{SL}(2, \mathbb{C})$ where $\sigma_n$ means the homomorphism from $\text{SL}(2, \mathbb{C})$ to $\text{GL}(V_n)$. It is known that each representation $(V_n, \sigma_n)$ turns into an irreducible $\text{SL}(n, \mathbb{C})$-representation of $\text{SL}(2, \mathbb{C})$ and that every irreducible $n$-dimensional representation of $\text{SL}(2, \mathbb{C})$ is equivalent to $(V_n, \sigma_n)$. Composing $\text{Hol}_{(M, \eta)}$ with $\sigma_n$, we obtain the following representation:

$$\rho_n : \pi_1((M, \eta), \ast) \rightarrow \text{SL}(n, \mathbb{C}).$$

In the following sections, we will discuss Reidemeister torsions associated with this representation $\rho_n$. Note that there are several computations of the Reidemeister torsions associated with $\sigma_{2k}$ in [14, 15].

4. The results of Menal-Ferrer and Porti. In this note, we focus on a knot complement. We introduce the results of Menal-Ferrer and Porti ([4, 5]) in this setting.

Let $K$ be a hyperbolic knot in the 3-sphere $S^3$, that is, $S^3 - K$ is an oriented, complete, finite-volume hyperbolic manifold with only one cusp. Then, $S^3 - K$ may be regarded as the interior of a compact manifold $M_K$ such that $\partial M_K = T$ where $T$ is homeomorphic to a torus $T^2$. In what follows, we consider the compact manifold $M_K$ instead of $S^3 - K$.

By Corollary 3.7 in [4], we have that $\dim_C H^i(M_K, \rho_n) = 0$ if $i$ is even, and that $\dim_C H^i(M_K, \rho_n) = 0$. Further, in [5], Menal-Ferrer and Porti proved the following. (Note that Poincaré duality with coefficients in $\rho_n$ holds (Corollary 3.7 in [5]).)

**Proposition 4.1** (Proposition 4.6 in [5]). Suppose that $H_\ast(T; \rho_n) \neq 0$. Let $G < \pi_1(M_K, \ast)$ be some fixed realization of the fundamental group of $T$ as a subgroup of $\pi_1(M_K, \ast)$. Choose a non-trivial cycle $\theta \in H_1(T; \mathbb{Z})$, and a non-trivial vector $v \in V_n$ fixed by $\rho_n(G)$. Then the following holds:

(a) A basis for $H_1(T; \mathbb{Z})$ is given by $i_\ast([v \otimes \theta])$.

(b) A basis for $H_2(M_K, \rho_n)$ is given by $i_\ast([v \otimes T])$.

Here, $i : T \rightarrow M_K$ denotes the inclusion.

Set $h^1 = i_\ast([v \otimes \theta])$, $h^2 = i_\ast([v \otimes T])$, and $h = \{h^1, h^2\}$. On the other hand, Menal-Ferrer and Porti (Theorem 0.2 in [4]) proved that $H^*(M_K, \rho_{2k}) = 0$ for $k \geq 1$. Therefore, we may define the following quotients:

$$T_{2k+1}(M, \eta) := \frac{\text{tor}(M_K, \rho_{2k+1}, h)}{\text{tor}(M_K, \rho_{2k}, h)} \in C^*/\{\pm 1\},$$

$$T_{2k}(M, \eta) := \frac{\text{tor}(M_K, \rho_{2k})}{\text{tor}(M_K, \rho_{2k})} \in C^*/\{\pm 1\}.$$

The quantity $T_{2k+1}$ is independent of the spin structure because of the fact that an odd-dimen-
sional irreducible complex representation of SL(2, C) factors through PSL(2, C). Since $S^3 - K$ has only one cusp, then all spin structures on $M_K$ are acyclic (Corollary 3.4 in [5]). This means that the limit of $T_{2k}$ is also independent of the spin structure (Theorem 7.1 in [5]). Thus it is not necessary to consider a spin structure on $M_K$ in our setting. Hence, the above definition may be simplified to the following form deleting $\eta$.

**Definition 4.2.**

$$T_{2k+1}(M_K) := \frac{\text{tor}(M_K, \rho_{2k+1}, h)}{\text{tor}(M_K, \rho_3, h)} \in \mathbf{C}^*/\{\pm 1\},$$

$$T_{2k}(M_K) := \frac{\text{tor}(M_K, \rho_{2k})}{\text{tor}(M_K, \rho_2)} \in \mathbf{C}^*/\{\pm 1\}.$$

Note that it is proved that the quotient is independent of the choices $h$ (Proposition 4.2 in [5]). Then, we can reduce Theorem 7.1 in [5] to the following statement:

**Theorem 4.3** (Theorem 7.1 in [5]).

$$\lim_{k \to \infty} \frac{\log |T_{2k+1}(M_K)|}{(2k + 1)^2} = \lim_{k \to \infty} \frac{\log |T_{2k}(M_K)|}{(2k)^2} = \frac{\text{Vol}(K)}{4\pi}.$$  

As in Remark 2.2, the sign of the right-hand side is plus.

5. **Proof of Theorem 1.1.**

**Case 1.** Even-dimensional representation $\rho_{2k}$ case.

By Theorem 0.2 in [4], $H^*(M_K, \rho_{2k}) = 0$ for $k \geq 1$. Then, by Theorem 2.5, we can prove that $\text{tor}(M_K, \rho_{2k}) = \text{tor}(M_K, \alpha \otimes \rho_{2k})|_{\rho_3} = \Delta_K|_{\rho_3}(1)$ from the map at the chain level $C_*(M_K, \alpha \otimes \rho_{2k}) \to C_*(M_K, \rho_{2k})$ induced by evaluation at $t = 1$. Then, we have:

$$T_{2k}(M_K) = \frac{\text{tor}(M_K, \rho_{2k})}{\text{tor}(M_K, \rho_2)} = \frac{\Delta_K|_{\rho_3}(1)}{\Delta_K|_{\rho_3}(1)} = A_{2k}(1).$$

Hence we have done in the case of $\rho_{2k}$ in Theorem 1.1: $\lim_{k \to \infty} \frac{\log |A_{2k}(1)|}{(2k)^2} = \frac{\text{Vol}(K)}{4\pi}$ by Theorem 4.3.

**Case 2.** Odd-dimensional representation $\rho_{2k+1}$ case.

Although the idea of the proof is the same as Yamaguchi’s one in [12,13], I think it is worth outlining it here for the convenience of readers. He investigated the case of the adjoint representation of SL(2, C), which is essentially equivalent to $\rho_3$ in our setting.

The homology group $H_*(M_K; \mathbb{Z}) = H_0(M_K; \mathbb{Z}) \oplus H_1(M_K; \mathbb{Z})$ has the basis $\{[p], [\mu]\}$, where $[p]$ is the homology class of a point and $[\mu]$ is that of the meridian of $K$. Further, $H_1(\partial M_K; \mathbb{Z})$ has the basis $\{[\nu], [\lambda]\}$, where $[\lambda]$ is the homology class of a longitude of $K$. By Proposition 4.1, we may define $h^1 = i_*(v \otimes [\lambda]), h^2 = i_*(v \otimes [\lambda])$ and $\mathbf{h} = \{h^1, h^2\}$.

It is known that $M_K$ collapses to a 2-dimensional CW-complex $W$ with only one vertex. We call $\varphi$ this deformation. Thus $M_K$ is simple homotopy equivalent to $W$. It is enough to prove the theorem for $W$ since a Reidemeister torsion is a simple homotopy invariant.

By Proposition 3.5 in [1], we have $H_0(W, \alpha \otimes \rho_{2k+1}) = 0$. Further, we have the next lemma by the same argument as Proposition 7 in [12] or Proposition 3.1.1 in [13].

**Lemma 5.1.** For $* = 1, 2$, we have:

$H_*(M_K, \alpha \otimes \rho_{2k+1}) = 0$.

**Proposition 5.2.** $\text{tor}(M_K, \alpha \otimes \rho_{2k+1})$ has a simple zero at $t = 1$. Moreover the following holds:

$$\text{tor}(M_K, \rho_{2k+1}, h) = \lim_{t \to 1} \text{tor}(M_K, \alpha \otimes \rho_{2k+1}).$$  

**Proof.** We define the subchain complex $C'_*(W, \rho_{2k+1})$ of the chain complex $C_*(W, \rho_{2k+1})$ by $C'_0(W, \rho_{2k+1}) = \text{span}(v \otimes \varphi(T))$, $C'_1(W, \rho_{2k+1}) = \text{span}(C(v \otimes \varphi(\lambda)))$ and $C'_i(W, \rho_{2k+1}) = 0$ ($i \neq 1, 2$). Note that $v$ is fixed by $\rho_{2k+1}(G)$, and the boundary operators of $C'_*(W, \rho_{2k+1})$ are zero by the definition. The modules of this subchain complex are lifts of homology groups $H_*(W, \rho_{2k+1})$. Similarly, we define the subcomplex $C'_*(W, \alpha \otimes \rho_{2k+1})$ of $C_*(W, \alpha \otimes \rho_{2k+1})$ by $C'_0(W, \alpha \otimes \rho_{2k+1}) = \text{span}(v \otimes \varphi(T))$, $C'_1(W, \alpha \otimes \rho_{2k+1}) = \text{span}(v \otimes \varphi(\lambda))$ and $C'_i(W, \alpha \otimes \rho_{2k+1}) = 0$ for $i \neq 1, 2$. Since $v$ is an invariant vector of $\rho_{2k+1}(G)$, we have:

$$\partial(v \otimes \varphi(T)) = 1 \otimes v \otimes \partial(\varphi(T)) = 1 \otimes v \otimes (\mu \cdot \varphi(\lambda)) - 1 \otimes v \otimes \varphi(\lambda)$$

$$= t \otimes \rho_{2k+1}^{-1}(\mu) \otimes \varphi(\lambda) - 1 \otimes v \otimes \varphi(\lambda)$$

$$= t \otimes v \otimes \varphi(\lambda) - 1 \otimes v \otimes \varphi(\lambda)$$

$$= (t - 1) (1 \otimes v \otimes \varphi(\lambda)).$$
Thus the boundary operators of $C'_s(W, \alpha \otimes \rho_{2k+1})$ are given by
$$0 \to C'_s(W, \alpha \otimes \rho_{2k+1}) \xrightarrow{i_{-1}} C'_s(W, \alpha \otimes \rho_{2k+1}) \to 0.$$ This means that the homology of $C'_s(W, \alpha \otimes \rho_{2k+1})$ is zero.

By the definition, the chain complex $C'_s(W, \rho_{2k+1})$ has the natural basis:
$$c' = \{\delta \otimes \varphi(T), \delta \otimes \varphi(\lambda)\}.$$ Let $C''_{c'}(W, \rho_{2k+1})$ be the quotient of $C'_s(W, \rho_{2k+1})$ by $C''_s(W, \rho_{2k+1})$, $c''$ a basis of $C''_{c'}(W, \rho_{2k+1})$, and $c''$ a lift of $c''$ to $C(W, \rho_{2k+1})$. By Lemma 5.1, we can apply Proposition 3.3.1 in [13] to this setting, then we have:
$$\lim_{t \to 1} \text{tor}(C_s(W, \alpha \otimes \rho_{2k+1}), 1 \otimes c' \cup 1 \otimes c'') \quad \text{tor}(C_s(W, \alpha \otimes \rho_{2k+1}), 1 \otimes c'').$$
By the calculation above, we have $\text{tor}(C'_s(W, \alpha \otimes \rho_{2k+1}), 1 \otimes c') = t-1$, thus we have this proposition.

**Proof of Theorem 1.1.** By Theorem 2.5 and Lemma 5.1, we have $\text{tor}(M_K, \alpha \otimes \rho_{2k+1}) = \Delta_{K, \rho_{2k+1}}(t)$. We also have $\Delta_{K, \rho_{2k+1}}(t) = (t-1)\Delta_{K, \rho_{2k+1}}(t)$ and $\text{tor}(M_K, \rho_{2k+1}, \mathbf{h}) = \Delta_{K, \rho_{2k+1}}(1)$ by Proposition 5.2, where $\Delta_{K, \rho_{2k+1}}(t)$ is a rational function. Then,
$$A_{K, 2k+1}(1) = \frac{\Delta_{K, \rho_{2k+1}}(1)}{\Delta_{K, \rho_{2k+1}}(1)} = \frac{\text{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\text{tor}(M_K, \rho_{3}, \mathbf{h})} = T_{2k+1}(M_K).$$
Thus we have Theorem 1.1 by Theorem 4.3. □

### 6. Some calculations on the figure eight knot complement

Let $K$ be the figure eight knot 41. Note that it is known that the volume of $K$ is 2.02988\ldots. The knot group $G(K)$ has the following presentation:
$$G(K) = \langle a, b | ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b \rangle,$$
where $a$ and $b$ correspond to the meridians of $K$. Consider the representation of this fundamental group:
$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$
where $u$ is a complex value satisfying $u^2 + u + 1 = 0$. This representation is the holonomy representation of $G(K)$. By the definition, we have
$$p\left(\rho(a)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right) = p\left(\begin{pmatrix} x - y \\ y \end{pmatrix} \right), \quad \text{and} \quad (x - y)^2 = x^2 - 2xy + y^2, \quad (x - y)y = xy - y^2.$$ Hence, we have:
$$\rho_2(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$ By the same calculations, we have:
$$\rho_3(b) = \begin{pmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_4(a) = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & 1 \\ -1 & 1 & -1 \end{pmatrix}, \quad \rho_4(b) = \begin{pmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 1 \end{pmatrix}.$$ Set $A = \rho_2(a) = \rho_4(a)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $B = \rho_2(b) = \rho_4(b)^{-1} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Via Fox calculus for $G(K)$, we obtain the denominator of $\Delta_{K, \rho_2}(t) = \det(tB - I) - (t - 1)^2$. On the other hand, the numerator of $\Delta_{K, \rho_2}(t) = \det(I - t^{-1}AB^{-1}A^{-1} + AB^{-1}A^{-1}B - tB + BAB^{-1}A^{-1}) = \frac{1}{t^2} (t - 1)^2(t^2 - 4t + 1)$. Here we use the value $u = -\frac{1+\sqrt{3}}{2}$. Continuing in this way, we have obtained the following data:
$$\Delta_{K, \rho_2}(t) = \frac{1}{t^2} (t^2 - 4t + 1), \quad \Delta_{K, \rho_2}(t) = -\frac{1}{t^2} (t^2 - 4t + 1)^2, \quad \Delta_{K, \rho_2}(t) = \frac{1}{t^2} (t^2 - 4t + 1)^2, \quad \Delta_{K, \rho_2}(t) = -\frac{1}{t^2} (t^2 - 4t + 1)^2 - 4t^2.$$ For $A_{K, 2k+1}(t)$, we have
$$\frac{4\pi \log |A_{K, 2k+1}(t)|}{4t^2} = \frac{4\pi \log |t^2 - 4t + 1|}{4t^2} = \frac{\pi \log |t^2 - 4t + 1|}{4t^2} \approx 0.544397, \quad \frac{4\pi \log |A_{K, 2k+1}(t)|}{5t^2} = \frac{4\pi \log |t^2 - 4t + 1|}{5t^2} \approx 4.7106. $$
These calculations were done by using Wolfram Mathematica.

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