A remark on the decomposition theorem for direct images of canonical sheaves tensorized with semipositive vector bundles

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(Communicated by Masaki KASHIWARA, M.J.A., June 13, 2016)

Abstract: The purpose of this short note is to give a remark on the decomposition theorem for direct images of canonical sheaves tensorized with Nakano semipositive vector bundles. Although our result is a direct consequence of Takegoshi's work in [3], it was not stated explicitly in [3]. Here we give the precise statement and the proof.

Key words: Decomposition theorem; semipositive vector bundles.

The decomposition theorem for direct images of canonical sheaves was proved by J. Kollár [1, Theorem 3.1]. Inspired by the work of S. Matsumura [2], here we note that the decomposition theorem also holds for direct images of canonical sheaves tensorized with Nakano semipositive vector bundles. Although Theorem 1 below is a direct consequence of Takegoshi's results in [3], it was not stated explicitly there. Therefore we give the precise statement of the decomposition theorem and prove it explicitly here. We remark that Theorem 1 below immediately implies the weaker form of the decomposition theorem [3, I Decomposition Theorem] (cf. Corollary 2).

Theorem 1. *Let X be a Kähler manifold of pure dimension, Y a complex analytic space and $f : X \longrightarrow Y$ a proper surjective morphism such that all the connected components of X are mapped surjectively to Y. For a Nakano semipositive vector bundle (E, h) on X, we have an isomorphism*

$$\bigoplus_q R^q f_*(\omega_X \otimes E)[-q] \simeq Rf_*(\omega_X \otimes E)$$

in the derived category of $\mathcal{O}_Y$-modules.

Proof. The sheaf of E-valued C^∞ (p, q)-forms on X is denoted by $\mathcal{A}_X^{p,q}(E)$. Then we have the Dolbeault quasi-isomorphism

$$\omega_X \otimes E \longrightarrow (\mathcal{A}_X^{n,\bullet}(E), \overline{\partial}),$$

which is an f_*-acyclic resolution of $\omega_X \otimes E$. Therefore we have an isomorphism

$$Rf_*(\omega_X \otimes E) \simeq (f_*\mathcal{A}_X^{n,\bullet}(E), \overline{\partial})$$

in the derived category of $\mathcal{O}_Y$-modules.

In the proof of Theorem 6.4 in [3], Takegoshi defined an $\mathcal{O}_Y$-subsheaf $R^0 f_*\mathcal{H}^{n,q}(E)$ of $\mathrm{Ker}(\overline{\partial} : f_*\mathcal{A}_X^{n,q}(E) \to f_*\mathcal{A}_X^{n,q+1}(E))$ such that the canonical inclusion

$$R^0 f_*\mathcal{H}^{n,q}(E) \longrightarrow \mathrm{Ker}(\overline{\partial} : f_*\mathcal{A}_X^{n,q}(E) \to f_*\mathcal{A}_X^{n,q+1}(E))$$

induces an isomorphism of $\mathcal{O}_Y$-modules

$$R^0 f_*\mathcal{H}^{n,q}(E) \xrightarrow{\simeq} R^q f_*(\omega_X \otimes E) \tag{1.1}$$

for every q. The composite of the inclusions

$$R^0 f_*\mathcal{H}^{n,q}(E) \longrightarrow \mathrm{Ker}(\overline{\partial} : f_*\mathcal{A}_X^{n,q}(E) \to f_*\mathcal{A}_X^{n,q+1}(E)) \longrightarrow f_*\mathcal{A}_X^{n,q}(E)$$

is denoted by φ^q. Then φ^q defines a morphism of complexes

$$R^0 f_*\mathcal{H}^{n,q}(E)[-q] \longrightarrow f_*\mathcal{A}_X^{n,\bullet}(E)$$

for every q. Since we have the isomorphism (1.1) for every q, we obtain a quasi-isomorphism

$$\bigoplus_q \varphi^q : \bigoplus_q R^0 f_*\mathcal{H}^{n,q}(E)[-q] \longrightarrow f_*\mathcal{A}_X^{n,\bullet}(E)$$

by taking the direct sum for all q. Combining with the isomorphism

$$\bigoplus_q R^q f_*(\omega_X \otimes E)[-q] \longleftarrow \bigoplus_q R^0 f_*\mathcal{H}^{n,q}(E)[-q],$$

we obtain an isomorphism

2010 Mathematics Subject Classification. Primary 14F05; Secondary 32J25.

doi: 10.3792/pjaa.92.84

$$\begin{array}{ccc} & \bigoplus_q R^0 f_* \mathcal{H}^{n,q}(E)[-q] & \\ \swarrow & & \searrow \\ \bigoplus_q R^q f_*(\omega_X \otimes E)[-q] & & f_*\mathcal{A}_X^{n,\bullet}(E) \simeq Rf_*(\omega_X \otimes E) \end{array}$$

in the derived category as desired. □

As a corollary of the theorem above, we have the following:

Corollary 2. *In addition to the situation in Theorem* 1*, let* $g : Y \longrightarrow Z$ *be any morphism of complex analytic spaces. Then we have*

$$\bigoplus_{p+q=n} R^p g_* R^q f_*(\omega_X \otimes E) \simeq R^n(g \cdot f)_*(\omega_X \otimes E)$$

for every n*. In particular, we have*

$$\bigoplus_{p+q=n} \mathrm{H}^p(Y, R^q f_*(\omega_X \otimes E)) \simeq \mathrm{H}^n(X, \omega_X \otimes E) \tag{2.1}$$

for every n*.*

Remark 3. For the case of X being compact, the decomposition (2.1) of the cohomology groups is proved by S. Matsumura [2, Corollary 1.2].

Acknowledgments. The author would like to thank Professor Osamu Fujino and Professor Shin-ichi Matsumura for their helpful discussion.

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