On some Hasse principles for algebraic groups over global fields. III

Dedicated to Prof. Carl Riehm on his 80th birthday

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Abstract: We establish some new local–global principles related with some splitting problems for connected linear algebraic groups over infinite algebraic extensions of global fields and give some applications to the isotropy problems. The main tools are certain new Hasse principles established for quadratic, (skew-)hermitian forms, and homogeneous projective spaces of reductive groups over such fields.

Key words: Hasse principle; splitting field; Tits index; tori; unipotent groups; reductive groups.

1. Introduction. This paper continues our study of arithmetic of linear algebraic groups defined over (possibly infinite) algebraic extensions of global fields begun in [NT1], [NT2], [NT3] via the so-called local–global principles. In this note, we are interested in some phenomena related with the notion of splitting.

Let k be a global field, V_k the set of all places of k. In [NT2] we extended some well-known local–global principles in the case of global fields to their infinite algebraic extensions, where one replaces the usual completions by the so-called localization fields. In [NT1], we discussed in detail whether or not there is any corresponding result for algebraic groups with a suitable notion of splitting. Recall that (cf. [Bo, Chap. V, 15.1], [CGP, A.1.2]) for a given field k, a connected solvable linear algebraic k-group G is *k-split* if there exists a composition series $G = G_0 > G_1 > \cdots > G_{n-1} > G_n = \{1\}$ such that $G_i/G_{i+1} \simeq \mathbf{G_a}$ or $\mathbf{G}_m$, for all $0 \leq i \leq n-1$. Also (cf. [Bo, Chap. V, 18.6], [CGP, A.4]), a connected reductive k-group G is *k-split* if G has a maximal torus which is defined and split over k. More generally, one says that a smooth connected affine algebraic k-group G is *pseudo-k-split* (or *pseudo-split over k*) if G has a maximal torus which is defined and split over k, see [CGP, Def. 2.3.1]. In [NT1] we introduced the notion of splitting which really combines the case of solvable and reductive groups. We say that a connected linear algebraic k-group G is *k-split*, or *split over k*, if its unipotent radical $R_u(G)$ is defined and split over k, and the reductive quotient group $G/R_u(G)$ is defined and split over k. Likewise, we say that a smooth affine k-group G is *quasi-split over k* (or *k-quasi-split*) if $R_u(G)$ is defined over k and $G/R_u(G)$ has a Borel subgroup B defined over k. It is clear that this notion is stronger than the property that G has a Borel subgroup defined over k (just consider a solvable algebraic group G defined over a non-perfect field, such that $R_u(G)$ is not defined over k).

This paper is a sequel to our previous papers [NT1], [NT2], which deal with the splitting phenomena and Hasse principle for homogeneous spaces over global fields and also with the extensions of known Hasse principles over global fields to the case of infinite global fields (cf. also expanded version in [NT3]). By using results of [NT2], we consider some extensions of results obtained in [NT1] to the case of infinite global fields and also some applications. Here we are interested in certain local–global principles related with the splitting property of a given reductive group defined over an infinite algebraic extension of a global field. Our main tools are some new Hasse principles established for quadratic, (skew-)hermitian forms, and homogeneous projective spaces of reductive groups over infinite algebraic extensions of global fields. Full details will appear elsewhere.

Notation and convention. For a field k with a place v, we denote by k_v the completion of k at v,

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$\mathscr{O}_v$ the ring of v-integers of k_v. A k-variety always means a geometrically reduced, integral and separated scheme of finite type over k.

2. Preliminaries.

2.1. Localization fields versus completions. Let L be a field, k an algebraic extension of L contained in an algebraic closure of L with the set of all places V_k. For each place $v \in V_k$, let v also denote the restriction of v to any intermediate field $L \subset K \subset k$ and let K_v be the completion of K at v. We denote by $k(v)$ the direct limit of all subfields K_v, where $[K : L] < \infty$, all of which are considered as subfields of k_v. This field was considered for the first time by Moriya [Mo] in the 1930's and we call it (after Neukirch, cf. [Ne, p. 160]) *the localization field of* k *at* v. In the sequel, we will call infinite algebraic extensions of local (resp. global) fields just by infinite local (resp. global) fields for short.

We say that some algebraic objects defined over an infinite global field k *locally have a property* P if for all places $v \in V_k$, considered as object over $k(v)$, it has property P. In particular, we say that *the Hasse principle in the new sense* holds for a k-variety X, if the following implication holds

$$(X(k(v)) \neq \emptyset, \forall v \in V_k) \Rightarrow (X(k) \neq \emptyset).$$

The usual Hasse principle is then referred to as the classical Hasse principle. One should note that if a variety X satisfies the classical Hasse principle then it automatically satisfies the Hasse principle in the new sense. However, it is still a widely open problem to see if the converse also holds.

2.2. A consequence of Hensel Lemma. Let X be an irreducible variety defined over a global field k. We consider the following class $\mathscr{V}_k$ of k-varieties. We say that $X \in \mathscr{V}_k$ if for almost all places v of k, $X(k_v) \neq \emptyset$. The class of such varieties is very large.

In fact, as a consequence of Hensel Lemma, it includes also the class of all geometrically irreducible varieties X defined over a global field k (cf. [La, Remark 1.6, p. 249]).

2.3. Hasse principle for projective homogeneous spaces. We recall the following important result related to the Hasse principle for homogeneous spaces over global fields which will be used frequently in the sequel.

2.3.1. Theorem. *Let* k *be a global field,* X *a projective homogeneous* k*-space under a connected reductive* k*-group* G*. Then the* (*classial*) *Hasse principle holds for* V*, i.e., the following implication holds*

$$(X(k_v) \neq \emptyset, \forall v \in V_k) \Rightarrow (X(k) \neq \emptyset).$$

In the case of number fields the proof was given by Harder [Ha] and in the case of global function fields, it was given in [NT1].

3. A general Hasse principle for varieties over infinite global fields. For a global field L, recall that $\mathscr{V}_L$ denotes the class of all irreducible L-varieties X which have for almost all places $v \in V_L$ local points (i.e., $X(L_v) \neq \emptyset$ for almost all v). Notice that we have $\mathscr{V}_L \subset \mathscr{V}_K$, if $L \subset K$. We have the following general result regarding the Hasse principle for varieties over infinite global fields.

3.1. Theorem. *Let* k *be an algebraic extension of a global field* L*,* $k = \cup_n L_n$*, where* $L = L_0 \subset L_1 \subset \cdots \subset L_i \subset L_{i+1} \subset k$ *is a tower of increasing finite extensions of* L*. Let* $X \in \mathscr{V}_L$ *be an irreducible* L*-variety, such that for each* n*,* X *being considered as* L_n*-variety, satisfies the classical Hasse principle over* L_n*. Then* X *also satisfies the Hasse principle over* k *in the new sense.*

Proof. The proof follows the same scheme as given in [KK] while they proved the Hasse principle in the new sense for quadratic forms in ≥ 3 variables which has been applied to other situations as in [NT2]. We recall briefly the idea of the proof as follows.

We may assume that $k = \cup L_n$ is the union of an increasing tower of finite extensions of a global field L and our variety X is defined over L. The main point here is that by assumption, our variety X, considered over each field L_n, locally over the fields $L_{n,v}$, has $L_{n,v}$-points for almost all $v \in V_{L_n}$. Then we may apply the König's Lemma to confirm that $X(k)$ is not empty. $\square$

Theorem 3.1 implies that all those varieties $\in \mathscr{V}_L$ defined over a global field L, which universally satisfy the Hasse principle, i.e., those also satisfy the Hasse principle over any finite extension of L, also satisfy the Hasse principle (in the new sense) over any algebraic extension k of L.

3.2. Corollary. *Let* L*,* k *be as above. Then the following varieties satisfy the Hasse principle in the new sense over* k*.*

(1) *Any principal homogeneous space* (*torsor*) *under a connected linear algebraic group* G *defined over* k*, which satisfies the cohomological Hasse principle over each global subfield contained in* k*.*

(2) *Any projective homogeneous space under a connected linear algebraic (supposed to be reductive if char. $k > 0$) group defined over k.*

4. Some splitting problems. In this and the next sections we consider some problems related with some local–global approaches related with splitting problems.

4.1. Let k be an infinite algebraic extension of a global field. We consider the following problem.

4.1.1. Problem. (*a*) *Assume that a connected smooth affine algebraic group G is K_v-split (resp., K_v-quasi-split) or more generally, G possesses K_v-subgroups of some given type, for all $v \in V_k$, where K_v/k_v is a Galois extension with its Galois group Γ_v belonging to a certain class of groups $\mathscr{C}$. Is it true that G is also split (resp. quasi-split), or more generally, does G possess a subgroup of a given type defined over a Galois extension K/k with its Galois group Γ also belonging to $\mathscr{C}$? If not, what is the obstruction ?*

(*b*) *Similar questions where k_v is replaced by $k(v)$?*

In this note we consider the last question (*b*) in the simplest case, where $\Gamma_v = \{1\}$ for all v, i.e., $k(v)$ are the (quasi-)splitting fields for G for all v. In other words, the first question we try to answer is the following

4.1.2. *Given that a smooth affine algebraic k-group G is (quasi-)split over $k(v)$ locally everywhere for all $v \in V_k$. Is G already (quasi-)split over k ? If not, what is the obstruction ?*

Closely related to 4.1.2 are the following questions

4.1.3. *Given that a smooth affine algebraic k-group G is (quasi-)split over k_v locally everywhere for all $v \in V_k$. Is G already (quasi-)split over k ? If not, what is the obstruction ?*

4.1.4. *Given that a smooth affine algebraic k-group G is (quasi-)split over a quadratic (or cyclic) extensions K_v/k_v (or extension $K_v/k(v)$) locally everywhere. Is G already (quasi-)split over a quadratic (or cyclic) extension K/k ? If not, what is the obstruction ?*

Some other questions and applications will be discussed later after we have given an answer to 4.1.2.

5. Local–global principle for splitting property. In this and next section, we give an answer to Question 4.1.2. Before going to the general case, we consider some partial cases as follows.

5.1. Solvable case. The first class of groups we are considering is that of solvable algebraic groups. By [Co], there exists a unique maximal connected normal k-split subgroup G_{split} for a given connected solvable k-group G. Thus G is k-split if and only if $G = G_{split}$.

We prove the following

5.1.1. Theorem. *Let k be an infinite global field, G a solvable k-group. Assume that k has at least one discrete place, if $R_u(G) \neq \{1\}$. Then G is split over k if and only if G is so over all $k(v)$, $v \in V_k$.*

5.1.2. Remark. One may extend the definition of isotropic torus to the case of solvable groups as follows. For a solvable affine algebraic group G defined over a field k, we say that G is *k-isotropic* if it contains a non-trivial k-split subgroup. Then the proof of Theorem 4.1.1 says that the following holds.

5.1.3. Theorem. *A connected solvable affine algebraic group defined over an infinite global field k (which is assumed to have at least one discrete valuation if $R_u(G) \neq 1$), is k-isotropic if and only if it is $k(v)$-isotropic over all localizations $k(v)$ of k.*

We will see below (Section 6) that this is not true any more for semisimple groups.

5.2. Reductive case. We have the following local–global principle for the splitting and quasi-splitting property of reductive groups.

5.2.1. Theorem. *Let k be an infinite global field, G a connected reductive k-group. Then*

(1) *G is quasi-split over k if and only if G is so over $k(v)$, for all $v \in V_k$.*

(2) *G is split over k if and only if G is so over $k(v)$, for all $v \in V_k$.*

5.3. General case. Let k be an infinite global field and let G be a smooth connected affine algebraic k-group. We want to know what happens to G if we assume that G is quasi-split (resp. split) over $k(v)$ for all places v of k. By combning the results of Sections 5.1 and 5.2, we arrive at the following

5.3.1. Theorem. *Let k be an infinite global field and let G be a smooth connected affine algebraic k-group. Assume that G is quasi-split (resp. split) over $k(v)$ for all places v of k and that k has at least one discrete valuation if G has non-trivial unipotent radical. Then G is also quasi-split (resp. split) over k.*

5.3.2. Remark. Notice that in the case of global fields, the proof we provided in [NT1] by using the reciprocity law approach due to Prasad and Rapinchuk [PR] (char. 0 case) and [Th] (char. $p > 0$ case) does not work here, simply because we lack a lot of information provided by the class field theory in the case of infinite global fields. It is worthwhile to investigate this case in more detail.

6. Some further applications.

6.1. In this section, we consider some applications related to the local–global behavior of *relative rank* (i.e., the dimension of a maximal split subtorus) of a given connected reductive group G defined over an infinite algebraic extensions k of a global field.

Let T be a maximal k-torus of G, T_s the maximal k-split subtorus of T, $T = T_a T_s$, an almost direct product, where T_a is anisotropic k-subtorus of T. Let $s := dim(T_s)$, $a := dim(T_a)$, $r := rank_k(G)$, the k-rank of G, $n := s + a = dim(T)$, the rank of G. We say that T is of *type* (a, s). It is clear that $r \geq s$. For each place v of k, denote $r_v := rank_{k(v)}(G)$. Then it is clear that $r_v \geq r$ for all v. There are natural questions related with the behavior of r_v:

6.1.1. (a) *Is it true that if for some non-negative integer c and for all v, we have $r_v = c$, then $r = c$?*
(b) *Is it true that if $r_v > 0$ for all v then so is r?*
(c) *Is it true that if k is a global field and if G has a maximal $k(v)$-torus of type (a, s) over $k(v)$ for all places v of k, then so does G over k ?*
(d) *Is it true that $min_v r_v = r$?*

6.1.2. Remark. (1) It should be mentioned that this question is closely related to questions we considered in previous sections. Namely, if G has a maximal torus T of type $(0, n)$ over a field k, then it means that G is split over k. Therefore the question has an affirmative answer in this case.
(2) If G has maximal $k(v)$-tori of type $(1, n-1)$ for all places, then perhaps at best we can say is that G is isotropic over $k(v)$ for all places v.
(3) If G is semisimple and it has at least two almost simple components, then we can construct without difficulty an example of a semisimple group G defined over a global field k such that G is isotropic over $k(v)$ for all places v but G is anisotropic over k. Therefore 6.1.1 truly makes sense when we restrict ourselves to the case where G is an absolutely almost simple k-group. We have the following local–global principle for isotropy of almost simple algebraic groups over infinite global fields.

6.2. Theorem. *Let k be an infinite global field, G an absolutely almost simple k-group and let c be a non-negative integer.*
(a) *If $r_v = c$ for all v, then $r = c$.*
(b) *Let G be of Dynkin type different from ${}^1\mathrm{A}_n$, or ${}^1\mathrm{E}_6$ (and k has a real embedding into $\mathbf{R}$). For each place v of k, denote $r_v := rank_{k(v)}(G)$. If $r_v > 0$ for all v, then $r > 0$.*
(c) *In the remaining cases ${}^1\mathrm{A}_n$ or ${}^1\mathrm{E}_6$, there are infinite global fields k and almost simple k-groups G of the corresponding type, for which the local–global principle for isotropy does not hold.*

Here we use the notation of Tits indices as in [Ti].

6.2.1. Remark. (1) It follows from above that questions 6.1.1 (c), 6.1.1 (d) also have negative answers.
(2) There are many open problems to what extent the classical local–global principles still hold for infinite global fields. It will be our study in the future.

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