Applications of the Laurent-Stieltjes constants for Dirichlet *L*-series

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Abstract: The Laurent-Stieltjes constants $\gamma_n(\chi)$ are, up to a trivial coefficient, the coefficients of the Laurent expansion of the usual Dirichlet L-series: when χ is non-principal, $(-1)^n\gamma_n(\chi)$ is simply the value of the n-th derivative of $L(s,\chi)$ at $s=1$. In this paper, we give an approximation of the Dirichlet L-functions in the neighborhood of $s=1$ by a short Taylor polynomial. We also prove that the Riemann zeta function $\zeta(s)$ has no zeros in the region $|s-1|\le 2.2093$, with $0\le\Re(s)\le 1$. This work is a continuation of [24].

Key words: The Laurent-Stieltjes constants; Dirichlet L-function; Riemann zeta function.

1. Introduction and main results. Let $\gamma_n(\chi)$ denote the n-th Laurent-Stieltjes coefficients around $s=1$ of the associated Dirichlet L-series for a given primitive Dirichlet character χ modulo q. These constants are defined by

$$(1)\quad L(s,\chi)=\frac{\delta_\chi}{s-1}+\sum_{n\ge 0}\frac{(-1)^n\gamma_n(\chi)}{n!}(s-1)^n,$$

where $\delta_\chi=1$ when χ is principal and $\delta_\chi=0$ otherwise. We may regard $\zeta(s)$ as the Dirichlet L-functions to the principal character χ_0 modulo 1. Then, we call the coefficients $\gamma_n(\chi_0)=\gamma_n$ in this series the *Laurent-Stieltjes constants for the Riemann zeta function*. When χ is non-principal, $(-1)^n\gamma_n(\chi)$ is simply the value of the n-th derivative of $L(s,\chi)$ at $s=1$. In this case, we call these derivatives *Laurent-Stieltjes constants for the Dirichlet L-functions*.

The interest in Laurent-Stieltjes constants has a long history, started by Dirichlet in 1837. For a nice survey on these constants see [25] or [23]. When χ is non-principal, Dirichlet produced a finite expansion for $L(1,\chi)$. Berger [3], Lerch [20], Gut [11] and Deninger [9] gave representations $\gamma_1(\chi)$ by elementary functions. In 1989, Kanemitsu [15] obtained similar results for $\gamma_n(\chi)$ with $n\ge 2$. Toyoizumi [26] and Ishikawa [12] gave explicit upper bounds for these constants.

When χ is a principal character modulo 1, Stieltjes in 1885 was the first to propose the following definition of γ_n

$$\gamma_n=\lim_{T\to\infty}\left(\sum_{m=1}^{T}\frac{(\log m)^n}{m}-\frac{(\log T)^{n+1}}{(n+1)}\right).$$

These constants have been studied by many authors, among them, Ramanujan [22], Jensen [14], Verma [27], Ferguson [10], Briggs and Chowla [6], Kluyver [16], Zhang and Williams [28], and more recently, Adell [1], Adell and Lekuona [2], Coffey [7], [8], Knessl and Coffey [17]. The first explicit upper bound for $|\gamma_n|$ has been given by Briggs [5], that is later improved by Berndt [4] and Israilov [13]. In 1985, the theory made a huge progress via an asymptotic expansion produced by Matsuoka [21], for these constants. Matsuoka gave the best upper bound for $|\gamma_n|$ for $n\ge 10$. He proved that

$$|\gamma_n|\le 10^{-4}e^{n\log\log n}.$$

Thanks to this result, Matsuoka showed that zeta function $\zeta(s)$ has no zeros in the region $|s-1|\le\sqrt{2}$, with $0\le\Re(s)\le 1$.

Many authors have tried to improve on the Matsuoka bound, with few success. Matsuoka's work relied on a formula that is essentially a consequence of Cauchy's Theorem and the functional equation. More recently, the author, in [24] and [25], extended this formula to Dirichlet L-functions. We gave the following upper bound for $|\gamma_n(\chi)|$ with $1\le q<\frac{\pi}{2}\frac{e^{(n+1)/2}}{n+1}$.

Theorem 1. *Let χ be a primitive Dirichlet character to modulus q. Then, for every $1\le q<\frac{\pi}{2}\frac{e^{(n+1)/2}}{n+1}$ and $n\ge 2$, we have*

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$$\frac{|\gamma_n(\chi)|}{n!} \le q^{-1/2}\ C(n,q)\ \min\left(1 + D(n,q), \frac{\pi^2}{6}\right),$$

with

$$C(n,q) = 2\sqrt{2}\ \times$$
$$\exp\left\{-(n+1)\log\theta(n,q) + \theta(n,q)\log\left(\frac{2q\theta(n,q)}{\pi e}\right)\right\},$$

and

$$\theta(n,q) = \frac{n+1}{\log\left(\frac{2q(n+1)}{\pi}\right)} - 1,$$

$$D(n,q) = 2^{-\theta(n,q)-1}\,\frac{\theta(n,q)+1}{\theta(n,q)-1}.$$

In the case when $\chi = \chi_0$ and $q = 1$, this leads to a sizable improvement of the Matsuoka bound and of previous results. The aim of this paper is to use this result to give applications of the Laurent-Stieltjes constants. This work is a continuation of [24]. We shall show that this result enables us to approximate $L(s,\chi)$ in the neighborhood of $s = 1$ by a short Taylor polynomial. We have

Application A. Let χ be a primitive Dirichlet character to modulus q. For $N = 4\log q$ and $q \ge 150$, we have

$$\left|L(s,\chi) - \sum_{n\le N}\frac{(-1)^n\gamma_n(\chi)}{n!}(s-1)^n\right| \le \frac{32.3}{q^{2.5}},$$

where $|s-1| \le e^{-1}$.

We also prove that

Application B. $\zeta(s)$ has no zeros in the region $|s-1| \le 2.2093$ with $0 \le \Re(s) \le 1$.

This result is an improvement on the Matsuoka result. To do that we apply the same technique used in [19] and [21] by giving the best possible choice of the radius of $|s-1|$ in which $\zeta(s)$ has no zeros.

2. Proofs.

2.1. Proof of Application A. From Theorem 1, for $n+1 \ge 4\log q$, we note that the function $\theta(n,q)$ is non-decreasing function of n, it follows that the function $D(n,q)$ is decreasing function of θ. For $n+1 \ge 4\log q$ and $q \ge 150$ we find that

$$\theta(n,q) \ge \frac{4\log q}{\log\left(\frac{8q\log q}{\pi}\right)} - 1 \ge \frac{4\log 150}{\log\left(\frac{1200\log 150}{\pi}\right)} - 1,$$

that is $\theta(n,q) \ge 1.65$. From the above, we note that

$$D(n,q) = 2^{-\theta(n,q)-1}\,\frac{\theta(n,q)+1}{\theta(n,q)-1} \le 0.65. \tag{2}$$

On the other hand, we have

$$\log\theta(n,q) + \log\frac{2q}{\pi e} \le \log\left(\frac{\frac{2q(n+1)}{\pi e}}{\log\left(\frac{2q(n+1)}{\pi}\right)}\right).$$

Putting $H = 2q(n+1)/\pi$, we obtain that

$$\theta(n,q)\left(\log\theta(n,q) + \log\left(\frac{2q}{\pi e}\right)\right) \le \frac{n+1}{\log H}\log\left(\frac{H/e}{\log H}\right).$$

For $H \ge e^{1/e}$, we infer that

$$\theta(n,q)\left(\log\theta(n,q) + \log\left(\frac{2q}{\pi e}\right)\right) \le n+1.$$

Hence

$$C(n,q) \le 2\sqrt{2}\exp\{-(n+1)\log\theta(n,q) + (n+1)\}.$$

That is

$$C(n,q) \le 2\sqrt{2}\left(\frac{e}{\theta(n,q)}\right)^{n+1}.$$

For $n+1 \ge N$, we have $\theta(n,q) \ge \theta(N,q)$ and then

$$\frac{|\gamma_n(\chi)|}{n!} \le \frac{2\sqrt{2}}{\sqrt{q}}(1 + D(n,q))\left(\frac{e}{\theta(N,q)}\right)^{n+1}.$$

Now, we recall that

$$L(s,\chi) = \sum_{n\ge 1}\frac{(-1)^n\gamma_n(\chi)}{n!}(s-1)^n.$$

Put

$$\left|L(s,\chi) - \sum_{n\le N-2}\frac{(-1)^n\gamma_n(\chi)}{n!}(s-1)^{n+1}\right| = I_1,$$

and let $\varepsilon > 0$ such that $|s-1| \le \varepsilon$. Then, for $n+1 \ge N = 4\log q$, we get

$$\begin{aligned}I_1 &\le \sum_{n\ge N-1}\frac{|\gamma_n(\chi)|}{n!}|s-1|^n\\ &\le \frac{2\sqrt{2}\ \varepsilon^{-1}}{\sqrt{q}}(1 + D(n,q))\sum_{n\ge N-1}\left(\frac{e\varepsilon}{\theta(N,q)}\right)^{n+1}\\ &\le \frac{2\sqrt{2}\ \varepsilon^{-1}}{\sqrt{q}}(1 + D(n,q))\left(\frac{e\varepsilon}{\theta(N,q)}\right)^{N}\left(\frac{1}{1-\frac{\varepsilon e}{\theta(N,q)}}\right).\end{aligned}$$

Taking $\varepsilon = e^{-1}$, we get

$$I_1 \le \frac{2\sqrt{2}\ e}{\sqrt{q}}(1 + D(n,q))\left(\frac{1}{q^{4\log\left(\frac{4\log q}{\log(8q\log q/\pi)}-1\right)}}\right)$$
$$\times\left(\frac{1}{1-\frac{1}{1.65}}\right).$$

Using Eq. (2), for $q \geq 150$, we conclude that

$$I_1 \leq \frac{32.3}{q^{2.5}}.$$

This completes the proof. □

2.2. Proof of Application B. For χ is a principal Dirichlet character modulo 1, Eq. (1) is rewritten as

$$\zeta(s) = \frac{1}{s-1} + \sum_{n\geq 0} \frac{(-1)^n}{n!}\gamma_n (s-1)^n. \tag{3}$$

Multiplying both sides of this equation by $s-1$, we get

$$|(s-1)\zeta(s)| \geq |1+\gamma_0(s-1)| - \sum_{n\geq 1} \frac{|\gamma_n|}{n!}|s-1|^{n+1}. \tag{4}$$

Put

$$|1+\gamma_0(s-1)| - \sum_{1\leq n\leq 11} \frac{|\gamma_n|}{n!}|s-1|^{n+1} = I_2.$$

Here, the above summation is taken over $1 \leq n \leq 11$, that the bound in Theorem 1 is numerically better than Matsuoka's bound as soon as $n \geq 11$.

Now, let $|s-1| \leq T_0$, where T_0 is a positive real number to be chosen later such that $|(s-1)\zeta(s)| > 0$. Using the fact that $0 \leq \Re(s) \leq 1$, then I_2 is estimated by

$$I_2 \geq 1 - \gamma_0 - \sum_{1\leq n\leq 11} \frac{|\gamma_n|}{n!} T_0^{n+1}. \tag{5}$$

Since the function $\theta(n,1)$ in Theorem 1 is non-decreasing function of $n \geq 4$, it follows that the function $D(n,1)$ is decreasing function of θ. For $n \geq 12$ we find that

$$\theta(n,1) \geq \theta(12,1),$$

and

$$D(n,1) \leq D(12,1).$$

Thus, we have

$$\log\theta(n,1) + \log\frac{2}{\pi e} \leq \log\left(\frac{\frac{2(n+1)}{\pi e}}{\log\left(\frac{2(n+1)}{\pi}\right)}\right).$$

Putting $M = 2(n+1)/\pi$, we obtain that

$$\theta(n,1)\log\left(\frac{2\theta(n,1)}{\pi e}\right) \leq \frac{n+1}{\log M}\log\left(\frac{M/e}{\log M}\right).$$

For $M \geq 2(12+1)/\pi$, we infer that

$$\theta(n,1)\log\left(\frac{2\theta(n,1)}{\pi e}\right) \leq 0.1728(n+1).$$

Hence, we get

$$C(n,1) \leq 2\sqrt{2}\left(\frac{e^{0.1728}}{\theta(n,1)}\right)^{n+1},$$

and then

$$\frac{|\gamma_n|}{n!} \leq 2\sqrt{2}(1+D(12,1))\left(\frac{e^{0.1728}}{\theta(12,1)}\right)^{n+1}.$$

It follows that

$$\sum_{n\geq 12} \frac{|\gamma_n|}{n!}|s-1|^{n+1} \leq 2\sqrt{2}(1+D(12,1))\sum_{n\geq 12}\left(\frac{T_0 e^{0.1728}}{\theta(12,1)}\right)^{n+1}. \tag{6}$$

From Eqs. (5) and (6), we write

$$|(s-1)\zeta(s)| \geq 1-\gamma_0 - \sum_{1\leq n\leq 11}\frac{|\gamma_n|}{n!}T_0^{n+1} - 2\sqrt{2}(1+D(12,1))\sum_{n\geq 12}\left(\frac{T_0 e^{0.1728}}{\theta(12,1)}\right)^{n+1}.$$

Using numerical values of γ_n for $1 \leq n \leq 11$ of [18], we find that the best possible choice of T_0 is 2.2093 in which

$$|(s-1)\zeta(s)| > 0.000941198 - 0.000924993 > 0.$$

This completes the proof. □

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References

[1] J. A. Adell, Asymptotic estimates for Stieltjes constants: a probabilistic approach, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2011), no. 2128, 954–963.
[2] J. A. Adell and A. Lekuona, Fast computation of

the Stieltjes constants, Math. Comp. **86** (2017), no. 307, 2479–2492.

[3] A. Berger, Sur une sommation de quelques séries, Nova. Acta Reg. Soc. Ups. **12** (1883), 31 pp.

[4] B. C. Berndt, On the Hurwitz zeta-function, Rocky Mountain J. Math. **2** (1972), no. 1, 151–157.

[5] W. E. Briggs, Some constants associated with the Riemann zeta-function, Michigan Math. J. **3** (1955–1956), 117–121.

[6] W. E. Briggs and S. Chowla, The power series coefficients of $\zeta(s)$, Amer. Math. Monthly **62** (1955), 323–325.

[7] M. W. Coffey, Hypergeometric summation representations of the Stieltjes constants, Analysis (Munich) **33** (2013), no. 2, 121–142.

[8] M. W. Coffey, Series representations for the Stieltjes constants, Rocky Mountain J. Math. **44** (2014), no. 2, 443–477.

[9] C. Deninger, On the analogue of the formula of Chowla and Selberg for real quadratic fields, J. Reine Angew. Math. **351** (1984), 171–191.

[10] R. P. Ferguson, An application of Stieltjes integration to the power series coefficients of the Riemann zeta function, Amer. Math. Monthly **70** (1963), 60–61.

[11] M. Gut, Die Zetafunktion, die Klassenzahl und die Kronecker'sche Grenzformel eines beliebigen Kreiskörpers, Comment. Math. Helv. **1** (1929), no. 1, 160–226.

[12] H. Ishikawa, On the coefficients of the Taylor expansion of the Dirichlet L-function at $s = 1$, Acta Arith. **97** (2001), no. 1, 41–52.

[13] M. I. Israilov, The Laurent expansion of the Riemann zeta function, Trudy Mat. Inst. Steklov. **158** (1981), 98–104, 229.

[14] J. L. W. V. Jensen, Sur la fonction $\zeta(s)$ de Riemann, C. R. Acad. Sci. Paris **104** (1887), 1156–1159.

[15] S. Kanemitsu, On evaluation of certain limits in closed form, in *Théorie des nombres* (*Quebec, PQ, 1987*), 459–474, de Gruyter, Berlin, 1989.

[16] J. C. Kluyver, On certain series of Mr. Hardy, Quart. J. Pure Appl. Math. **50** (1927), 185–192.

[17] C. Knessl and M. W. Coffey, An effective asymptotic formula for the Stieltjes constants, Math. Comp. **80** (2011), no. 273, 379–386.

[18] R. Kreminski, Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants, Math. Comp. **72** (2003), no. 243, 1379–1397.

[19] E. Lammel, Ein Beweis, dass die Riemannsche Zetafunktion $\zeta(s)$ in $|s-1| \leqq 1$ keine Nullstelle besitzt, Univ. Nac. Tucumán Rev. Ser. A **16** (1966), 209–217.

[20] M. Lerch, Sur quelques formules relatives au nombre des classes, Bull. Sci. Math. **21** (1897), 290–304.

[21] Y. Matsuoka, Generalized Euler constants associated with the Riemann zeta function, in *Number theory and combinatorics. Japan 1984* (*Tokyo, Okayama and Kyoto, 1984*), 279–295, World Sci. Publishing, Singapore, 1985.

[22] S. Ramanujan, *Collected papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, Providence, RI, 2000.

[23] S. Saad Eddin, *Two problems with Laurent-Stieltjes coefficients*, LAP Lambert Academic Publishing, Saarbrücken, 2017.

[24] S. Saad Eddin, Explicit upper bounds for the Stieltjes constants, J. Number Theory **133** (2013), no. 3, 1027–1044.

[25] S. Saad Eddin, On two problems concerning the Laurent-Stieltjes coefficients of Dirichlet L-series, 2013, Ph. D thesis. (University of Lille 1, France).

[26] M. Toyoizumi, On the size of $L^{(k)}(1,\chi)$, J. Indian Math. Soc. (N.S.) **60** (1994), no. 1–4, 145–149.

[27] D. P. Verma, Laurent's expansion of Riemann's zeta-function, Indian J. Math. **5** (1963), 13–16.

[28] N. Y. Zhang and K. S. Williams, Some results on the generalized Stieltjes constants, Analysis **14** (1994), no. 2–3, 147–162.