Entire functions sharing one finite value with their derivatives in some angular domains

By Xiao-Min LI,[*)] Xue-Feng LIU[*)] and Hong-Xun YI[**)]

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Abstract: We study the uniqueness question of transcendental entire functions sharing one finite nonzero value with their derivatives in some angular domains instead of the whole complex plane. The results in the present paper improve and extend the corresponding results from Chang-Fang [2] and extend Theorem 3 from Zheng [15]. An example is provided to show that the results in this paper, in a sense, are best possible.

Key words: Nevanlinna theory on an angular domain; entire functions; shared values; uniqueness theorems.

1. Introduction and main results. Let $f: \mathbf{C} \to \mathbf{C} \cup \{\infty\}$ be a transcendental meromorphic function, where $\mathbf{C}$ is the complex plane. We assume that the readers are familiar with the Nevanlinna theory of meromorphic functions and the standard notations such as Nevanlinna's deficiency $\delta(a, f)$ of f with respect to $a \in \mathbf{C}$ and Nevanlinna characteristic $T(r, f)$ of f. Moreover, the definitions of the lower order $\mu(f)$ and the order $\rho(f)$ can be found in Hayman [7] and Laine [10]. An $a \in \mathbf{C} \cup \{\infty\}$ is called an IM (CM) shared value in a domain $X \subseteq \mathbf{C}$ of two meromorphic functions f and g, if in X, $f(z) = a$ if and only if $g(z) = a$ ignoring multiplicities (counting multiplicities). Throughout this paper, we denote by $n(r, \mathbf{C} \setminus \overline{X}, a, f)$ the number of a-points of f in $\{z : z \in \mathbf{C} \setminus \overline{X}\} \cap \{z : |z| < r\}$, where each a-point of f in $n(r, \mathbf{C} \setminus \overline{X}, a, f)$ is counted according its multiplicity. We also denote by $\overline{n}(r, \mathbf{C} \setminus \overline{X}, a, f)$ the reduced form of $n(r, \mathbf{C} \setminus \overline{X}, a, f)$. Nevanlinna [11] proved that if two meromorphic functions f and g have five distinct IM shared values in $X = \mathbf{C}$, then $f = g$. Later on, many mathematicians in the world treated some uniqueness questions of meromorphic functions with shared values in the whole complex plane (cf. Yang-Yi [12]). In 1986, Jank-Mues-Volkmann [9] proved that if a nonconstant meromorphic function f shares a nonzero finite complex value a CM with f' and f'', then $f = f'$. Later on, Yang [13] and Chang-Fang [2] studied the uniqueness question of entire functions share one finite nonzero complex number with their derivatives, and dealt with a question posed by Yang and Yi (cf. [12, p. 398]). An example from Yang [13] shows that this question is negative. In 2004, Zheng [15] first took into the uniqueness question of meromorphic functions with shared values in some angular domains. Next, we consider q pairs of real numbers $\{\alpha_j, \beta_j\}$ and let a positive number ω such that

(1.1) $$-\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_q < \beta_q \le \pi,$$

(1.2) $$\omega = \max\left\{\frac{\pi}{\beta_1 - \alpha_1}, \frac{\pi}{\beta_2 - \alpha_2}, \ldots, \frac{\pi}{\beta_q - \alpha_q}\right\}.$$

In this paper, we will study the following question: Let f be a nonconstant meromorphic function, a a nonzero finite complex number, and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM in some angular domains instead of the whole complex plane, can we have $f^{(k)} = f^{(m)}$? In this direction, we will prove the following result:

Theorem 1.1. *Let f be a transcendental entire function with finite lower order μ such that for some $b \in \mathbf{C}$, $\delta = \delta(b, f) > 0$. Suppose that $q \ge 3$ pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfy* (1.1) *and*

(1.3) $$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}},$$

where $\sigma = \max\{\omega, \mu\}$, ω is defined as in (1.2)*, that f, $f^{(k)}$ and $f^{(m)}$ share a CM in $\overline{X} = \bigcup_{j=1}^{q}\{z : \alpha_j \le$*

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*) Department of Mathematics, Ocean University of China, Qingdao, Shandong 266100, P. R. China.
**) Department of Mathematics, Shandong University, Jinan, Shandong 250199, P. R. China.

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$\arg z \le \beta_j\}$, and that $f^{(k)}$ and $f^{(m)}$ share a IM n $\mathbf{C} \setminus \overline{X}$, where k and m are two distinct positive integers satisfying $k > m$, a is a finite nonzero complex number. If there exists some positive integer N_1 such that

$$\overline{n}(r, \mathbf{C} \setminus \overline{X}, a, f) \le N_1, \tag{1.4}$$

as $r \to \infty$, then $f^{(k)} = f^{(m)}$.

If we remove the assumption "$\mu(f) < \infty$" in Theorem 1.1, we have the following result:

Theorem 1.2. *Let f be a transcendental entire function such that for some $b \in \mathbf{C}$, $\delta = \delta(b, f) > 0$. Assume that for $q \ge 3$ radii $\arg z = \alpha_j$ $(1 \le j \le q)$ satisfying*

$$-\pi \le \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \alpha_{q+1} = \alpha_1 + 2\pi, \tag{1.5}$$

that f, $f^{(k)}$ and $f^{(m)}$ share a CM in $X = \mathbf{C} \setminus \bigcup_{j=1}^{q} \{z : \arg z = \alpha_j\}$, and that $f^{(k)}$ and $f^{(m)}$ share a IM in A_ε, where k and m are two distinct positive integers satisfying $k > m$, a is a finite nonzero complex number, and A_ε is the union of the sectors

$$A_\varepsilon = \bigcup_{j=1}^{q} \{z : |\arg z - \alpha_j| < \varepsilon\} \tag{1.6}$$

for $\varepsilon > 0$. If there exists some positive integer N_2 such that $\overline{n}(r, n(r, A_\varepsilon, a, f) \le N_2$, then $f^{(k)} = f^{(m)}$.

The following example shows that the assumption "$\delta = \delta(b, f) > 0$" in Theorems 1.1 and 1.2, the assumption (1.4) in Theorems 1.1 and the assumption "$\overline{n}(r, n(r, A_\varepsilon, a, f) \le N_2$" in Theorems 1.2 can not be simply dropped:

Example 1.1 ([15, Remark A]). For each real number a satisfying $0 \le a \le 1$, we let $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = a$, where $z = x + yi$ and $x, y \in \mathbf{R}$. Then we have $e^{-y} \cos x = a$ and $e^{-y} \sin x = \sqrt{1 - a^2}$, and so $e^{-2y} = 1$, which implies $y = 0$. Hence $z = x$ is a real number. Similarly, if $\cos z = \frac{e^{iz} + e^{-iz}}{2} = a$, where $z = x + yi$ and $x, y \in \mathbf{R}$, then we can deduce $y = 0$ and so $z = x$ is also a real number. Therefore, for each real number a satisfying $0 \le a \le 1$, $f(z) = \sin z$ and $f'(z)$ and $f^{(4)}(z)$ can take over a only on the real axis, and so they share a CM in the domain $X = \mathbf{C} \setminus \mathbf{R}$. Obviously, $\omega = \rho(\sin z) = \rho(\cos z) = 1$ and $A_\varepsilon = \{z : |\arg z + \pi| < \varepsilon\} \cup \{z : |\arg z| < \varepsilon\}$ for $\varepsilon > 0$. Moreover, $\delta(b, \sin z) = 0$ for all $b \in \mathbf{C}$, and that $\overline{n}(r, A_\varepsilon, a, f) \to \infty$, as $r \to \infty$. But $f' \not\equiv f^{(4)}$.

2. Preliminaries. In this section, we introduce some important lemmas to prove the main results in this paper.

First, we introduce the Nevanlinna theory on an angular domain, which can be found in Goldberg-Ostrovskii [5, pp. 23–26]:

Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \le \arg z \le \beta\}$, where $\alpha, \beta \in [0, 2\pi]$ and so $0 \le \beta - \alpha < 2\pi$. Following Goldberg-Ostrovskii [5, pp. 23–26], we define

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right) (\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|) \frac{dt}{t},$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta \quad \text{and} \tag{2.1}$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{1 < |b_m| < r} \left(\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{r^{2\omega}}\right) \sin \omega(\theta_m - \alpha), \tag{2.2}$$

where $\omega = \pi/(\beta - \alpha)$, $1 \le r < +\infty$ and $b_m = |b_m| e^{i\theta_m}$ are the poles of f on $\overline{\Omega}(\alpha, \beta)$ appearing often according to their multiplicities. The function $C_{\alpha,\beta}$ is called the angular counting function of the poles of f on $\overline{X}(\alpha, \beta)$ and the Nevanlinna angular characteristic function is defined as $S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f)$. Similarly, for any finite value a, we define $A_{\alpha,\beta}(r, f_a)$, $B_{\alpha,\beta}(r, f_a)$, $C_{\alpha,\beta}(r, f_a)$ and $S_{\alpha,\beta}(r, f_a)$, where $f_a = 1/(f - a)$. For the sake of simplicity, next we omit the subscript of all the above notations and respectively use the notations $A(r, f_a)$, $B(r, f_a)$, $C(r, f_a)$ and $S(r, f_a)$ instead of $A_{\alpha,\beta}(r, f_a)$, $B_{\alpha,\beta}(r, f_a)$, $C_{\alpha,\beta}(r, f_a)$ and $S_{\alpha,\beta}(r, f_a)$ for any finite complex value a.

Lemma 2.1 ([5, pp. 23–26] and [5, Theorem 3.1]). *Let f be meromorphic on $\overline{\Omega}(\alpha, \beta)$. Then, for an arbitrary complex number $a \in \mathbf{C}$ and any integer $k \ge 0$ we have*

$$S\left(r, \frac{1}{f - a}\right) = S(r, f) + O(1),$$

$$S(r, f^{(k)}) \le 2^k S(r, f) + R(r, f)$$

and $A(r, \frac{f^{(k)}}{f}) + B(r, \frac{f^{(k)}}{f}) = R(r, f)$, where and in what follows, $R(r, f)$ is such a quantity that if $\rho(f) < \infty$, then $R(r, f) = O(1)$, as $r \to \infty$, if $\rho(f) = \infty$, then $R(r, f) = O(\log(rT(r, f)))$, as $r \notin E$ and $r \to \infty$, here and in what follows, E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context.

Lemma 2.2 ([5, p. 112, Theorem 3.3]). *Let f be a meromorphic function on $\overline{\Omega}(\alpha,\beta)$. Then, for arbitrary q distinct values $a_j \in \mathbf{C}\cup\{\infty\}$ $(1\le j\le q)$,*

$$(q-2)S(r,f) \le \sum_{j=1}^{q} \overline{C}\left(r,\frac{1}{f-a_j}\right) + R(r,f),$$

where $\overline{C}(r,\frac{1}{f-a_j})$ is the reduced form of $C(r,\frac{1}{f-a_j})$.

Lemma 2.3 ([4, 14]). *Let f be a transcendental meromorphic function in $\mathbf{C}$ with the lower order $0\le\mu<\infty$ and the order $0<\rho<\infty$. Then, for an arbitrary positive number σ satisfying $\mu\le\sigma\le\rho$ and a set E with finite linear measure, there exist a sequence of positive numbers $\{r_n\}$ such that*

(i) $r_n\notin E$ and $\lim_{n\to\infty}\frac{r_n}{n}=\infty$, (ii) $\liminf_{r\to\infty}\frac{\log T(r_n,f)}{\log r_n}\ge\sigma$ and (iii) $T(t,f)<(1+o(1))(\frac{t}{r_n})^{\sigma}T(r_n,f)$.

A sequence $\{r_n\}$ in Lemma 2.3 is called a Pólya peak of order σ outside E in this paper. For $r>0$ and $a\in\mathbf{C}$, we define (iv) $D(r,a) := \{\theta\in[-\pi,\pi) : \log^+\frac{1}{|f(re^{i\theta})-a|} > \frac{1}{\log r}T(r,f)\}$ *and* $D(r,\infty) := \{\theta\in[-\pi,\pi) : \log^+|f(re^{i\theta})|\,\frac{1}{\log r}T(r,f)\}$.

Lemma 2.4 ([1]). *Let f be a transcendental meromorphic function in $\mathbf{C}$ with the finite lower order μ and the order $0<\rho\le\infty$, and for some $a\in\mathbf{C}\cup\{\infty\}$, $\delta(a,f)=\delta>0$. Then, for an arbitrary Pólya peak $\{r_n\}$ of order $\sigma>0$, $\mu\le\sigma\le\rho$, we have*

$$\liminf_{n\to\infty}\operatorname{mes}D(r_n,a)\ge\min\left\{2\pi,\frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\}.$$

Remark 2.1. Lemma 2.4 was proved in [1] for the Pólya peak of order μ, the same argument of Baernstein [1] can derive Lemma 2.4 for the Pólya peak of order σ with $\mu\le\sigma\le\rho$.

Lemma 2.5 ([8]). *Let f be a transcendental meromorphic function in $\mathbf{C}$. Then, for each $K>1$, there exists a set $M(K)$ of the lower logarithmic density at most $d(K)=1-(2e^{K-1}-1)^{-1}>0$, that is*

$$\underline{\log\operatorname{dens}}\,M(K)=\liminf_{r\to\infty}\frac{1}{\log r}\int_{M(K)\cap[1,r]}\frac{dt}{t}\le d(K),$$

such that, for every positive integer k,

$$\limsup_{\substack{r\to\infty\\ r\notin M(K)}}\frac{T(r,f)}{T(r,f^{(k)})}\le 3eK.$$

Lemma 2.6 ([3]). *Let f be a meromorphic function with $\delta(\infty,f)=\delta>0$. Then, given $\varepsilon>0$,*

$$\operatorname{mes}E(r,f)>\frac{1}{(T(r,f))^{\varepsilon}(\log r)^{1+\varepsilon}}\quad(r\notin F),$$

where

$$E(r,f)=\left\{\theta\in[-\pi,\pi):\log^+|f(re^{i\theta})|>\frac{\delta}{4}T(r,f)\right\}$$

and that F is a set of positive real numbers with finite logarithmic measure depending on ε.

3. Proof of Theorems.

Proof of Theorem 1.1. Suppose that $f^{(k)}\not\equiv f^{(m)}$. Set

$$\varphi=\frac{f^{(k)}-f^{(m)}}{f-a}. \tag{3.1}$$

Then, $\varphi\not\equiv 0$. Moreover, by (3.1), Lemma 2.1 and the assumption that f, $f^{(k)}$ and $f^{(m)}$ share a CM in $\overline{X}=\bigcup_{j=1}^{q}\{z:\alpha_j\le\arg z\le\beta_j\}$, we have

$$S_{\alpha_j,\beta_j}(r,\varphi)=A_{\alpha_j,\beta_j}\left(r,\frac{f^{(k)}-f^{(m)}}{f-a}\right)+B_{\alpha_j,\beta_j}\left(r,\frac{f^{(k)}-f^{(m)}}{f-a}\right)\le R_{\alpha_j,\beta_j}(r,f),\ 1\le j\le q. \tag{3.2}$$

Set

$$h_1=\frac{f^{(k)}-a}{f-a}\ \text{ and }\ h_2=\frac{f^{(m)}-a}{f-a}. \tag{3.3}$$

Then, by (3.1) and (3.3), we have $h_1-h_2=\varphi$, and so

$$\frac{h_1}{\varphi}-\frac{h_2}{\varphi}=1. \tag{3.4}$$

Thus, by (3.2), (3.4), Lemmas 2.1 and 2.2, we have

$$S_{\alpha_j,\beta_j}(r,h_1)\le R_{\alpha_j,\beta_j}(r,f),\quad 1\le j\le q, \tag{3.5}$$

$$S_{\alpha_j,\beta_j}(r,h_2)\le R_{\alpha_j,\beta_j}(r,f),\quad 1\le j\le q. \tag{3.6}$$

By (3.3) we have

$$\frac{1}{f-a}=\frac{1}{a}\left(\frac{f^{(k)}}{f-a}-h_1\right),\qquad \frac{1}{f-a}=\frac{1}{a}\left(\frac{f^{(m)}}{f-a}-h_2\right). \tag{3.7}$$

By (3.5), Lemma 2.1 and the left equality of (3.7), we deduce

$$A_{\alpha_j,\beta_j}\left(r,\frac{1}{f-a}\right)+B_{\alpha_j,\beta_j}\left(r,\frac{1}{f-a}\right)=R_{\alpha_j,\beta_j}(r,f) \tag{3.8}$$

for $1\le j\le q$.

If $b\ne a$, by the left equality of (3.1), we have

$$(3.9)\qquad \frac{1}{f-b}=\frac{1}{(a-b)\varphi}\cdot\left(\frac{f^{(k)}}{f-a}-\frac{f^{(k)}}{f-b}\right) -\frac{1}{(a-b)\varphi}\cdot\left(\frac{f^{(m)}}{f-a}-\frac{f^{(m)}}{f-b}\right).$$

By (3.8), (3.9) and Lemma 2.1, we deduce

$$(3.10)\qquad A_{\alpha_j,\beta_j}\left(r,\frac{1}{f-b}\right)+B_{\alpha_j,\beta_j}\left(r,\frac{1}{f-b}\right) \leq O(\log r+\log T(r,f)),\quad 1\leq j\leq q,$$

as $r\notin E$ and $r\to\infty$. Now we prove

$$(3.11)\qquad \rho(f)\leq\omega.$$

Suppose that (3.11) does not hold. Then

$$(3.12)\qquad \rho(f)>\omega.$$

We consider the following two cases:

Case 1. Suppose that $\rho(f)>\mu(f)$. Then, by the fact $\sigma=\max\{\omega,\mu\}$ we have

$$(3.13)\qquad \rho(f)>\sigma\geq\mu(f).$$

By (1.3), we can find some sufficiently small positive number ε such that

$$(3.14)\qquad \sum_{j=1}^{q}(\alpha_{j+1}-\beta_j)+2\varepsilon<\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}}$$

and

$$(3.15)\qquad \rho(f)>\sigma+2\varepsilon>\mu(f).$$

Applying Lemma 2.3 to f, we can find that there exists a Pólya peak of order $\sigma+2\varepsilon$ outside E. Combining this with Lemma 2.4 and

$$(3.16)\qquad \sigma+2\varepsilon\geq\omega+2\varepsilon\geq\omega_j+2\varepsilon\geq 1+2\varepsilon,$$

we have

$$(3.17)\qquad \operatorname{meas}D(r_n,b)\geq\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}}-\varepsilon.$$

Without loss of generality, we can assume that (3.16) holds for all the n. Set

$$(3.18)\qquad K_n=\operatorname{meas}\left(D(r_n,b)\cap\bigcup_{j=1}^{q}(\alpha_j+\varepsilon,\beta_j-\varepsilon)\right).$$

Then, by (3.14), (3.17) and (3.18), we have

$$(3.19)\qquad \begin{aligned} K_n &\geq \operatorname{meas}D(r_n,b) - \operatorname{meas}\left([0,2\pi)\setminus\bigcup_{j=1}^{q}(\alpha_j+\varepsilon,\beta_j-\varepsilon)\right)\\ &= \operatorname{meas}D(r_n,b) - \operatorname{meas}\left(\bigcup_{j=1}^{q}(\beta_j-\varepsilon,\alpha_{j+1}+\varepsilon)\right)\\ &= \operatorname{meas}D(r_n,b)-\sum_{j=1}^{q}(\alpha_{j+1}-\beta_j+2\varepsilon)\geq\varepsilon. \end{aligned}$$

By (3.18) and (3.19), we can find that there exists some positive integer j_0 satisfying $1\leq j_0\leq q$ such that for infinitely many positive integers n,

$$(3.20)\qquad \operatorname{meas}\bigl(D(r_n,b)\cap(\alpha_{j_0}+\varepsilon,\beta_{j_0}-\varepsilon)\bigr)\geq\frac{K_n}{q}>\frac{\varepsilon}{q}.$$

Without loss of generality, we can assume that (3.20) holds for all the positive integers n. Next we set $E_n=D(r_n,b)\cap(\alpha_{j_0}+\varepsilon,\beta_{j_0}-\varepsilon)$. Then, from (3.20) and the definition of $D(r,b)$ in (iv) of Lemma 2.3 we have

$$(3.21)\qquad \int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|f(r_ne^{i\theta})-b|}\,d\theta \geq\frac{T(r_n,f)}{\log r_n}\operatorname{meas}E_n>\frac{\varepsilon}{q}\frac{T(r_n,f)}{\log r_n}.$$

On the other hand, by (3.8), (3.10), Lemma 2.1, Lemma 2.5 and the definition of $B_{\alpha,\beta}(r,f)$ in (2.1), we have

$$(3.22)\qquad \begin{aligned} \int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|f(r_ne^{i\theta})-b|}\,d\theta &\leq\frac{\pi}{2\omega_{j_0}\sin(\varepsilon\omega_{j_0})}r_n^{\omega_{j_0}}B_{\alpha_{j_0},\beta_{j_0}}\left(r_n,\frac{1}{f(r_ne^{i\theta})-b}\right)\\ &\leq K_{j_0,\varepsilon}r_n^{\omega_{j_0}}\log(r_nT(r_n,f))\\ &=K_{j_0,\varepsilon}r_n^{\omega_{j_0}}(\log r_n+\log T(r_n,f))+O(1), \end{aligned}$$

where $r_n\notin E$ and $\omega_{j_0}=\frac{\pi}{\beta_{j_0}-\alpha_{j_0}}$, $K_{j_0,\varepsilon}$ is a positive constant depending only on j_0 and ε. From (3.21) and (3.22) we have

$$(3.23)\qquad \log T(r_n,f)\leq\log\log T(r_n,f)+\omega_{j_0}\log r_n +3\log\log r_n+O(1),$$

where $r_n\notin E$ and $r_n\to\infty$. Noting that $\{r_n\}$ is a Pólya peak of order $\sigma+2\varepsilon$ of f outside E, we can get from (3.23) that

$$\sigma+2\varepsilon\leq\lim_{r_n\to\infty}\frac{\log T(r_n,f)}{\log r_n}\leq\omega_{j_0}\leq\omega,$$

which contradicts the assumption $\sigma=\max\{\omega,\mu\}$.

Case 2. Suppose that $\rho(f)=\mu(f)$. Then, by the same argument as in Case 1 with all $\sigma+2\varepsilon$ replaced with $\sigma=\mu(f)=\rho(f)$, we can derive $\rho(f)=$

$\sigma \le \omega$, which contradicts (3.12). This completes the proof of (3.11).

Finally, we will complete the proof of Theorem 1.1. First of all, by (3.1) and (1.4), we have $\varphi = \frac{h_3}{P_1}$, where h_3 is an entire function, P_1 is a nonzero polynomial. Then, by (3.1), (3.11), Corollary 3 from Gundersen [6] and the assumptions of Theorem 1.1, there exists arbitrarily large positive number r such that for any $\varepsilon > 0$, we deduce

$$(3.24)\quad |h_3(z)| \le |P_1(z)| \cdot \left(\left|\frac{f^{(k)}(z)}{f(z)-a}\right| + \left|\frac{f^{(m)}(z)}{f(z)-a}\right|\right) \le r^{\deg(P_1)+k\omega-k+k\varepsilon}(1+o(1)),$$

as $r = |z| \notin E$ and $r \to \infty$, where $E \subset (0,+\infty)$ is a subset that has finite linear measure. By (3.24) we can see that h_3 is a polynomial of degree not more than $\deg(P_1) + k\omega - k$. Combining this with (3.1), (3.24) and the assumption that $f^{(k)}$ and $f^{(m)}$ share a IM in $\mathbf{C} \setminus \overline{X}$, we deduce

$$(3.25)\quad \overline{n}(r, \mathbf{C} \setminus \overline{X}, a, f^{(k)}) + \overline{n}(r, \mathbf{C} \setminus \overline{X}, a, f^{(m)}) \le N_3,$$

where N_3 is some positive integer. Suppose that $\delta(a,f) > 0$. Then, by (3.11), the left equality of (3.7), and the lemma of logarithmic derivative (cf. [10, Corollary 2.3.4]), we can see that there exists some subset $I \subset (0,\infty)$ with linear measure $\operatorname{mes} I = \infty$, and there exist some infinite sequences of positive numbers $\{r_n\} \subset I$ such that

$$(3.26)\quad T(r_n, f) \le (\delta(a,f)+\varepsilon) m\left(r_n, \frac{1}{f-a}\right) \le (1+\varepsilon)T(r_n, h_1) + O(\log r_n),$$

as $r_n \notin E$, $r_n \in I$ and $r_n \to \infty$. By (3.26), Theorem 1.5 [12] and the assumption that f is a transcendental meromorphic function, we deduce that h_1 is a transcendental meromorphic function. Similarly, we can deduce that h_2 is also a transcendental meromorphic function. Therefore, by (3.3), (3.25) and the assumption that f, $f^{(k)}$ and $f^{(m)}$ share a CM in $\overline{X}$, we deduce

$$(3.27)\quad N(r, h_j) + N\left(r, \frac{1}{h_j}\right) = o(T(r, h_j)), \quad j = 1, 2.$$

By $\varphi = h_3/P_1 \not\equiv 0$, we know that φ is a nonzero function. Therefore, by (3.4), (3.27) and Theorem 1.62 [12] we can get a contradiction. Suppose that $\delta(b,f) > 0$. Then, by (3.9) and the lemma of logarithmic derivative, we can find that there exists some subset $I \subset (0,\infty)$ with linear measure $\operatorname{mes} I = \infty$, and there exist some infinite sequences of positive numbers $\{r_n\} \subset I$ such that $T(r_n, f) \le O(\log r_n)$, as $r_n \notin E$, $r_n \in I$ and $r_n \to \infty$. This implies that f is a rational function, which is impossible. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Suppose that $f^{(k)} \not\equiv f^{(m)}$. Then, in the same manner as in Case 1 of the proof of Theorems 1.1, we have from Lemma 2.1 that

$$(3.28)\quad B_{\alpha_j,\alpha_{j+1}}\left(r, \frac{1}{f-b}\right) \le R_{\alpha_j,\beta_j}(r,f) \le O(\log r + \log T(r,f)), \quad 1 \le j \le q,$$

as $r \notin E$ and $r \to \infty$. Next we prove $\mu(f) < \infty$. Indeed, for the exceptional set F in Lemma 2.6 and the exceptional set E in (3.22), we have

$$\overline{\log \operatorname{dens}}(F \cup E) = 0,$$

and so for $M(K)$ in Lemma 2.5, where $K \ge 2$ is a positive number, we have

$$\underline{\log \operatorname{dens}}(M(K) \cup F \cup E) \le \underline{\log \operatorname{dens}}(M(K)) \le d(K),$$

where $d(K) = 1 - (2e^{K-1} - 1)^{-1}$. Applying this and Lemma 2.6 to f, we can find that there exist a sequence of positive numbers $r_n \notin M(K) \cup F \cup E$ such that

$$(3.29)\quad \operatorname{meas} E\left(r_n, \frac{1}{f-b}\right) > \frac{1}{(T(r_n,f))^{\varepsilon}(\log r_n)^{1+\varepsilon}},$$

as $r_n \to \infty$. Set

$$(3.30)\quad \varepsilon_n = \frac{1}{2q+1}\frac{1}{(T(r_n,f))^{\varepsilon}(\log r_n)^{1+\varepsilon}}.$$

Then, by (3.29) and (3.30) we have

$$\operatorname{meas}\left(E\left(r_n, \frac{1}{f-b}\right) \cap \bigcup_{j=1}^{q}(\alpha_j+\varepsilon_n, \alpha_{j+1}-\varepsilon_n)\right) \ge \operatorname{meas} E\left(r_n, \frac{1}{f-b}\right) - \operatorname{meas}\left(\bigcup_{j=1}^{q}(\alpha_j-\varepsilon_n, \alpha_j+\varepsilon_n)\right) > (2q+1)\varepsilon_n - 2q\varepsilon_n = \varepsilon_n.$$

This implies that there exists some j_0 satisfying $1 \le j_0 \le q$ such that

$$(3.31)\quad \operatorname{meas}\left(E\left(r_n, \frac{1}{f-b}\right) \cap (\alpha_{j_0}+\varepsilon_n, \alpha_{j_0+1}-\varepsilon_n)\right) \ge \frac{\varepsilon_n}{q}.$$

Without loss of generality, we can assume that

(3.31) holds for all n. Next, we set

$$(3.32)\quad \tilde{E}_n = E\left(r_n, \frac{1}{f-b}\right) \cap \left(\alpha_{j_0} + \varepsilon_n, \alpha_{j_0+1} - \varepsilon_n\right).$$

By (3.32) and the definition of $\tilde{E}(r_n, \frac{1}{f-b})$, we have

$$(3.33)\quad \int_{\alpha_{j_0}+\varepsilon_n}^{\alpha_{j_0+1}-\varepsilon_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - b|}\, d\theta \geq$$

$$\int_{\tilde{E}_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - b|}\, d\theta \geq \operatorname{meas}(\tilde{E}_n)\, \frac{\delta(b,f)}{4}\, T(r_n, f)$$

$$\geq \frac{\varepsilon_n \delta(b,f)}{4q}\, T(r_n, f).$$

On the other hand, by (3.28), Lemma 2.1, Lemma 2.5 and the definition of $B_{\alpha,\beta}(r,f)$ in (2.1), we have

$$(3.34)\quad \int_{\alpha_{j_0}+\varepsilon_n}^{\alpha_{j_0+1}-\varepsilon_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - b|}\, d\theta$$

$$\leq \frac{\pi r_n^{\omega_{j_0}}}{2\omega_{j_0} \sin(\varepsilon_n \omega_{j_0})}\, B_{\alpha_{j_0}, \alpha_{j_0+1}}\left(r_n, \frac{1}{f(r_n e^{i\theta}) - b}\right)$$

$$\leq \tilde{K}_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n + \log T(r_n, f)),$$

as $r_n \notin M(K) \cup F \cup E$ and $r_n \to \infty$, where $\omega_{j_0} = \frac{\pi}{\alpha_{j_0+1} - \alpha_{j_0}}$, $\tilde{K}_{j_0,\varepsilon}$ is a positive constant depending only on j_0 and ε. By (3.33) and (3.34), we have

$$(3.35)\ \delta(b,f)(T(r_n,f))^{1-\varepsilon}$$

$$\leq 4q(2q+1)\tilde{K}_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n)^{1+\varepsilon} \log r_n$$

$$+ 4q(2q+1)\tilde{K}_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n)^{1+\varepsilon} \log T(r_n, f)$$

$$+ O(1),$$

as $r_n \notin M(K) \cup F \cup E$ and $r_n \to \infty$. By (3.35) we derive $\mu(f) \leq \omega_{j_0} \leq \omega$, which implies $\mu(f) < \infty$. Therefore, by (3.28) and in the same manner as in the proof of Theorem 1.1 we can get the conclusion of Theorem 1.2. □

4. Concluding remarks. Regarding Theorem 3 from Zheng [15], one may ask: Can we find some additional assumptions replaced with the assumption "$\rho(f) > \omega$" in Theorem 3 from Zheng [15], so as to make the conclusion of Theorem 3 from Zheng [15] hold for $q \geq 2$?

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