On variations of the Liouville constant which are also Liouville numbers

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Abstract: Let ℓ be the Liouville's constant, defined as a decimal with a 1 in each decimal place corresponding to $n!$ and 0 otherwise. This number is a classical example of a Liouville number. In this note, we give an optimal condition on the number of replacements of 0's by 1's between two consecutive 1's in the decimal expansion of ℓ in order to ensure that this new number is still a Liouville number.

Key words: Liouville numbers; Diophantine numbers; continued fraction.

1. Introduction. A real number ξ is called a *Liouville number*, if there exist a sequence of rational numbers $(p_k/q_k)_k$, with $q_k > 1$, and a sequence of positive real numbers $(\omega_k)_k$ such that $\limsup_{k\to\infty} \omega_k = +\infty$ and

$$0 < \left|\xi - \frac{p_k}{q_k}\right| < q_k^{-\omega_k}, \text{ for all } k = 1, 2, 3, \ldots.$$

The most classical example of a Liouville number is the *Liouville's constant* ℓ (see [1]), defined as a decimal with a 1 in each decimal place corresponding to $n!$ and 0 otherwise. It can be represented by the fast convergent series $\ell = \sum_{n=1}^{\infty} 10^{-n!} = 0.1100010\ldots$.

An irrational real number ξ is said to be *Diophantine* if there exist positive constants C and κ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{C}{q^{\kappa}}$$

for all rational numbers p/q, with $q \geq 1$. These numbers are very important in complex dynamics because of their failure to be closely approximated by rational numbers. In particular, an irrational number is a Liouville number if and only if it is not Diophantine.

The question which motivated this note is to see "how many" replacements of 0's by 1's one can make between two consecutive 1's in the decimal expansion of ℓ, in order to guarantee that this new number is still a Liouville number.

In order to make this more precise, let S be a subset of positive integer numbers. We define ℓ^S as the number obtained by replacing, for all $i \in S$, the number in the i-th position in the decimal expansion of ℓ, say s, by $1 - s$. For example, $\ell^{\mathcal{F}} = 0$ (where $\mathcal{F}$ is the set of factorial numbers), and $\ell^{\mathbf{N}} = 0.001110111111\ldots = 1/9 - \ell$ which is a Liouville number (since the difference between a Liouville number and a rational number is always a Liouville number). By using a similar argument, if S is finite, then ℓ^S is also a Liouville number.

In this note, we give the optimal condition on the cardinality of $S \cap (n!, (n+1)!)$ such that ℓ^S is a Liouville number. More precisely, we have

Theorem 1.1. *Let S be a subset of positive integer numbers which does not contain factorials. If*

$$\liminf_{n\to\infty} \frac{\#(S \cap (n!, (n+1)!))}{\log n} = 0,$$

then ℓ^S is a Liouville number. Moreover, this result is the best possible, in the sense that for any $\epsilon > 0$, there exists a set S with

$$\#(S \cap (n!, (n+1)!)) < \epsilon \log n \qquad (n > 1)$$

and such that ℓ^S is a Diophantine number (in particular, it is not a Liouville number).

2. The proof. By hypothesis, we can write $S \cup \mathcal{F} = \{y_1, y_2, \ldots\}$ with $y_1 < y_2 < \cdots$. Set $k_n = \#(S \cap (n!, (n+1)!))$ and (when $k_n \geq 1$) suppose that $n! < x_1^{(n)} < \cdots < x_{k_n}^{(n)} < (n+1)!$, where $x_i^{(n)} \in S$ (in the case of $k_n = 0$, we make the convention that the set $\{x_1^{(n)}, \ldots, x_{k_n}^{(n)}\}$ is empty). Define $x_0^{(n)} = n!$ and $x_{k_n+1}^{(n)} = (n+1)!$, then there exists $j_n \in [1, k_n + 1]$ such that $x_{j_n}^{(n)}/x_{j_n-1}^{(n)} = \max_{1\leq i\leq k_n+1}\{x_i^{(n)}/x_{i-1}^{(n)}\}$. Then, we have

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$$n+1 = \frac{x_1^{(n)}}{x_0^{(n)}} \cdot \frac{x_2^{(n)}}{x_1^{(n)}} \cdots \frac{x_{k_n+1}^{(n)}}{x_{k_n}^{(n)}} \leq \left(\frac{x_{j_n}^{(n)}}{x_{j_n-1}^{(n)}}\right)^{k_n+1}.$$

Therefore $\omega_n := x_{j_n}^{(n)}/x_{j_n-1}^{(n)} \geq (n+1)^{1/(k_n+1)}$. Moreover, we suppose that $x_{j_n}^{(n)} = y_{\ell_n}$ and $x_{j_n-1}^{(n)} = y_{\ell_n-1}$.

Now, for all $n \geq 1$, define integers

$$q_n = 10^{x_{j_n-1}^{(n)}} \text{ and } p_n = \sum_{i=1}^{\ell_n-1} 10^{x_{j_n-1}^{(n)} - y_i}.$$

Thus

$$0 < \ell^S - \frac{p_n}{q_n} = \frac{1}{10^{x_{j_n}^{(n)}}}\left(1 + \sum_{i=\ell_n+1}^{\infty} \frac{1}{10^{y_i - y_{\ell_n}}}\right) < \frac{1.2}{10^{x_{j_n}^{(n)}}} = \frac{1.2}{q_n^{\omega_n}}.$$

By using that $\liminf k_n/\log n = 0$, we get $\limsup \omega_n = \infty$. This implies that ℓ^S is a Liouville number as desired.

For the second part, for $\epsilon > 0$, choose an integer $k \geq 3$ such that $2/\log k < \epsilon$. For any $n > 0$, define $a_n^{(j)} = k^j n!$, for $j \in [0, s_n]$ and $a_n^{(s_n+1)} = (n+1)!$, where s_n is the largest integer such that $k^{s_n} n! < (n+1)!$, that is, $s_n < \log(n+1)/\log k \leq 2\log n/\log k < \epsilon \log n$, for $n > 1$. Consider the set $S := \{a_n^{(j)} : n > 0 \text{ and } j \in [0, s_n]\} \backslash \mathcal{F}$, then

$$\#(S \cap (n!, (n+1)!)) = s_n < \epsilon \log n.$$

Now, we shall prove that ℓ^S is a Diophantine number. In fact, note that

$$\ell^S = \sum_{n=1}^{\infty} \sum_{j=0}^{s_n} \frac{1}{10^{a_n^{(j)}}}.$$

For $j < s_n$, let $U_{n,j}/10^{a_n^{(j)}}$ be the truncation of the above series with denominator $10^{a_n^{(j)}}$ (observe that this fraction is irreducible). Then

$$\ell^S - \frac{U_{n,j}}{10^{a_n^{(j)}}} < \frac{1.2}{10^{a_n^{(j+1)}}} = \frac{1.2}{10^{k a_n^{(j)}}} \leq \frac{1}{2(10^{a_n^{(j)}})^2},$$

where we used that $k \geq 3$. Hence $U_{n,j}/10^{a_n^{(j)}}$ is a convergent, say $p_{r(n,j)}/q_{r(n,j)}$, of the continued fraction of $\ell^S = [b_0; b_1, b_2, \ldots]$. Note that $q_{r(n,j)} = 10^{a_n^{(j)}}$. Also, for $r(n,j-1) < m \leq r(n,j)$, it holds that

$$b_m \leq \prod_{i=r(n,j-1)+1}^{r(n,j)} b_i < \frac{q_{r(n,j)}}{q_{r(n,j-1)}} = q_{r(n,j-1)}^{k-1} \leq q_{m-1}^{k-1}.$$

Thus, for any convergent p_m/q_m of the continued fraction of ℓ^S, we have

$$\left|\ell^S - \frac{p_m}{q_m}\right| > \frac{1}{3b_{m+1}q_m^2} \geq \frac{1}{3q_m^{k+1}}. \tag{1}$$

Aiming for a contradiction, suppose that ℓ^S is not a Diophantine number, then there exists a rational number p/q, with $q \geq 3$ and such that

$$\left|\ell^S - \frac{p}{q}\right| < \frac{1}{q^{k+2}}. \tag{2}$$

Since $1/q^{k+2} \leq 1/(2q^2)$, then p/q must be a convergent of the continued fraction of ℓ^S. Therefore, by combining (1) and (2), we arrive at the contradiction that $q < 3$. The proof is complete. □

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References

[1] J. Liouville, Nouvelle démonstration d'un théorème sur les irrationnelles algébriques, inséré dans le Compte rendu de la dernière séance, C. R. Acad. Sci. Paris **18** (1844), 910–911.