An estimate on volumes of trajectory-balls for Kähler magnetic fields

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Abstract: By applying a comparison theorem on trajectory-harps, we give an estimate of volumes of trajectory-balls for Kähler magnetic fields from below under an assumption that sectional curvatures of the underlying Kähler manifold are bounded from above.

Key words: Kähler magnetic fields; trajectory-balls; trajectory-harps; comparison theorems.

1. Introduction. Let $(M, J, \langle\, ,\, \rangle)$ be a complete Kähler manifold with complex structure J. We say constant multiples of the Kähler form $\mathbf{B}_J$ on M to be *Kähler magnetic fields* ([1], see also [6,8]). Under the action of a Kähler magnetic field $\mathbf{B}_\kappa = \kappa \mathbf{B}_J$ $(\kappa \in \mathbf{R})$, the motion of a charged particle of unit mass and of unit speed is expressed as a smooth curve γ on M which is parameterized by its arc-length and that satisfies $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇. We call such a smooth curve a *trajectory* for $\mathbf{B}_\kappa$. Since trajectories for the trivial magnetic field $\mathbf{B}_0 = 0$ are geodesics, we may say that trajectories are extended objects of geodesics which are closely related with the complex structure. Needless to say that geodesics play quite an important role in the study of Riemannian manifolds. We hence consider that trajectories give us some clues to study Kähler manifolds from the Riemannian geometric point of view.

In this paper we study volumes of trajectory-balls which correspond to geodesic balls. In [4] Bai and the second author studied volume elements of trajectory-balls which are related with magnetic Jacobi fields along trajectories. By use of comparison theorems on magnetic Jacobi fields corresponding to Rauchi's comparison theorem ([2]), they gave estimates of volume elements from above and from below. We here study volumes of trajectory-balls by another way. We consider the relationship between trajectories and geodesics through trajectory-harps. A trajectory-harp is a variation of geodesics which consists of a trajectory-segment and geodesics joining initial point and points of the segment. We give an estimate on volumes of trajectory-balls from below with the aid of the comparison theorem on string-lengths of trajectory-harps given in [3] and of estimates on volumes of geodesic balls.

2. Trajectory-balls. Let M be a complete Kähler manifold. For a unit tangent vector $v \in UM$, we denote by γ_v the trajectory for a Kähler magnetic field $\mathbf{B}_\kappa$ with initial condition $\dot\gamma(0) = v$. Given a point $p \in M$, we define a magnetic exponential map $\mathbf{B}_\kappa \exp_p : T_pM \to M$ on the tangent space by

$$\mathbf{B}_\kappa \exp_p(w) = \begin{cases} \gamma_{w/\|w\|}(\|w\|), & \text{if } w \neq 0_p, \\ p, & \text{if } w = 0_p. \end{cases}$$

We call $B_r^\kappa(p) = \{\mathbf{B}_\kappa \exp_p(tv) \mid 0 \le t < r, v \in U_pM\}$ a *trajectory-ball* of arc-radius r centered at p. Since $\mathbf{B}_0 \exp_p$ is the ordinary exponential map $\exp_p$, we see that $B_r^0(p)$ is a geodesic ball of radius r.

At an arbitrary point $p \in M$, we define the $\mathbf{B}_\kappa$-injectivity radius $\iota_\kappa(p)$ at p by

$$\iota_\kappa(p) = \sup\{r > 0 \mid \mathbf{B}_\kappa \exp_p|_{B_r(0_p)} \text{ is injective}\}.$$

Clearly, $\iota_0(p)$ is the ordinary injectivity radius at p. On a simply connected Kähler manifold M whose sectional curvatures satisfy $\mathrm{Riem}_M \le c < 0$, it is known that $\iota_\kappa(p) = \infty$ for κ with $|\kappa| < \sqrt{|c|}$ (see [3]).

Our interest lies on giving an estimation on volumes of trajectory-balls of arc-radius less than the $\mathbf{B}_\kappa$-injectivity radius. For constants κ and c, we define two functions

$$\mathfrak{s}_\kappa(t;c),\ \mathfrak{c}_\kappa(t;c) : \left[0, 2\pi/\sqrt{\kappa^2 + c}\right] \to \mathbf{R}$$

in the following manner:

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$$\mathfrak{s}_\kappa(t;c) = \left(2/\sqrt{\kappa^2+c}\right)\ \sin\left(\sqrt{\kappa^2+c}\,t/2\right),$$
$$\mathfrak{c}_\kappa(t;c) = \cos\left(\sqrt{\kappa^2+c}\,t/2\right),$$

when $\kappa^2+c>0$,

$$\mathfrak{s}_\kappa(t;c) = t, \quad \mathfrak{c}_\kappa(t;c) = 1,$$

when $\kappa^2+c=0$, and

$$\mathfrak{s}_\kappa(t;c) = \left(2/\sqrt{|c|-\kappa^2}\right)\ \sinh\left(\sqrt{|c|-\kappa^2}\,t/2\right),$$
$$\mathfrak{c}_\kappa(t;c) = \cosh\left(\sqrt{|c|-\kappa^2}\,t/2\right),$$

when $\kappa^2+c<0$. Here, we regard $2\pi/\sqrt{\kappa^2+c}$ as infinity when $\kappa^2+c\le 0$. These functions satisfy the relation

$$(\kappa^2+c)\{\mathfrak{s}_\kappa(t;c)\}^2 + 4\{\mathfrak{c}_\kappa(t;c)\}^2 = 4.$$

Our result is the following

Theorem 1. *Let M be a complete Kähler manifold of complex dimension n whose sectional curvatures satisfy $\mathrm{Riem}_M \le c$ with some constant c. Then at an arbitrary point $p\in M$, for an arbitrary r with $0<r\le \iota_0(p)$, the volume of a trajectory-ball $B_r^\kappa(p)$ of arc-radius r for a non-trivial Kähler magnetic field $\mathbf{B}_\kappa$ is estimated from below as follows:*

$$(\omega_{2n-1})^{-1}\,\mathrm{vol}(B_r^\kappa(p))$$
$$\ge \int_0^r \{\mathfrak{s}_\kappa(t;c)\}^{2n-1}\mathfrak{c}_\kappa(t;c)\left\{1-\frac{c}{4}\{\mathfrak{s}_\kappa(t;c)\}^2\right\}^{n-1}dt,$$

where ω_{2n-1} denotes the volume of a unit sphere S^{2n-1} in $\mathbf{R}^{2n}$.

3. Trajectory-harps and proof of Theorem 1. We study trajectory-balls by studying the relationship between trajectories and geodesics. Let $\gamma:[0,T]\to M$ be a trajectory-segment for a Kähler magnetic field $\mathbf{B}_\kappa$. Here, we may take T to be either a positive constant or infinity. We suppose $\gamma(t)\ne\gamma(0)$ for all t with $0<t\le T$. We say a smooth variation of geodesics $\alpha_\gamma:[0,T]\times\mathbf{R}\to M$ to be a *trajectory-harp* associated with γ if it is defined as follows:

i) $\alpha_\gamma(t,0)=\gamma(0)$,

ii) when $t=0$, the curve $s\mapsto\alpha_\gamma(0,s)$ is the geodesic of initial vector $\dot\gamma(0)$,

iii) when $t\ne 0$, the curve $s\mapsto\alpha_\gamma(t,s)$ is the geodesic of unit speed joining $\gamma(0)$ and $\gamma(t)$.

We note that even if the trajectory-segment γ is not contained in the geodesic ball of the injectivity radius at $\gamma(0)$ centered at $\gamma(0)$ we can define a smooth variation of geodesics satisfying the above conditions. For small t $(0\le t<T_0)$, the geodesic $s\mapsto\alpha_\gamma(t,s)$ is the minimal geodesic joining $\gamma(0)$ and $\gamma(t)$. We have a smooth curve $[0,T_0)\to U_pM$ into the unit tangent space. When $\gamma(T_0)$ is not a conjugate point of $\gamma(0)$ along the geodesic of initial vector $\lim_{t\to T_0}\frac{\partial\alpha_\gamma}{\partial s}(t,0)\in T_{\gamma(0)}M$, as the exponential map $\exp_{\gamma(0)}:T_{\gamma(0)}M\to M$ is a local diffeomorphism at this vector, we can extend this curve smoothly.

In [3], the second author showed a comparison theorem on trajectory-harps by applying Rauchi's comparison theorem on Jacobi fields. We say the length $\ell_\gamma(t)$ of the geodesic segment $s\mapsto\alpha_\gamma(t,s)$ of $\gamma(0)$ to $\gamma(t)$ the *string-length* of the trajectory-harp α_γ at $\gamma(t)$. On a complex space form $\mathbf{C}M^n(c)$ of constant holomorphic sectional curvature c, which is a complex projective space $\mathbf{C}P^n(c)$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $\mathbf{C}H^n(c)$ according as c is positive, zero or negative, the function $\ell_\kappa(t;c)$ of string-length of trajectory-harps for a Kähler magnetic field $\mathbf{B}_\kappa$ is given by the relation $\mathfrak{s}_0(\ell_\kappa(t;c);c)=\mathfrak{s}_\kappa(t;c)$ for $0<t\le\pi/\sqrt{\kappa^2+c}$.

That is,

1) when $c>0$,

$$\frac{1}{\sqrt{c}}\sin\frac{1}{2}\sqrt{c}\,\ell_\kappa(t;c) = \frac{1}{\sqrt{\kappa^2+c}}\sin\frac{1}{2}\sqrt{\kappa^2+c}\,t,$$

2) when $c=0$ and $\kappa\ne 0$,

$$\ell_\kappa(t;c) = \frac{2}{|\kappa|}\sin\frac{1}{2}|\kappa|\,t,$$

3) when $c<0$,

$$\frac{1}{\sqrt{|c|}}\sinh\frac{1}{2}\sqrt{|c|}\,\ell_\kappa(t;c) = \frac{1}{\sqrt{|c|-\kappa^2}}\sinh\frac{1}{2}\sqrt{|c|-\kappa^2}\,t,$$

if $|\kappa|<\sqrt{|c|}$,

$$\frac{1}{\sqrt{|c|}}\sinh\frac{1}{2}\sqrt{|c|}\,\ell_\kappa(t;c) = \frac{t}{2},$$

if $|\kappa|=\sqrt{|c|}$, and

$$\frac{1}{\sqrt{|c|}}\sinh\frac{1}{2}\sqrt{|c|}\,\ell_\kappa(t;c) = \frac{1}{\sqrt{\kappa^2+c}}\sin\frac{1}{2}\sqrt{\kappa^2+c}\,t,$$

if $|\kappa|>\sqrt{|c|}$.

By these relations we find that the string-cosine

$d\ell_\kappa(t;c)/dt$ of trajectory-harps for $\mathbf{B}_\kappa$ on $\mathbf{C}M^n(c)$ is given as

$$\frac{d\ell_\kappa(t;c)}{dt} = \frac{\sqrt{\kappa^2+c}\,\cos\frac{1}{2}\sqrt{\kappa^2+c}\,t}{\sqrt{\kappa^2+c\cos^2\frac{1}{2}\sqrt{\kappa^2+c}\,t}},$$

when $\kappa^2+c>0$,

$$\frac{d\ell_\kappa(t;c)}{dt} = \frac{2}{\sqrt{|c|t^2+4}},$$

when $\kappa^2+c=0$, and

$$\frac{d\ell_\kappa(t;c)}{dt} = \frac{\sqrt{|c|-\kappa^2}\,\cosh\frac{1}{2}\sqrt{\kappa^2+c}\,t}{\sqrt{|c|\cosh^2\frac{1}{2}\sqrt{\kappa^2+c}\,t-\kappa^2}},$$

when $\kappa^2+c>0$. Therefore, we have

$$\frac{d\ell_\kappa(t;c)}{dt} = \sqrt{\frac{4-(\kappa^2+c)\{\mathfrak{s}_\kappa(t;c)\}^2}{4-c\{\mathfrak{s}_\kappa(t;c)\}^2}} = \frac{2\mathfrak{c}_\kappa(t;c)}{\sqrt{4-c\{\mathfrak{s}_\kappa(t;c)\}^2}}.$$

The function of string-length of a trajectory-harp is estimated from below by the function of string-length of a trajectory-harp on a complex space form.

Proposition 1 ([3]). *Let $\gamma:[0,T]\to M$ be a trajectory-segment γ for a Kähler magnetic field $\mathbf{B}_\kappa$ on a Kähler manifold M. If sectional curvatures of M satisfy $\mathrm{Riem}_M\le c$ with some constant c and γ lies in the geodesic ball $B^0_{\iota_0(\gamma(0))}(\gamma(0))$ centered at $\gamma(0)$ and of injectivity radius at $\gamma(0)$, then the function $\ell_\gamma(t)$ of string-length of the trajectory-harp associated with γ is estimated as $\ell_\gamma(t)\ge\ell_\kappa(t;c)$ for $0\le t\le\min(T,\pi/\sqrt{\kappa^2+c})$.*

Even if γ is not contained in the closure of the geodesic ball $B^0_{\iota_0(\gamma(0))}(\gamma(0))$, the argument in [3] goes through if we have a smooth trajectory-harp associated with γ. Thus, we can weaken a condition on the trajectory-segment γ.

We now show Theorem 1. Proposition 1 guarantees that the trajectory-ball $B^\kappa_r(p)$ contains the geodesic ball $B^0_{\ell_\kappa(r;c)}(p)$. Since $r\le\iota_0(p)$, we see $\ell_\kappa(r;c)\le\iota_0(p)$, hence we have

$$\mathrm{vol}(B^\kappa_r(p))\ge\mathrm{vol}(B^0_{\ell_\kappa(r;c)}(p)).$$

On the other hand, as we have $\mathrm{Riem}_M\le c$, Bishop's comparison theorem on volumes of geodesic balls shows that

$$\mathrm{vol}(B^0_\ell(p))\ge\omega_{2n-1}\int_0^\ell\{\mathfrak{s}_0(s;4c)\}^{2n-1}ds$$

for $0<\ell<\iota_0(p)$ (see [5,7], for example). Here, we note that

$$\begin{aligned}\mathfrak{s}_0(s;4c) &= \begin{cases}(1/\sqrt{c})\ \sin\sqrt{c}\,s, & \text{if } c>0,\\ s, & \text{if } c=0,\\ (1/\sqrt{|c|})\ \sinh\sqrt{|c|}\,s, & \text{if } c<0,\end{cases}\\ &= \mathfrak{s}_0(s;c)\mathfrak{c}_0(s;c)\\ &= \mathfrak{s}_0(s;c)\sqrt{1-\frac{c}{4}\{\mathfrak{s}_0(s;c)\}^2}.\end{aligned}$$

Therefore, by putting $s=\ell_\kappa(t;c)$ we obtain

$$\begin{aligned}&(\omega_{2n-1})^{-1}\,\mathrm{vol}(B^\kappa_r(p))\\ &\ge(\omega_{2n-1})^{-1}\,\mathrm{vol}(B^0_{\ell_\kappa(r;c)}(p))\\ &\ge\int_0^{\ell_\kappa(r;c)}\{\mathfrak{s}_0(s;c)\}^{2n-1}\left\{1-\frac{c}{4}(\mathfrak{s}_0(s;c))^2\right\}^{\frac{2n-1}{2}}ds\\ &=\int_0^r\{\mathfrak{s}_0(\ell_\kappa(t;c);c)\}^{2n-1}\\ &\qquad\times\left\{1-\frac{c}{4}\{\mathfrak{s}_0(\ell_\kappa(t;c);c)\}^2\right\}^{(2n-1)/2}\frac{d\ell_\kappa}{dt}\,dt\\ &=\int_0^r\{\mathfrak{s}_\kappa(t;c)\}^{2n-1}\left\{1-\frac{c}{4}\{\mathfrak{s}_\kappa(t;c)\}^2\right\}^{\frac{2n-1}{2}}\frac{d\ell_\kappa}{dt}\,dt\\ &=\int_0^r\{\mathfrak{s}_\kappa(t;c)\}^{2n-1}\left\{1-\frac{c}{4}\{\mathfrak{s}_\kappa(t;c)\}^2\right\}^{n-1}\\ &\qquad\times\left\{1-\frac{\kappa^2+c}{4}\{\mathfrak{s}_\kappa(t;c)\}^2\right\}^{1/2}dt,\end{aligned}$$

and get the assertion of Theorem 1.

When M is compact, we can give the following by making use of Gromov's comparison theorem on volumes of geodesic balls (see [7], for example).

Theorem 2. *Let M be a compact Kähler manifold of diameter R and of complex dimension n. Suppose its sectional curvatures satisfy $\mathrm{Riem}_M\le c$ with some constant c. Then at an arbitrary point $p\in M$, for an arbitrary r with $0<r\le\iota_\kappa(p)$, the volume of a trajectory-ball $B^\kappa_r(p)$ of arc-radius r for a non-trivial Kähler magnetic field $\mathbf{B}_\kappa$ is estimated from below as follows*:

$$\mathrm{vol}(B_r^{\kappa}(p)) \geq \frac{\mathrm{vol}(M)}{\displaystyle\int_0^R \{\mathfrak{s}_0(s;4c)\}^{2n-1}\,ds} \times \left(\int_0^r \{\mathfrak{s}_\kappa(t;c)\}^{2n-1}\,\mathfrak{c}_\kappa(t;c) \times \left\{1-\frac{c}{4}\{\mathfrak{s}_\kappa(t;c)\}^2\right\}^{n-1} dt\right).$$

We should note that in this theorem we can take r so that $r \leq \iota_\kappa(p)$. This point is different from Theorem 1.

For the sake of comparison, we here recall a result in [4].

Proposition 2 ([4]). *Let M be a complete Kähler manifold of complex dimension n. Suppose its sectional curvatures satisfy* $\mathrm{Riem}_M \leq c$ *with some constant c. Then at an arbitrary point $p \in M$, for an arbitrary r with $0 < r \leq \iota_\kappa(p)$, we have*

$$\mathrm{vol}(B_r^{\kappa}(p)) \geq \omega_{2n-1} \int_0^r \mathfrak{s}_\kappa(t;c)\,\mathfrak{c}_\kappa(t;c)\,\{\mathfrak{s}_\kappa(t;4c)\}^{2n-2}dt.$$

We note that the assumption on r for Proposition 2 is weaker than the assumption in Theorem 1. But we can not say clearly which estimate is sharper. When $c=0$ or when $n=1$, these two estimates are equivalent. For the function $\mathfrak{s}_\kappa(t;c)$ we have the following: If $c_1 < c_2$, we see $\mathfrak{s}_\kappa(t;c_1) > \mathfrak{s}_\kappa(t;c_2)$ for $0 < t < \pi/\sqrt{\kappa^2 + c_2}$. Also, [4] gives an estimate of volumes of trajectory-balls from above. But we can not give a similar estimate from above when $n \geq 2$, because we do not have a comparison theorem on string-length from above.

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