Hardy's inequality on Hardy spaces

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Abstract: We extend the Hardy inequalities to the classical Hardy spaces and the rearrangement-invariant Hardy spaces.

Key words: Hardy's inequality; Hardy space; rearrangement-invariant; atomic decomposition; interpolation.

1. Introduction. The main theme of this paper is the Hardy inequalities on rearrangement-invariant Hardy spaces including the classical Hardy spaces, the Hardy-Lorentz spaces and the Hardy-Orlicz spaces.

The Hardy inequality is one of the important inequalities in analysis. It is a crucial tool in real interpolation theory [2] and its high dimension generalization provides inspiration on the Hardy inequality for Sobolev functions.

It is impossible to give a detailed review on Hardy's inequality in this short paper, the reader is referred to [4,19,24] for a detailed reference for Hardy's inequality and its applications on analysis.

One of the extensions on the Hardy inequality is the validity of the Hardy inequalities on some non-Lebesgue spaces. For instance, we have the Hardy inequalities on rearrangement-invariant Banach function spaces in [20].

The Hardy inequalities on the Morrey spaces built on rearrangement-invariant Banach function spaces are obtained [13]. In addition, the Hardy inequalities on block spaces are given in [14].

We have the Hardy inequalities on Lebesgue spaces with variable exponents in [3,8,9,21,25,26].

The Hardy inequalities on the Hardy-Morrey spaces, Hardy-Morrey spaces with variable exponents and weak Hardy-Morrey spaces are presented in [16–18], respectively.

In this paper, we extend the Hardy inequalities to the classical Hardy spaces and the rearrangement-invariant Hardy spaces in the form given in [3] and [19, p. 6] which are generalizations of the Hardy inequalities in [13,17,18].

We use the atomic decompositions of Hardy spaces to obtain the Hardy inequalities on the classical Hardy spaces. With these Hardy inequalities, the Hardy inequalities on rearrangement-invariant Hardy space are established by using the interpolation functor introduced in [15].

2. Hardy's inequality. We establish the Hardy inequalities on the classical Hardy spaces in this section. We begin with the Hardy operator used in this paper.

Let $\mathbf{Z}_-$ denote the set of non-positive integers. For any $\mu \in \mathbf{R}$ and $\alpha \in \mathbf{Z}_-$, write

$$T_{\alpha,\mu} f(x) = x^{\alpha+\mu-1} \int_0^x \frac{f(y)}{y^\alpha}\, dy.$$

We present the main result of this paper in the following theorem.

Theorem 2.1. *Let $0 < p \le 1$ and $0 \le \mu < 1$ and $\alpha \in \mathbf{Z}_-$. If*

$$\frac{1}{p} = \frac{1}{r} + \mu,$$

then there exists a constant $C > 0$ such that for any $f \in H^p(\mathbf{R})$ with $\operatorname{supp} f \subseteq [0,\infty)$,

$$\|T_{\alpha,\mu} f\|_{L^r(0,\infty)} \le C\|f\|_{H^p(\mathbf{R})}.$$

As we prove the above theorem by using the atomic decompositions of Hardy spaces, we recall the atomic decompositions in the followings.

Let $B(z,r) = \{x \in \mathbf{R} : |x - z| < r\}$ denote the open ball with center $z \in \mathbf{R}$ and radius $r > 0$. Let $\mathbf{B} = \{B(z,r) : z \in \mathbf{R}, r > 0\}$ and $\mathbf{B}_+ = \{B \in \mathbf{B} : B \subseteq (0,\infty)\}$.

Definition 2.1. Let $1 < q \le \infty$ and $N \in \mathbf{N}$. A Lebesgue measurable function A is a (q,N)-atom

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for $H^p(\mathbf{R})$ if there exists a $B \in \mathbf{B}$ such that

$$\operatorname{supp} A \subseteq \bar{B},$$
$$\|A\|_{L^q} \le |B|^{\frac{1}{q}-\frac{1}{p}} \text{ and}$$
$$\int x^\gamma A(x)dx = 0 \text{ and } \forall \gamma \in \mathbf{N},\ 0 \le \gamma \le N.$$

Theorem 2.2. *Let $0 < p \le 1 < q \le \infty$. For any $N \in \mathbf{N}$ with $N \ge [\frac{1}{p} - 1]$ and $f \in H^p(\mathbf{R})$, we have a family of (q,N)-atoms $\{a_j\}$ and scalars $\{\lambda_j\}$ such that $f = \sum \lambda_j a_j$ in $H^p(\mathbf{R})$ and*

$$\left(\sum |\lambda_j|^p\right)^{1/p} \le C\|f\|_{H^p(\mathbf{R})} \tag{2.1}$$

for some $C > 0$. Furthermore,

$$\|f\|_{H^p(\mathbf{R})} \approx \inf\left\{ \left(\sum |\lambda_j|^p\right)^{1/p} : f = \sum \lambda_j a_j,\ a_j \text{ are } (q,N)\text{-atoms}\right\}.$$

The reader is referred to [5, Theorem 7.4] for the proof of the above result.

We now study the action of $T_{\alpha,\mu}$ on the (q,N)-atom.

Lemma 2.1. *Let $0 < r < \infty$, $1 < q \le \infty$, $\mu \in \mathbf{R}$ and $\alpha \in \mathbf{Z}_-$. If $\frac{1}{q} - \frac{1}{r} < \mu \le \frac{1}{q}$, then for any Lebesgue measurable function a satisfying*

$$\operatorname{supp} a \subseteq \bar{B},\ B \in \mathbf{B}_+, \tag{2.2}$$

$$\|a\|_{L^q} \le |B|^{\frac{1}{q}-\frac{1}{p}} \text{ and} \tag{2.3}$$

$$\int x^{-\alpha} a(x)dx = 0, \tag{2.4}$$

we have

$$\|T_{\alpha,\mu} a\|_{L^r} \le C|B|^{\mu+\frac{1}{r}-\frac{1}{p}}$$

for some $C > 0$.

Proof. Let $\operatorname{supp} a = [c,d] = \bar{B}$. In view of the support condition (2.2) and the vanishing moment condition (2.4) satisfied by a, we find that

$$\int_0^x y^{-\alpha} a(y)dy = 0, \quad x < c \text{ and}$$
$$\int_0^x y^{-\alpha} a(y)dy = 0, \quad x > d.$$

Therefore, $\operatorname{supp}(T_{\alpha,\mu} a) \subseteq [c,d]$.

By the Hölder inequality, we have

$$\left|\int_0^x \frac{a(y)}{y^\alpha} dy\right| \le \|a\|_{L^q} \left(\int_0^x y^{-\alpha q'} dy\right)^{1/q'}$$
$$= C\|a\|_{L^q} x^{-\alpha+\frac{1}{q'}}$$

for some $C > 0$.

Consequently,

$$|T_{\alpha,\mu} a(x)| = x^{\alpha+\mu-1} \left|\int_0^x \frac{a(y)}{y^\alpha} dy\right|$$
$$\le C\|a\|_{L^q} x^{\mu-\frac{1}{q}}.$$

As $\operatorname{supp}(T_{\alpha,\mu} a) \subseteq [c,d]$, we find that

$$\|T_{\alpha,\mu} a(x)\|_{L^r}$$
$$\le C\|a\|_{L^q} \left(\int_c^d x^{r\mu-\frac{r}{q}} dx\right)^{1/r}$$
$$= C\|a\|_{L^q} \left(d^{r\mu-\frac{r}{q}+1} - c^{r\mu-\frac{r}{q}+1}\right)^{1/r}.$$

As $\frac{1}{q} - \frac{1}{r} < \mu \le \frac{1}{q}$, we have $0 < r\mu - \frac{r}{q} + 1 \le 1$. Hence,

$$d^{r\mu-\frac{r}{q}+1} - c^{r\mu-\frac{r}{q}+1} \le (d-c)^{r\mu-\frac{r}{q}+1}.$$

The size condition (2.3) assures that

$$\|T_{\alpha,\mu} a(x)\|_{L^r} \le C|B|^{\mu+\frac{1}{r}-\frac{1}{p}}.$$

□

In Theorem 2.1, we consider $f \in H^p(\mathbf{R})$ with $\operatorname{supp} f \subseteq [0,\infty)$. Notice that the atomic decomposition given in Theorem 2.2 does not guarantee that the supports of the atoms for the atomic decomposition of f are subsets of $[0,\infty)$. In order to tackle this problem, we consider the even part and the odd part of tempered distributions and modify the atomic decomposition obtained in Theorem 2.2.

For any $f \in \mathcal{S}'(\mathbf{R})$, define $f(-\cdot)$ as $\langle f, \varphi\rangle = \langle f(-\cdot), \varphi(-\cdot)\rangle$, $\varphi \in \mathcal{S}(\mathbf{R})$. For any $f \in H^p(\mathbf{R})$, the even part and the odd part of f is defined as $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$, respectively.

Proposition 2.1. *Let $0 < p \le 1 < q \le \infty$ and $\alpha \in \mathbf{Z}_-$. For any $f \in H^p(\mathbf{R})$ with $\operatorname{supp} f \subseteq [0,\infty)$, we have a family of Lebesgue measurable functions $\{a_j\}$ satisfying (2.2)–(2.4) and scalars $\{\lambda_j\}$ such that $f = \sum \lambda_j a_j$ and*

$$\left(\sum |\lambda_j|^p\right)^{1/p} \le C\|f\|_{H^p(\mathbf{R})} \tag{2.5}$$

for some $C > 0$.

Proof. We first consider the case when $|\alpha|$ is even.

According to Theorem 2.2, we have $f = \sum_{j\in\mathbf{Z}} \lambda_j a_j$ where $\{a_j\}_{j\in\mathbf{Z}}$ are (p,N) atoms with $N > |\alpha|$.

We consider the even part of f and find that

$$f_e(x) = \sum_{j\in\mathbf{Z}} \lambda_j \frac{a_j(x) + a_j(-x)}{2}.$$

As a_j satisfies the vanishing moment condition up to order N and $N > |\alpha|$, we find that

$$\frac{1}{2}\int_{\mathbf{R}} x^{-\alpha} a_j(x)dx = \frac{1}{2}\int_{\mathbf{R}} x^{-\alpha} a_j(-x)dx = 0.$$

If $\operatorname{supp} a_j \subset [0,\infty)$, $a_j(-x) \equiv 0$ on $(0,\infty)$. If $\operatorname{supp} a_j \subset (-\infty, 0]$, $a_j(x) \equiv 0$ on $(0,\infty)$ and $a_j(-x)$ is a (p,N) atom. Therefore, they satisfy (2.2)–(2.4).

If 0 is an interior point of $\operatorname{supp} a_j$, we get

$$\int_{\mathbf{R}} x^{-\alpha} \frac{\chi_{[0,\infty)}(x)a_j(x) + \chi_{[0,\infty)}(x)a_j(-x)}{2}dx = \int_{\mathbf{R}} x^{-\alpha} a_j(x)dx = 0.$$

Therefore,

$$\frac{\chi_{(0,\infty)}(x)a_j(x) + \chi_{(0,\infty)}(x)a_j(-x)}{2}$$

satisfies (2.2)–(2.4).

As $\operatorname{supp} f \subseteq [0,\infty)$, we have

$$\begin{aligned} f(x) &= 2\chi_{[0,\infty)}(x)f_e(x) \\ &= 2\sum_{j\in\mathbf{Z}} \lambda_j \frac{\chi_{[0,\infty)}(x)a_j(x) + \chi_{[0,\infty)}(x)a_j(-x)}{2}. \end{aligned}$$

Finally, (2.5) is inherited from (2.1). Therefore, we obtain our desired decomposition for f.

For the case when $|\alpha|$ is odd, we consider the odd part of f. The rest of the proof for this case is almost identical to the proof of the case when $|\alpha|$ is even. The only modification is that for the odd part

$$f_o(x) = \sum_{j\in\mathbf{Z}} \lambda_j \frac{a_j(x) - a_j(-x)}{2},$$

when 0 is the interior point of $\operatorname{supp} a_j$, we have

$$\int_{\mathbf{R}} x^{-\alpha} \frac{\chi_{[0,\infty)}(x)a_j(x) - \chi_{[0,\infty)}(x)a_j(-x)}{2}dx = \int_{\mathbf{R}} x^{-\alpha} a_j(x)dx = 0.$$

This is similar to the case when $|\alpha|$ is even, we find that

$$\begin{aligned} f(x) &= 2\chi_{[0,\infty)}(x)f_o(x) \\ &= 2\sum_{j\in\mathbf{Z}} \lambda_j \frac{\chi_{[0,\infty)}(x)a_j(x) - \chi_{[0,\infty)}(x)a_j(-x)}{2} \end{aligned}$$

which is our desired decomposition. □

We are now ready to present the proof of Theorem 2.1.

Proof of Theorem 2.1. In view of Proposition 2.1, we have a family of Lebesgue measurable functions $\{a_j\}$ and scalars $\{\lambda_j\}$ satisfying (2.2)–(2.5) such that $f = \sum \lambda_j a_j$.

We consider $F = \sum \lambda_j T_{\alpha,\mu} a_j$. As $0 < p \leq 1$ and $0 \leq \mu < 1$, there exists a $q > 1$ such that

$$\frac{1}{q} - \frac{1}{r} < \frac{1}{p} - \frac{1}{r} = \mu < \frac{1}{q}.$$

When $r \leq 1$, $\|\cdot\|^r_{L^r(0,\infty)}$ satisfies the triangle inequality. According to Lemma 2.1, we have

$$\begin{aligned} \|F\|^r_{L^r(0,\infty)} &\leq \sum |\lambda_j|^r \|T_{\alpha,\mu} a_j\|^r_{L^r(0,\infty)} \\ &\leq C\sum |\lambda_j|^r \leq C\Big(\sum |\lambda_j|^p\Big)^{r/p} \end{aligned}$$

for some $C > 0$ because $p \leq r$.

When $r > 1$, as $0 < p \leq 1$ and $\|\cdot\|_{L^r(0,\infty)}$ is a norm, we find that

$$\begin{aligned} \|F\|_{L^r(0,\infty)} &\leq \sum |\lambda_j| \|T_{\alpha,\mu} a_j\|_{L^r(0,\infty)} \\ &\leq C\sum |\lambda_j| \leq C\Big(\sum |\lambda_j|^p\Big)^{1/p}. \end{aligned}$$

Therefore, (2.5) yields that

$$\|F\|_{L^r(0,\infty)} \leq C\|f\|_{H^p(\mathbf{R})}.$$

□

In the proof of Theorem 2.1, we find that we need to use the atomic decompositions of Hardy spaces with (q,N)-atoms satisfying $1 < q < \frac{1}{\mu}$. Notice that a substantial amount of applications of the atomic decomposition can be achieved by considering (∞, N)-atoms.

The above result shows that the atomic decompositions of Hardy spaces by (q,N)-atoms with q close to 1 also yield some valuable application which cannot be obtained by (∞, N)-atoms.

For the atomic decompositions with atoms defined by non-Lebesgue spaces, the reader is referred to [10,12].

3. Rearrangement-invariant Hardy spaces. In this section, we extend the Hardy inequalities to rearrangement-invariant Hardy spaces. We first recall the definition of rearrangement-invariant quasi-Banach function space (r.i.q.B.f.s.) from [11, Definition 4.1].

Let $\mathcal{M}(\mathbf{R})$ be the set of Lebesgue measurable functions on $\mathbf{R}$.

Definition 3.1. A quasi-Banach space $X \subset \mathcal{M}(\mathbf{R})$ is called a rearrangement-invariant quasi-Banach function space if there exists a quasi-norm $\rho_X : \mathcal{M}(0,\infty) \to [0,\infty]$ satisfying
(a) $\rho_X(f) = 0 \Leftrightarrow f = 0$ a.e.,
(b) $|g| \le |f|$ a.e. $\Rightarrow \rho_X(g) \le \rho_X(f)$,
(c) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho_X(f_n) \uparrow \rho_X(f)$ and
(d) $\chi_E \in \mathcal{M}(0,\infty)$ and $|E| < \infty \Rightarrow \rho_X(\chi_E) < \infty$,
so that

$$\|f\|_X = \rho_X(f^*), \quad \forall f \in X. \tag{3.1}$$

For any $s \ge 0$ and $f \in \mathcal{M}(0,\infty)$, define $(D_s f)(t) = f(st)$, $t \in (0,\infty)$. Let $\|D_s\|_{\bar{X}\to\bar{X}}$ be the operator norm of D_s on $\bar{X}$. We recall the definition of Boyd's indices for r.i.q.B.f.s. from [22].

Definition 3.2. Let X be a r.i.q.B.f.s. on $\mathbf{R}$. Define the lower Boyd index of X, p_X and the upper Boyd index of X, q_X as

$$\begin{aligned} p_X &= \sup\{p > 0 : \exists C > 0 \text{ such that} \\ &\quad \forall\, 0 \le s < 1,\ \|D_s\|_{\bar{X}\to\bar{X}} \le Cs^{-1/p}\} \text{ and} \\ q_X &= \inf\{q > 0 : \exists C > 0 \text{ such that} \\ &\quad \forall\, 1 \le s,\ \|D_s\|_{\bar{X}\to\bar{X}} \le Cs^{-1/q}\}, \end{aligned}$$

respectively.

As the definition of interpolation functor involves the notion of category and compatible couples, for simplicity, we refer the reader to [27, Section 1.2] for details of category and compatible couples.

We recall the definition of the K-functional from [2, Section 3.1] and [27, Section 1.3.1].

Definition 3.3. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. For any $f \in X_0 + X_1$, the K-functional is defined as

$$\begin{aligned} &K(f,t,X_0,X_1) \\ &\quad = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\} \end{aligned}$$

where the infimum is taking over all $f = f_0 + f_1$ for which $f_i \in X_i$, $i = 0,1$.

The following interpolation functor is introduced in [15, Definition 4.2].

Definition 3.4. Let $0 < \theta, r < \infty$ and X be a r.i.q.B.f.s. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. The space $(X_0,X_1)_{\theta,r,X}$ consists of all f in $X_0 + X_1$ such that

$$\begin{aligned} &\|f\|_{(X_0,X_1)_{\theta,r,X}} \\ &\quad = \rho_X(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, X_0, X_1)) < \infty \end{aligned}$$

where ρ_X is the quasi-norm given in (3.1).

The above interpolation functor is an extension of the interpolation functor given in Marcinkiewicz real interpolation functor and the interpolation functors in [6,7] for the studies of Lorentz-Karamata spaces and Orlicz spaces, respectively.

We recall a function space associated with the above interpolation from [15, Section 3.1].

Definition 3.5. Let $\alpha \ge 0$. For any r.i.q.B.f.s. X, X_α consists of those $f \in \mathcal{M}(\mathbf{R})$ such that

$$\|f\|_{X_\alpha} = \rho_X(t^{-\alpha} f^*(t)) < \infty.$$

Obviously, from (3.1), we have $X_0 = X$. In [15], we find that X_α is related to the mapping properties of the fractional integral operators, the convolution operators and the Fourier integral operators in r.i.q.B.f.s.

We find that whenever X is a r.i.q.B.f.s., X_α is also a r.i.q.B.f.s.

Proposition 3.1. *Let $\alpha > 0$ and X be a r.i.q.B.f.s. If $0 < p_X \le q_X < \frac{1}{\alpha}$, then X_α is a r.i.q.B.f.s.*

For the proof of the above proposition, the reader is referred to [15, Proposition 3.1].

We have the following theorem from [15, Theorem 4.2] which assures that X_α is an interpolation space from Lebesgue spaces by using the functor $(\cdot,\cdot)_{\theta,r,X}$.

Theorem 3.1. *Let $0 \le \alpha < \infty$, $0 < p_0 < p_1 < \infty$ and X be a r.i.q.B.f.s. with $0 < p_X \le q_X < \frac{1}{\alpha}$. Let r, θ satisfy*

$$\frac{1}{\theta} = \frac{1}{p_0} - \frac{1}{p_1} \quad \text{and} \quad \frac{1}{r} = \frac{1}{p_0} + \alpha. \tag{3.2}$$

Suppose that $p_1 > q_X$, $p_0 < p_X$ and

$$\frac{1}{p_1} + \frac{\alpha}{n} < \frac{1}{q_X} \le \frac{1}{p_X} < \frac{1}{p_0} + \alpha. \tag{3.3}$$

Then

$$(L^{p_0}, L^{p_1})_{\theta,r,X} = X_\alpha.$$

The reader may consult [15, Theorem 4.2] for the proof of the preceding theorem.

We now turn to the definition of rearrangement-invariant Hardy spaces. Let $\mathcal{P}$ denote the class of polynomials on $\mathbf{R}$.

Definition 3.6. Let X be a r.i.q.B.f.s with $0 < p_X \le q_X < \infty$. The rearrangement-invariant Hardy space associated with X, H_X, consists of those $f \in \mathcal{S}'(\mathbf{R})/\mathcal{P}$ such that

$$\|f\|_{H_X} = \left\| \left(\sum_{j\in\mathbf{Z}} |\varphi_j * f|^2 \right)^{1/2} \right\|_X < \infty$$

where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, $j \in \mathbf{Z}$ and $\varphi \in \mathcal{S}(\mathbf{R})$ satisfy

$$\operatorname{supp}\hat{\varphi} \subseteq \{\xi \in \mathbf{R}^n : 1/2 \le |\xi| \le 2\} \text{ and}$$
$$|\hat{\varphi}(\xi)| \ge C, \quad 3/5 \le |\xi| \le 5/3$$

for some $C > 0$.

Notice that H_X is not rearrangement-invariant in terms of the condition given in [2, Chapter 2, Definition 1.2]. For simplicity, we use the absurd terminology "rearrangement-invariant" to name H_X.

If $X = L^p$ with $0 < p \le 1$, we write H_X by H_p.

When $X = L_{p,q}$ where $L_{p,q}$ is a Lorentz space, then H_X becomes the Hardy-Lorentz spaces $H_{p,q}$ studied in [1].

If X is generated by a growth function of lower type Φ (see [28, p. 403]), then H_X is the Hardy type Orlicz spaces H_Φ considered in [23,28].

Theorem 3.2. *Let X be a r.i.q.B.f.s. with $0 < p_X \le q_X < \frac{1}{\alpha}$. Suppose that $0 < p_0 < p_X \le q_X < p_1 < \infty$ and r, θ satisfy* (3.2) *and* (3.3)*. Then,*

$$(H_{p_0}, H_{p_1})_{\theta,r,X} = H_{X_\alpha}.$$

For the proof of Theorem 3.2, the reader is referred to [15, Corollary 8.5].

We are now ready to extend the Hardy inequalities to rearrangement-invariant Hardy spaces.

Theorem 3.3. *Let $0 \le \mu < 1$, $\alpha \in \mathbf{Z}_-$ and X be a r.i.q.B.f.s. with $0 < p_X \le q_X < 1$. Then there exists a constant $C > 0$ such that for any $f \in H_X$ with* $\operatorname{supp} f \subseteq [0, \infty)$*,*

$$\|T_{\alpha,\mu} f\|_{X_\mu(0,\infty)} \le C\|f\|_{H_X}.$$

Proof. In view of Theorem 2.1, we have

$$\|T_{\alpha,\mu} f\|_{L^r(0,\infty)} \le C\|f\|_{H^p(\mathbf{R})}$$

whenever

$$\frac{1}{p} = \frac{1}{r} + \mu.$$

As $0 < p_X \le q_X < 1$, there exist s_1, s_0 such that $q_X < s_1 < 1 < \frac{1}{\mu}$ and $0 < s_0 < p_X$.

The mappings $T_{\alpha,\mu} : H_{s_0} \to L^{q_0}(0,\infty)$ and $T_{\alpha,\mu} : H_{s_1} \to L^{q_1}$ with

$$\frac{1}{s_i} = \frac{1}{q_i} + \mu, \quad i = 0, 1$$

are bounded.

Let $\frac{1}{\theta} = \frac{1}{s_0} - \frac{1}{s_1} = \frac{1}{q_0} - \frac{1}{q_1}$. In addition, as

$$\frac{1}{q_1} + \mu = \frac{1}{s_1} < \frac{1}{q_X} \le \frac{1}{p_X} < \frac{1}{s_0} = \frac{1}{q_0} + \mu,$$

(3.2) and (3.3) are fulfilled for the interpolations $(L^{q_0}, L^{q_1})_{\theta,s_0,X}$.

Theorems 3.1 and 3.2 yield

$$\|T_{\alpha,\mu} f\|_{X_\alpha(0,\infty)} \le C\|T_{\alpha,\mu} f\|_{(L^{q_0},L^{q_1})_{\theta,s_0,X}}$$
$$\le C\|f\|_{(H_{s_0},H_{s_1})_{\theta,s_0,X}} = \|f\|_{H_X}.$$

□

As some special cases of the above theorem, we have Hardy inequalities on the Hardy-Lorentz spaces [1] and the Hardy-Orlicz spaces [23,28].

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