Toric 2-Fano manifolds and extremal contractions

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Abstract: We show that for a projective toric manifold with the ample second Chern character, if there exists a Fano contraction, then it is isomorphic to the projective space. For the case that the second Chern character is nef, the Fano contraction gives either a projective line bundle structure or a direct product structure. We also show that, for a toric weakly 2-Fano manifold, there does not exist a divisorial contraction to a point.

Key words: Toric variety; Mori theory; 2-Fano manifold.

1. Introduction. We assume all algebraic varieties are defined over $\mathbf{C}$. A *Fano* manifold X is a smooth projective algebraic variety with the ample first Chern class $\mathrm{c}_1(X) = -K_X$. In order to study rational surfaces on a Fano manifold, the following special class of Fano manifolds was introduced in de Jong-Starr [4]:

Definition 1.1. A Fano manifold X is a 2-*Fano* (resp. *weakly* 2-*Fano*) manifold if the second Chern character

$$\mathrm{ch}_2(X) = \frac{1}{2}\left((c_1(X))^2 - 2c_2(X)\right)$$

is ample (resp. nef). Here, an algebraic cycle F of codimension 2 on X is *ample* (resp. *nef*) if the intersection number $(F \cdot S)$ is positive (resp. $(F \cdot S) \geq 0$) for any surface S in X.

The classification is an important problem for 2-Fano manifolds, and Schrack [9] shows the following results:

Theorem 1.2 (Schrack [9]). *Let X be a 2-Fano manifold of $\dim X = 4$. Then, the following hold*:

(a) *If there exists a Fano contraction $\varphi_R : X \to \overline{X}$, then $\overline{X}$ is a point.*
(b) *If there exists a divisorial contraction $\varphi_R : X \to \overline{X}$ with the exceptional divisor E, then $\varphi_R(E)$ is not a point.*

The purpose of this paper is to generalize Theorem 1.2 for any dimension d when X is a *toric variety*. Namely, we obtain the following

Theorem 1.3. *Let X be a toric 2-Fano manifold of dimension $d = \dim X \geq 2$. Then, the following hold*:

(a) *If there exists a Fano contraction $\varphi_R : X \to \overline{X}$, then $\overline{X}$ is a point.*
(b) *Suppose that $d \geq 3$. If there exists a divisorial contraction $\varphi_R : X \to \overline{X}$ with the exceptional divisor E, then $\varphi_R(E)$ is not a point.*

We will prove (a) of Theorem 1.3 in Section 3 without the assumption that X is Fano. Moreover, we determine the structure of a projective toric manifold X with the nef second Chern character $\mathrm{ch}_2(X)$ and a Fano contraction. For (b), we need the assumption that X is Fano. Namely, we use the classification of toric Fano manifolds. The assertion (b) is generalized for toric weakly 2-Fano manifolds.

2. Preliminaries. In this section, for the proof of the main theorem, we explain the way to calculate intersection numbers for toric cycles, and the notion of primitive collections and primitive relations. For the fundamental properties of toric manifolds, we refer to Cox-Little-Schenck [2], Fulton [3] and Oda [6].

Let $X = X_\Sigma$ be a smooth complete toric d-fold associated to a fan Σ in $\mathbf{Z}^d$. We put $\mathrm{G}(\Sigma)$ as the set of the primitive generators for the 1-dimensional cones in Σ. Let $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_m\}$ and $D_i = D_{x_i}$ be the torus invariant divisor corresponding to x_i. For a torus invariant subvariety $Y \subset X$ of codimension l, we define the polynomial $I_{Y/X} = I_{Y/X}(X_1, \ldots, X_m) \in R_X := \mathbf{Z}[X_1, \ldots, X_m]$, by introducing the independent elements $X_1, \ldots, X_m$ associated to $x_1, \ldots, x_m$, respectively, as

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$$I_{X/Y} := \sum_{1\le i_1,\dots,i_l\le m} (D_{i_1}\cdots D_{i_l}\cdot Y)X_{i_1}\cdots X_{i_l}.$$

We can say that $I_{Y/X}$ has all the numerical information of Y on X.

Example 2.1. For a torus invariant curve $C = C_\sigma \subset X$ corresponding to a $(d-1)$-dimensional cone $\sigma \in \Sigma$, let

$$y_1 + y_2 + a_1x_1 + \cdots + a_{d-1}x_{d-1} = 0$$

be the *wall relation* associated to σ, where $y_1, y_2, x_1, \dots, x_{d-1} \in \mathrm{G}(\Sigma)$ and $a_1, \dots, a_{d-1} \in \mathbf{Z}$. Then, $I_{C/X} = Y_1 + Y_2 + a_1X_1 + \cdots + a_{d-1}X_{d-1} \in R_X$, where $X_1, \dots, X_{d-1}, Y_1, Y_2$ are the independent elements in R_X corresponding to $x_1, \dots, x_{d-1}, y_1, y_2$, respectively.

So, we can easily calculate $I_{C/X}$ for a torus invariant curve C. For the case of surfaces, the following holds:

Theorem 2.2 (Sato [8])**.** *Let $S \subset X$ be a torus invariant surface. Then the following hold*:

(a) *If $S \cong \mathbf{P}^2$, then $I_{S/X} = (I_{C/X})^2$ for a torus invariant curve $C \subset S$.*

(b) *If S is isomorphic to the Hirzebruch surface F_a of degree $a \ge 0$, then*

$$I_{S/X} = a(I_{C_{\mathrm{fib}}/X})^2 + 2(I_{C_{\mathrm{fib}}/X})(I_{C_{\mathrm{neg}}/X}),$$

where C_{fib} is the fiber of $S \to \mathbf{P}^1$, while C_{neg} is the negative section of S.

For a toric variety X, it is well known that

$$\mathrm{ch}_2(X) = \frac{1}{2}(D_1^2 + \cdots + D_m^2).$$

So, for a torus invariant surface $S \subset X$, we can calculate $(\mathrm{ch}_2(X)\cdot S)$ from $I_{S/X}$ easily. Therefore, Theorem 2.2 is crucial in our proof. We note that for the toric case, in order to check the ampleness or nefness for $\mathrm{ch}_2(X)$, it suffices to consider the intersection numbers with *torus invariant* surfaces. Namely, the following holds:

Proposition 2.3 (Nobili [5])**.** *For a projective toric manifold X, $\mathrm{ch}_2(X)$ is ample* (*resp. nef*) *if and only if* $(\mathrm{ch}_2(X)\cdot S) > 0$ (*resp.* $(\mathrm{ch}_2(X)\cdot S) \ge 0$) *for any torus invariant surface S in X.*

Next, we briefly introduce the notion of primitive collections and primitive relations. We will use these concepts to describe the fans for certain toric Fano manifolds in Section 4.

Definition 2.4. Let $X = X_\Sigma$ be the smooth complete toric d-fold associated to a fan Σ. A subset $P \subset \mathrm{G}(\Sigma)$ is called a *primitive collection* if it does not generate a cone in Σ, while any proper subset generates a cone in Σ.

For a primitive collection $P = \{x_1, \dots, x_s\} \subset \mathrm{G}(\Sigma)$, there exists the unique element $\sigma(P) \in \Sigma$ such that $x_1 + \cdots + x_s$ is contained in the relative interior of $\sigma(P)$. So, we have a linear relation

$$x_1 + \cdots + x_s = a_1y_1 + \cdots + a_ty_t,$$

where $\{y_1, \dots, y_t\}$ is the set of generators for $\sigma(P)$ and $a_1, \dots, a_t$ are positive integers. We call it the *primitive relation* corresponding to P.

We can recover the fan Σ from the data of all the primitive relations (see Proposition 3.6 in Sato [7]). So, we can describe a fan by giving all the primitive relations. We also remark that the primitive collections and primitive relations are convenient to deal with the toric Mori theory.

3. Fano contractions.

The following theorem is an assertion for not necessarily Fano toric varieties.

Theorem 3.1. *Let X be a smooth projective toric d-fold, and suppose that there exists a Fano contraction $\varphi_R : X \to \overline{X}$. Then, the following hold*:

(a) *If $\mathrm{ch}_2(X)$ is ample, then $\overline{X}$ is a point, that is, X is isomorphic to $\mathbf{P}^d$.*

(b) *If $\mathrm{ch}_2(X)$ is nef but not ample, then φ_R gives either a $\mathbf{P}^1$-bundle structure or a direct product structure.*

Proof. Since $X = X_\Sigma$ is a smooth toric variety, φ_R simply gives a projective space bundle structure. So, let $s-1$ be the dimension of a fiber of φ_R. There exists the primitive relation $x_1 + \cdots + x_s = 0$ associated to φ_R, where $\{x_1, \dots, x_s\} \subset \mathrm{G}(\Sigma)$. Suppose that $s - 1 < d$, that is, $\dim \overline{X} > 0$. Then, we can take a $(d-1)$-dimensional cone in Σ generated by $\{x_1, \dots, x_{s-1}, z_1, \dots, z_{d-s}\} \subset \mathrm{G}(\Sigma)$. Let

$$y_1 + y_2 + a_1x_1 + \cdots + a_{s-1}x_{s-1} + b_1z_1 + \cdots + b_{d-s}z_{d-s} = 0$$

be the associated wall relation, where both $\{y_1, x_1, \dots, x_{s-1}, z_1, \dots, z_{d-s}\}$ and $\{y_2, x_1, \dots, x_{s-1}, z_1, \dots, z_{d-s}\}$ generate maximal cones in Σ, for $y_1 \ne y_2 \in \mathrm{G}(\Sigma)$ and $a_1, \dots, a_{s-1}, b_1, \dots, b_{d-s} \in \mathbf{Z}$. If $\max\{a_1, \dots, a_{s-1}\} = a_i > 0$, then by the equality $x_i = -(x_1 + \cdots + \check{x}_i + \cdots + x_s)$, the above wall relation becomes

$$y_1 + y_2 + (a_1 - a_i)x_1 + \cdots + \check{x}_i + \cdots + (a_{s-1} - a_i)x_{s-1} + (-a_i)x_s + b_1z_1 + \cdots$$

$$+ b_{d-s}z_{d-s} = 0.$$

Therefore, by reordering $x_1, \ldots, x_s$, we can assume that $a_1 \le \cdots \le a_{s-1} \le 0$. Put τ as the $(d-2)$-dimensional cone in Σ whose generators are $x_1, \ldots, x_{s-2}, z_1, \ldots, z_{d-s}$. Since $x_{s-1} + x_s = 0$ and $y_1 + y_2 = (-a_{s-1})x_{s-1}$ in $\mathbf{R}^d/\mathrm{Span}\tau$, the torus invariant subsurface $S = S_\tau \subset X$ associated to τ is isomorphic to the Hirzebruch surface $F_{-a_{s-1}}$ of degree $-a_{s-1} \ge 0$. So, Theorem 2.2 says that

$$\begin{aligned} I_{S/X} = & -a_{s-1}(X_1 + \cdots + X_s)^2 \\ & + 2(X_1 + \cdots + X_s)(Y_1 + Y_2 + a_1X_1 + \cdots \\ & + a_{s-1}X_{s-1} + b_1Z_1 + \cdots + b_{d-s}Z_{d-s}), \end{aligned}$$

where $X_1, \ldots, X_s, Y_1, Y_2, Z_1, \ldots, Z_{d-s}$ are the independent elements in R_X corresponding to $x_1, \ldots, x_s, y_1, y_2, z_1, \ldots, z_{d-s}$, respectively. Therefore, we have

$$\begin{aligned} 2(\mathrm{ch}_2(X) \cdot S) &= -sa_{s-1} + 2(a_1 + \cdots + a_{s-1}) \\ &= (a_1 - a_{s-1}) + \cdots + (a_{s-1} - a_{s-1}) \\ &\quad + a_1 + \cdots + a_{s-2} \le 0. \end{aligned}$$

In particular, $\mathrm{ch}_2(X)$ is *not* ample.

If φ_R is a $\mathbf{P}^1$-bundle structure, that is, $s = 2$, then $(\mathrm{ch}_2(X) \cdot S) = 0$. If $s > 2$, then the above equality says that $a_1 = \cdots = a_{s-1} = 0$. In this case, X becomes a direct product of $\overline{X}$ and a fiber of φ_R. □

By assuming X to be a Fano manifold, we have the following. The former is (a) in Theorem 1.3:

Corollary 3.2. *Let X be a smooth toric 2-Fano d-fold, and suppose that there exists a Fano contraction $\varphi_R : X \to \overline{X}$. Then, $\dim \overline{X} = 0$, that is, X is isomorphic to the d-dimensional projective space $\mathbf{P}^d$.*

Corollary 3.3. *Let X be a smooth toric weakly 2-Fano d-fold, and suppose that there exists a Fano contraction $\varphi_R : X \to \overline{X}$ such that $\dim \overline{X} > 0$. Then, X is either a projective line bundle over $\overline{X}$ or the direct product of $\overline{X}$ and a fiber of φ_R.*

4. Divisorial contractions. In this section, we give the proof of (b) of Theorem 1.3. First, we suppose $d \ge 3$. The case $d = 2$ will be studied later. Let X be a toric Fano manifold equipped with a divisorial contraction $\varphi_R : X \to \overline{X}$ such that $\dim \varphi_R(E) = 0$, where E is the exceptional divisor. In this case, we need the condition where X is *Fano* for our proof. Toric Fano manifolds with such contractions are completely classified by Bonavero [1]. There exist the following two cases:

(b1) X is a $\mathbf{P}^1$-bundle over $\mathbf{P}^{d-1}$: $\mathbf{P}_{\mathbf{P}^{d-1}}(\mathcal{O} \oplus \mathcal{O}(\alpha))$ $(1 < \alpha < d)$.

(b2) The Picard number of $X = X_\Sigma$ is 3 and the primitive relations of Σ are

$$x_1 + \cdots + x_d = \alpha x_{d+1},$$
$$x_2 + \cdots + x_d + x_{d+3} = (\alpha - 1)x_{d+1},$$
$$x_1 + x_{d+2} = x_{d+3},\ x_{d+1} + x_{d+2} = 0$$
$$\text{and } x_{d+1} + x_{d+3} = x_1,$$

where $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_{d+3}\}$ and $1 \le \alpha \le d-1$.

In the case (b1), X has a Fano contraction. So, we can use the results of Section 3. Therefore, it suffices to consider the case (b2).

Put $\tau \in \Sigma$ as the $(d-2)$-dimensional cone generated by $\mathrm{G}(\tau) = \{x_2, \ldots, x_{d-2}, x_{d+3}\}$. By the above primitive relations, we see that there exist exactly 4 maximal cones in Σ which contain τ, i.e., the cones generated by $\mathrm{G}(\tau) \cup \{x_1, x_{d-1}\}$, $\mathrm{G}(\tau) \cup \{x_1, x_d\}$, $\mathrm{G}(\tau) \cup \{x_{d-1}, x_{d+2}\}$ and $\mathrm{G}(\tau) \cup \{x_d, x_{d+2}\}$, respectively. So, the associated torus invariant subsurface $S = S_\tau \subset X$ is isomorphic to a Hirzebruch surface. Since $x_1 + x_{d+2} = 0$ and $x_{d-1} + x_d = (\alpha - 1)x_1$ in $\mathbf{R}^d/\mathrm{Span}\tau$, its degree is $\alpha - 1$. Obviously, the wall relation corresponding to the fiber $C_{\mathrm{fib}} \subset X$ of $S \to \mathbf{P}^1$ is

$$x_1 + x_{d+2} - x_{d+3} = 0.$$

On the other hand, the wall corresponding to the negative section $C_{\mathrm{neg}} \subset X$ of S is generated by $\{x_1, \ldots, x_{d-2}, x_{d+3}\}$. Therefore, its wall relation is

$$-(\alpha - 1)x_1 + x_2 + \cdots + x_d + \alpha x_{d+3} = 0.$$

So, by Theorem 2.2, we have

$$\begin{aligned} I_{S/X} = & (\alpha - 1)(X_1 + X_{d+2} - X_{d+3})^2 \\ & + 2(X_1 + X_{d+2} - X_{d+3})(-(\alpha - 1)X_1 + X_2 + \cdots \\ & + X_d + \alpha X_{d+3}), \end{aligned}$$

where $X_1, \ldots, X_{d+3}$ are the independent elements in R_X corresponding to $x_1, \ldots, x_{d+3}$, respectively, and

$$\begin{aligned} (\mathrm{ch}_2(X) \cdot S) &= 3(\alpha - 1) - 2(\alpha - 1) - 2\alpha \\ &= -(\alpha + 1) < 0. \end{aligned}$$

Namely, X is *not* a weakly 2-Fano manifold. So, we obtain the following result which implies (b) of Theorem 1.3:

Theorem 4.1. *Let X be a toric weakly 2-Fano manifold. If there exists a divisorial contrac-*

tion $\varphi_R : X \to \overline{X}$*, then* $\dim \varphi(E) > 0$ *for the exceptional divisor* E *of* φ_R*.*

Remark 4.2. There exists a toric variety of Picard number 3 determined by the primitive relations in (b2) for every $\alpha \geq 1$. The argument above shows that the second Chern character of the toric variety is not ample for every α.

For a smooth projective toric surface $X = X_\Sigma$, its second Chern character is calculated as $\mathrm{ch}_2(X) = (12 - 3m)/2$, where m is the number of 1-dimensional cones in Σ. Thus, the following is obvious:

Proposition 4.3. *Let* X *be a smooth projective toric surface. If* $\mathrm{ch}_2(X)$ *is nef but not ample, then* X *is isomorphic to a Hirzebruch surface.*

Remark 4.4. Smooth toric weakly 2-Fano d-folds are completely classified for $d \leq 4$ by Nobili [5] and Sato [8].

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